Algebra and functions

AQA A-level Mathematics 7357, section B: quadratics, polynomials, partial fractions, modulus, functions, transformations

The section B toolkit

Tool	Key fact
Completed square	$a(x-h)^2+k$ has **vertex** (h,k)
Discriminant	b^2-4ac counts real roots
Factor theorem	$f(a)=0 \iff (x-a)$ is a factor
Function	one output for each input of the domain
Inverse	exists only if one-to-one; reflect in $y=x$

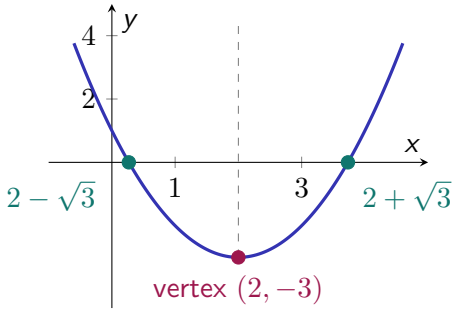
Surds kept exact

1. **Simplify** $\sqrt{48} = \sqrt{16 \times 3} = 4\sqrt{3}$
2. **Conjugate** $(3 - \sqrt{3})(3 + \sqrt{3}) = 9 - 3 = 6$
3. **Rationalise** $\dfrac{6}{3 - \sqrt{3}} \cdot \dfrac{3 + \sqrt{3}}{3 + \sqrt{3}} = \dfrac{6(3 + \sqrt{3})}{6}$
4. **Result** $3 + \sqrt{3} \approx 4.732$

$$(a - \sqrt{b})(a + \sqrt{b}) = a^2 - b$$

the **conjugate** clears the **surd** from the denominator

Completing the square

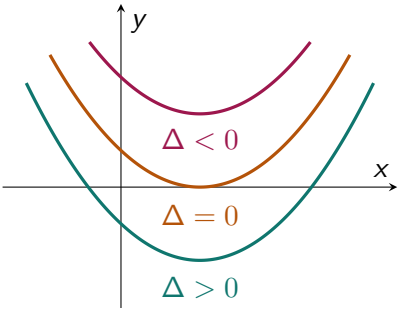


- $x^2 - 4x + 1 = (x-2)^2 - 3$.
- **Vertex** $(2,-3)$; roots $2 \pm \sqrt{3}$.
- Check: $(2-\sqrt{3})^2 - 4(2-\sqrt{3}) + 1 = 0$.

$$a(x-h)^2 + k$$

vertex (h,k), on the line of symmetry

The discriminant counts roots



Δ	Real roots
> 0	two distinct
$= 0$	one repeated
< 0	none

Worked: a range of k

1. **Given** $x^2 + kx + 9 = 0$ has two distinct real roots
2. **Coefficients** $a = 1$, $b = k$, $c = 9$
3. **Condition** $k^2 - 4(1)(9) > 0$, so $k^2 > 36$
4. **Critical values** $k = \pm 6$
5. **Region** $k^2 - 36$ is a parabola in k, positive outside its roots

Decision

$k < -6$ or $k > 6$ (not $-6 < k < 6$)

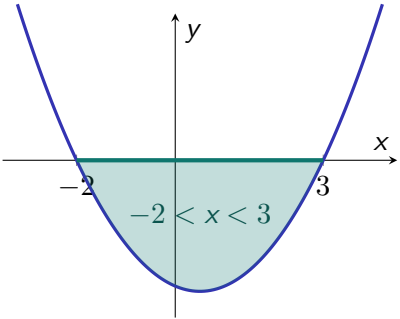
Worked: a line meets a circle

1. **Given** $y = x + 1$ and $x^2 + y^2 = 25$
2. **Substitute** $x^2 + (x+1)^2 = 25$
3. **Expand** $2x^2 + 2x - 24 = 0$, so $x^2 + x - 12 = 0$
4. **Factorise** $(x+4)(x-3) = 0$: $x = 3$ or $x = -4$
5. **Pair up** $y = x + 1$ gives $y = 4$ or $y = -3$

Decision

Two points: $(3, 4)$ and $(-4, -3)$; pair each x with its own y.

A quadratic inequality, drawn



- Solve $x^2 - x - 6 < 0$.
- $(x-3)(x+2)$: **critical values** -2 and 3.
- Below the axis between the roots.

$$-2 < x < 3$$

strict inequality: the end points are excluded

The factor theorem

1. **Test** $f(x) = x^3 - 2x^2 - 5x + 6$: $f(1) = 1 - 2 - 5 + 6 = 0$
2. **Factor** $(x - 1)$ is a factor
3. **Divide** $f(x) = (x - 1)(x^2 - x - 6)$
4. **Finish** $f(x) = (x - 1)(x - 3)(x + 2)$

$$f(a) = 0 \iff (x - a) \text{ is a factor of } f(x)$$

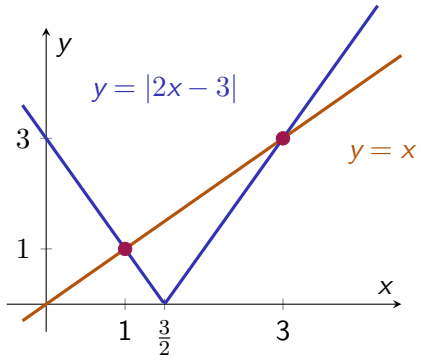
Partial fractions

1. **Form** $\dfrac{5x+1}{(x-1)(x+2)} \equiv \dfrac{A}{x-1} + \dfrac{B}{x+2}$
2. **Clear** $5x+1 \equiv A(x+2) + B(x-1)$
3. **x = 1** $6 = 3A$, so $A = 2$
4. **x = −2** $-9 = -3B$, so $B = 3$

Decision

$\dfrac{2}{x-1} + \dfrac{3}{x+2}$; check at $x = 0$: $-2 + \frac{3}{2} = -\frac{1}{2}$ ✓

A modulus equation, drawn



- $|2x-3| = x$.
- Branches: $2x-3 = x$ or $-(2x-3) = x$.
- $x = 3$ or $x = 1$; both satisfy $x \geq 0$.

Functions, domain and range

Many-to-one

$f(x) = x^2$, $x \in \mathbb{R}$: **range** $f(x) \geq 0$.

$f(2) = f(-2) = 4$: no inverse.

One-to-one

$f(x) = x^2$, $x \geq 0$: range $f(x) \geq 0$.

Each output has one input: an inverse exists.

Domain the set of inputs a function is defined on; restricting it can make a function one-to-one.

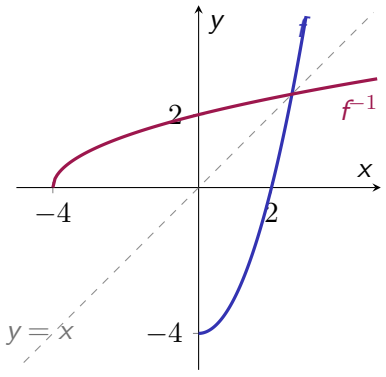
Composite functions

1. **Define** $f(x) = 2x + 1$, $g(x) = x^2$
2. **fg** $fg(3) = f(9) = 19$; $fg(x) = 2x^2 + 1$
3. **gf** $gf(3) = g(7) = 49$; $gf(x) = (2x + 1)^2$

$$fg(x) = f\big(g(x)\big)$$

the function nearest x acts first

The inverse is a reflection



- $f(x) = x^2 - 4,\ x \geq 0.$
- $y = x^2 - 4 \Rightarrow x = \sqrt{y+4}.$
- $f^{-1}(x) = \sqrt{x+4},\ x \geq -4.$

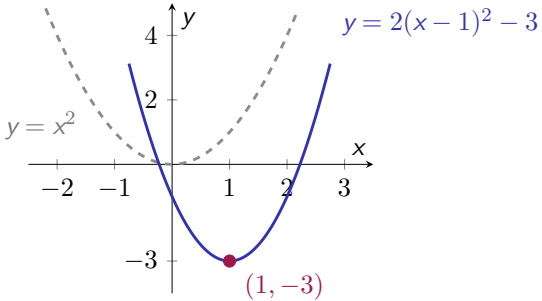
Domain of f^{-1} = **range** of f; range of f^{-1} = domain of f.

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Transformations of $y = f(x)$

Graph	Transformation
$y = f(x) + a$	**translation** $\begin{pmatrix} 0 \\ a \end{pmatrix}$
$y = f(x + a)$	translation $\begin{pmatrix} -a \\ 0 \end{pmatrix}$
$y = af(x)$	**stretch** parallel to the y-axis, factor a
$y = f(ax)$	stretch parallel to the x-axis, factor $\frac{1}{a}$
$y = -f(x)$, $y = f(-x)$	**reflection** in the x-axis; in the y-axis

Two transformations, one at a time



- $y = x^2 \to y = 2(x-1)^2 - 3$.
- Right 1, **stretch** $\times 2$ vertically, down 3.
- $(0,0) \to (1,0) \to (1,0) \to (1,-3)$.

Trap: squaring adds roots

1. **Square** $\sqrt{x+3} = x-3 \Rightarrow x+3 = x^2-6x+9$
2. **Solve** $x^2-7x+6=0$: $x=1$ or $x=6$
3. **Check 1** $\sqrt{4}=2$ but $1-3=-2$: reject
4. **Check 6** $\sqrt{9}=3=6-3$: accept

Decision

Only $x=6$. The arrow ran one way only: $\Rightarrow$, not $\Leftrightarrow$.

Check yourself

Question	**Answer**
Vertex of $(x+5)^2-7$?	$(-5,-7)$
Roots of x^2+2x+5?	none: $\Delta=-16$
$(x+2)$ a factor of f means...	$f(-2)=0$
$y=f(x-4)$ moves the graph...	right 4
Domain of f^{-1}?	the **range** of f