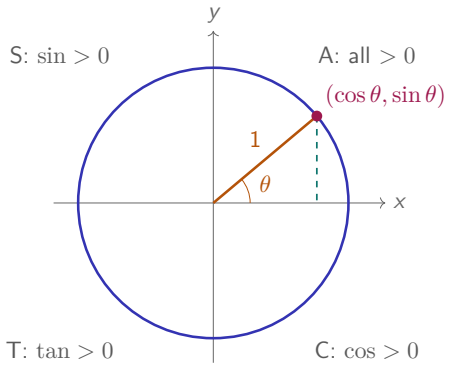
# Trigonometry

AQA A-level Mathematics 7357, section E: triangles, radians, graphs, identities, compound angles, equations

# Exact values to know

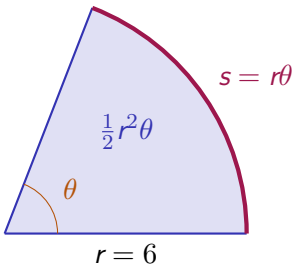
| $\theta$ | $\sin\theta$ | $\cos\theta$ | $\tan\theta$ |
|---|---|---|---|
| $0$ | $0$ | $1$ | $0$ |
| $30° = \frac{\pi}{6}$ | $\frac{1}{2}$ | $\frac{\sqrt{3}}{2}$ | $\frac{1}{\sqrt{3}}$ |
| $45° = \frac{\pi}{4}$ | $\frac{\sqrt{2}}{2}$ | $\frac{\sqrt{2}}{2}$ | $1$ |
| $60° = \frac{\pi}{3}$ | $\frac{\sqrt{3}}{2}$ | $\frac{1}{2}$ | $\sqrt{3}$ |
| $90° = \frac{\pi}{2}$ | $1$ | $0$ | undefined |

# The unit circle




- The point at angle $\theta$ is $(\cos\theta, \sin\theta)$.
- Signs follow the **quadrant**.
- $\sin^2\theta + \cos^2\theta = 1$ is Pythagoras.

# Radians, arcs and sectors

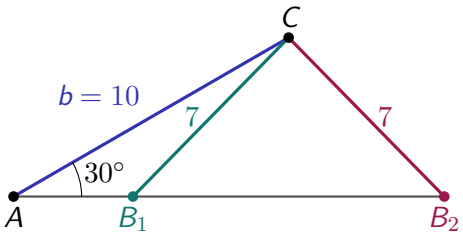



- $\pi$ radians $= 180°$.
- Arc and **sector** formulas need radians.
- $r = 6$ cm, $\theta = 1.2$: $s = 7.2$ cm, $A = 21.6$ cm$^2$.

$$s = r\theta, \qquad A = \tfrac{1}{2}r^2\theta$$

$\theta$ in radians

# The ambiguous case, drawn




- $A = 30°$, $b = 10$, $a = 7$.
- $\sin B = \frac{10 \sin 30°}{7} = \frac{5}{7}$.
- $B = 45.6°$ or $134.4°$: two triangles.

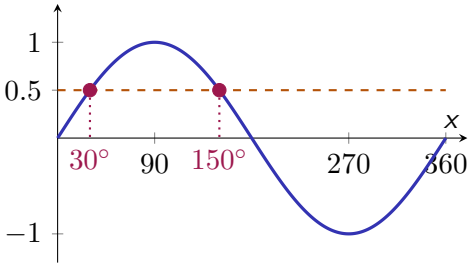
$$\frac{a}{\sin A} = \frac{b}{\sin B}$$

# Worked: cosine rule and area

1. **Given** sides 6 and 8 enclose $60^\circ$
2. **Choose** included angle known: **cosine rule**
3. **Substitute** $c^2 = 6^2 + 8^2 - 2(6)(8)\cos 60^\circ = 100 - 48 = 52$
4. **Side** $c = \sqrt{52} = 2\sqrt{13} \approx 7.21$
5. **Area** $\frac{1}{2}(6)(8)\sin 60^\circ = 24 \cdot \frac{\sqrt{3}}{2} = 12\sqrt{3}$

$$a^2 = b^2 + c^2 - 2bc\cos A, \qquad \text{Area} = \tfrac{1}{2}ab\sin C$$

# Solving on the graph




- $\sin x = 0.5$, $0° \le x < 360°$.
- **Principal value** $30°$; symmetry gives $180° - 30°$.
- $x = 30°$ or $150°$.

For $\sin 2x = 0.5$, solve on $0 \le 2x < 720°$ first: four solutions.

# The identities

| **Identity** | **Comes from** |
| --- | --- |
| $\tan\theta \equiv \dfrac{\sin\theta}{\cos\theta}$ | the unit-circle coordinates |
| $\sin^2\theta + \cos^2\theta \equiv 1$ | Pythagoras on the **unit circle** |
| $1 + \tan^2\theta \equiv \sec^2\theta$ | divide by $\cos^2\theta$ |
| $1 + \cot^2\theta \equiv \csc^2\theta$ | divide by $\sin^2\theta$ |

$\sec\theta = \frac{1}{\cos\theta}$, $\csc\theta = \frac{1}{\sin\theta}$, $\cot\theta = \frac{1}{\tan\theta}$.

# Worked: a hidden quadratic

1. **Replace** $2(1-\sin^2 x)+3\sin x=3$
2. **Rearrange** $2\sin^2 x-3\sin x+1=0$
3. **Factorise** $(2\sin x-1)(\sin x-1)=0$
4. **Solve** $\sin x=\frac{1}{2}$: $30^\circ, 150^\circ$; $\sin x=1$: $90^\circ$

## Decision

$x=30^\circ,\ 90^\circ,\ 150^\circ$

# Compound angles

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$
$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$

1. **Split** $\sin 75^\circ = \sin(45^\circ + 30^\circ)$
2. **Expand** $\sin 45^\circ \cos 30^\circ + \cos 45^\circ \sin 30^\circ$
3. **Exact** $\frac{\sqrt{2}}{2} \cdot \frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2} \cdot \frac{1}{2} = \frac{\sqrt{6}+\sqrt{2}}{4}$

## Decision

Check: $\frac{\sqrt{6}+\sqrt{2}}{4} = 0.9659 = \sin 75^\circ$ ✓

# Double angles: never divide by $\sin x$

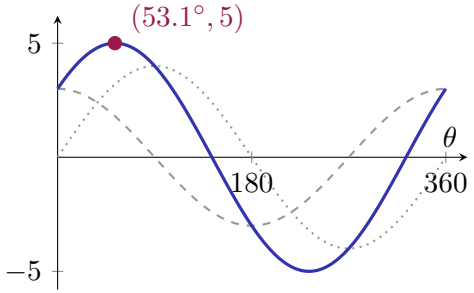
$$\sin 2A = 2\sin A\cos A, \quad \cos 2A = 2\cos^2 A - 1 = 1 - 2\sin^2 A$$

1. **Expand** $2\sin x\cos x = \sin x$
2. **Factorise** $\sin x(2\cos x - 1) = 0$
3. **Branch 1** $\sin x = 0$: $x = 0^\circ, 180^\circ$
4. **Branch 2** $\cos x = \frac{1}{2}$: $x = 60^\circ, 300^\circ$

**Decision**

Four solutions; dividing by $\sin x$ would lose $0^\circ$ and $180^\circ$.

# One wave: $3\cos\theta + 4\sin\theta$




- $R\cos\alpha = 3$, $R\sin\alpha = 4$.
- $R = 5$, $\tan\alpha = \frac{4}{3}$, $\alpha = 53.13°$.
- Maximum 5 at $\theta = 53.13°$.

$$3\cos\theta + 4\sin\theta = 5\cos(\theta - 53.13°)$$

# Small-angle approximations

| **Function** | **Approximation** | **At $\theta = 0.1$** |
|---|---|---|
| $\sin\theta$ | $\approx \theta$ | $0.09983$ vs $0.1$ |
| $\cos\theta$ | $\approx 1 - \frac{\theta^2}{2}$ | $0.995004$ vs $0.995$ |
| $\tan\theta$ | $\approx \theta$ | $0.10033$ vs $0.1$ |

$$\frac{1-\cos 4\theta}{\theta \sin\theta} \approx \frac{8\theta^2}{\theta^2} = 8 \text{ for small } \theta.$$

# Proving an identity

1. **Start LHS** $\dfrac{1-\cos 2x}{\sin 2x}$
2. **Double angles** $= \dfrac{1-(1-2\sin^2 x)}{2\sin x\cos x}$
3. **Simplify** $= \dfrac{2\sin^2 x}{2\sin x\cos x} = \dfrac{\sin x}{\cos x}$
4. **Finish** $= \tan x \equiv$ RHS

**Decision**

Never write the identity at the top and work on both sides.

# Inverse trig functions

| Function | Domain | Range (principal values) |
|---|---|---|
| $\arcsin x$ | $-1 \leq x \leq 1$ | $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$ |
| $\arccos x$ | $-1 \leq x \leq 1$ | $0 \leq y \leq \pi$ |
| $\arctan x$ | all real $x$ | $-\frac{\pi}{2} < y < \frac{\pi}{2}$ |

$\arcsin(\sin \frac{5\pi}{6}) = \frac{\pi}{6}$, not $\frac{5\pi}{6}$.

# Check yourself

| **Question** | **Answer** |
|---|---|
| $\cos \frac{2\pi}{3}$? | $-\frac{1}{2}$ |
| Arc, $r = 5$, $\theta = 0.8$? | $4$ |
| Solutions of $\tan x = 1$, $0 \leq x < 2\pi$? | $\frac{\pi}{4}, \frac{5\pi}{4}$ |
| $1 + \tan^2 \theta \equiv$? | $\sec^2 \theta$ |
| Max of $5\cos(\theta - \alpha) + 2$? | 7 |