Sequences and series

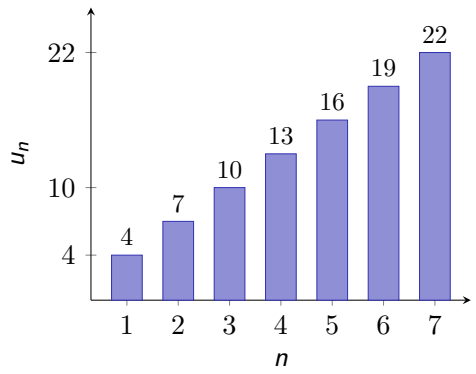
AQA A-level Mathematics 7357, section D: binomial expansion, arithmetic and geometric sequences, recurrences, sigma notation

The formulae of section D

Sequence	**nth term**	**Sum of n terms**
Arithmetic	$a + (n-1)d$	$\frac{n}{2}[2a + (n-1)d] = \frac{n}{2}(a + l)$
Geometric	ar^{n-1}	$\frac{a(1-r^n)}{1-r}$, $r \neq 1$
Geometric, $\lvert r \rvert < 1$	$\to 0$	$S_\infty = \frac{a}{1-r}$

a first term, d **common difference**, r **common ratio**, l last term.

An arithmetic sequence from two terms



- $u_3 = 10,\ u_7 = 22.$
- $a + 2d = 10,\ a + 6d = 22:$ $4d = 12.$
- $d = 3,\ a = 4.$

$$u_n = a + (n-1)d = 3n + 1$$

Summing by pairing the ends

5	8	11	14	17	20	23	26
26	23	20	17	14	11	8	5
31	31	31	31	31	31	31	31

$2S_8 = 8 \times 31$, so $S_8 = 124$

- $a = 5$, $d = 3$, $n = 8$: $u_8 = 26$.
- Each end-pair totals 31.
- $S_8 = \frac{8}{2}(5 + 26) = 124$.

$$S_n = \frac{n}{2}(a + l)$$

Sigma notation

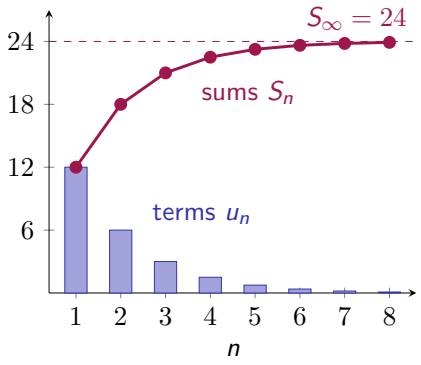
1. **Read** $\sum_{r=1}^{20}(3r+2) = 5+8+11+\cdots+62$
2. **Identify** arithmetic, $a = 5$, $l = 62$, $n = 20$
3. **Sum** $\frac{20}{2}(5+62) = 670$

Decision

$\sum_{r=5}^{20}$ has $20-5+1 = 16$ terms, not 15.

$$\sum_{r=1}^{n} 1 = n, \qquad \sum_{r=1}^{n} r = \tfrac{1}{2}n(n+1)$$

Geometric partial sums settle down



- $a = 12$, $r = 0.5$: $12, 6, 3, 1.5, \ldots$
- $S_n = 24(1 - 0.5^n)$.
- $S_\infty = \frac{12}{1-0.5} = 24$.

Worked: the first sum beyond 23.9

1. **Set up** $24(1-0.5^n) > 23.9$, so $0.5^n < \frac{1}{240}$
2. **Logs** $n \ln 0.5 < -\ln 240$; dividing by $\ln 0.5 < 0$ flips the sign
3. **Solve** $n > \frac{\ln 240}{\ln 2} = 7.91$
4. **Check** $S_7 = 23.8125 < 23.9$, $S_8 = 23.90625 > 23.9$

Decision

$n = 8$

When does a geometric series converge?

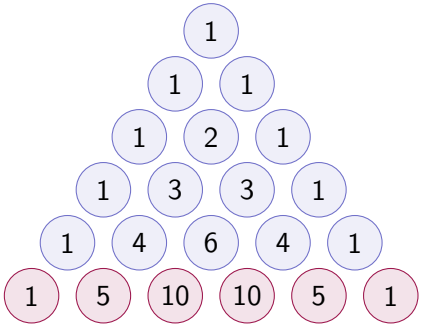
Ratio	**Terms**	**Sum to infinity**
$r = 0.5$	$12, 6, 3, 1.5, \ldots$	$\frac{12}{0.5} = 24$
$r = -0.5$	$12, -6, 3, -1.5, \ldots$	$\frac{12}{1.5} = 8$
$r = 1$	$12, 12, 12, \ldots$	none: grows without **limit**
$r = 2$	$12, 24, 48, \ldots$	none: **diverges**

Recurrence relations

Rule	Start	Terms	Behaviour
$u_{n+1} = 2u_n - 1$	$u_1 = 3$	$3, 5, 9, 17, \ldots$	**increasing**
$u_{n+1} = 0.5u_n + 3$	$u_1 = 10$	$10, 8, 7, 6.5, \ldots$	**decreasing**, to 6
$u_{n+1} = \dfrac{1}{1 - u_n}$	$u_1 = 2$	$2, -1, \frac{1}{2}, 2, \ldots$	**periodic**, order 3

A limit L satisfies $L = 0.5L + 3$, so $L = 6$.

Pascal's triangle and $\binom{n}{r}$



$n = 5$

- $(a+b)^n = \sum \binom{n}{r} a^{n-r} b^r$.
- $(2+x)^5 = 32 + 80x + 80x^2 + \ldots$

$$\binom{n}{r} = \frac{n!}{r!\,(n-r)!}$$

Binomial expansion for any n

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots$$

1. **n = ½** $(1-2x)^{1/2} = 1 - x - \frac{1}{2}x^2 - \frac{1}{2}x^3 - \ldots$
2. **Validity** $|-2x| < 1$, so $|x| < \frac{1}{2}$
3. **Estimate** $x = 0.01$: $\sqrt{0.98} \approx 0.9899495$

Decision

Calculator: $\sqrt{0.98} = 0.98994949\ldots$; the error is in the 8th decimal place.

Validity when the constant is not 1

1. **Factor** $(4+x)^{-1/2} = 4^{-1/2}\left(1+\frac{x}{4}\right)^{-1/2}$
2. **Expand** $\frac{1}{2}\left(1-\frac{x}{8}+\frac{3x^2}{128}-\dots\right)$
3. **Simplify** $\frac{1}{2}-\frac{x}{16}+\frac{3x^2}{256}-\dots$
4. **Validity** $\left|\frac{x}{4}\right|<1$, so $|x|<4$

$$(a+bx)^n = a^n\left(1+\frac{b}{a}x\right)^n, \quad |x|<\left|\frac{a}{b}\right|$$

A model: a salary rising 3% a year

1. **Model** year n salary $25\,000 \times 1.03^{n-1}$: geometric
2. **Year 10** $25\,000 \times 1.03^{9} = 32\,619$
3. **Total** $S_{10} = \dfrac{25\,000(1.03^{10} - 1)}{1.03 - 1} = 286\,597$

Decision

About 286 600 pounds; the model assumes the rise never changes.

Traps in sequences and series

Slips

$n = 7.91$ rounded to 8 without checking S_7 and S_8.

Forgetting to flip the inequality when dividing by $\ln 0.5$.

Quoting S_∞ for $r = 1.2$.

Habits

A term count is a positive integer; test both neighbours.

$\ln$ of a number below 1 is negative.

State the validity, $|x| < \ldots$, with every non-integer expansion.

Check yourself

Question	**Answer**
Sum $2 + 5 + 8 + \ldots$ to 10 terms?	$5(4 + 27) = 155$
S_∞ of $8, 4, 2, \ldots$?	16
$\sum_{r=1}^{4} 2^r$?	30
Validity of $(1 + 3x)^{-1}$?	$\lvert x \rvert < \frac{1}{3}$
Coefficient of x^2 in $(1 + x)^6$?	15