Proof

AQA A-level Mathematics 7357, section A: deduction, exhaustion, counterexample, contradiction

Four shapes of argument

Method	**What it settles**	**Its shape**
Deduction	a claim about every case	general variables, logical steps
Exhaustion	a claim about a finite set	a complete list of cases, each checked
Counterexample	disproves an all-values claim	one valid failing case
Contradiction	a claim that is hard to attack directly	assume the negation, reach an impossibility

Forty checks, one failure

prime for $n = 0$ to 39

0	1	2	3	4	5	6	7	8	9
10	11	12	13	14	15	16	17	18	19
20	21	22	23	24	25	26	27	28	29
30	31	32	33	34	35	36	37	38	39

40 $41^2 = 1681$**: not prime**

- $n^2 + n + 41$ is prime for $n = 0, 1, \ldots, 39$.
- At $n = 40$: $1600 + 40 + 41 = 1681 = 41^2$.
- Evidence is not proof.

One counterexample kills a universal claim.

The language of proof

Symbol	Read as	Example
$\Rightarrow$	implies	$x = 3 \Rightarrow x^2 = 9$
$\Leftarrow$	is implied by	$x^2 = 9 \Leftarrow x = -3$
$\Leftrightarrow$	**if and only if**	$2x + 1 = 7 \Leftrightarrow x = 3$
$\therefore$	therefore	$n = 2k, \therefore n^2 = 4k^2$
$\equiv$	is identically equal to	$(x+1)^2 \equiv x^2 + 2x + 1$

$x^2 = 9 \Rightarrow x = 3$ is false: $x = -3$ also squares to 9.

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Deduction: odd plus odd is even

1. **Define** Two odd integers: $2a+1$ and $2b+1$, with $a, b \in \mathbb{Z}$.
2. **Add** $(2a+1)+(2b+1) = 2a+2b+2$
3. **Factorise** $= 2(a+b+1)$, and $a+b+1$ is an **integer**.
4. **Conclude** A multiple of 2, so even, for **every** pair.

Decision

Using $2n+1$ for both would prove it only for two equal odd numbers.

$$(2a+1)+(2b+1) = 2(a+b+1)$$

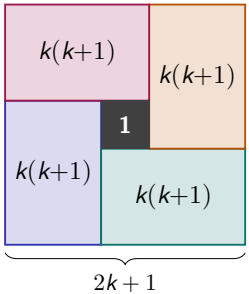
Deduction: $n^3 - n$ is a multiple of 6

1. **Factorise** $n^3 - n = n(n^2 - 1) = (n-1)\,n\,(n+1)$
2. **Structure** three **consecutive** integers
3. **Factor 2** at least one of them is even
4. **Factor 3** exactly one of them is a multiple of 3

$$n^3 - n = (n-1)\,n\,(n+1)$$

it contains a factor 2 and a factor 3, so 6 divides it for every **integer**

Deduction drawn: an odd square



- $(2k+1)^2 = 4k(k+1) + 1$.
- $k(k+1)$ is even, so $4k(k+1)$ is a multiple of 8.
- Drawn for $k = 2$: $25 = 4 \times 6 + 1$.

$$(2k+1)^2 = 8m + 1$$

every odd square is one more than a multiple of 8

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Proof by exhaustion

Case	$n^2 + 2$	**Divided by 4**
$n = 2k$ (even)	$4k^2 + 2$	remainder 2
$n = 2k + 1$ (odd)	$4k^2 + 4k + 3$	remainder 3

Even or odd covers every **integer**, and neither leaves remainder 0: $4 \nmid n^2 + 2$.

Disproof by counterexample

1. **Claim** If p is prime, $2^p - 1$ is prime.
2. **Test** $p = 2, 3, 5, 7$ give $3, 7, 31, 127$: all prime.
3. **Fail** $p = 11$: $2^{11} - 1 = 2047 = 23 \times 89$
4. **State** 11 is prime but 2047 is not, so the claim is false.

Decision

A counterexample must satisfy the **hypothesis**: here p must be prime.

Contradiction: $\sqrt{2}$ is irrational

1. **Assume** $\sqrt{2} = \frac{p}{q}$, integers in lowest terms.
2. **Square** $p^2 = 2q^2$, so p^2 is even, so p is even.
3. **Substitute** $p = 2r$: $4r^2 = 2q^2$, so $q^2 = 2r^2$ and q is even.
4. **Clash** p and q share the factor 2: not lowest terms.

$$p^2 = 2q^2 \Rightarrow p, q \text{ both even}$$

the assumption is false, so $\sqrt{2}$ is **irrational**

Contradiction: infinitely many primes

1. **Assume** finitely many primes $p_1, p_2, \ldots, p_n$
2. **Build** $N = p_1 p_2 \cdots p_n + 1$
3. **Divide** each p_i leaves remainder 1 when dividing N
4. **Clash** $N > 1$ has a prime factor, and it is not on the list.

$$N = p_1 p_2 \cdots p_n + 1 \equiv 1 \pmod{p_i}$$

no finite list holds every prime (N itself need not be prime: 30031 is 59 times 509)

Spot the flaw

Your turn

A student "proves" $\sqrt{2}+\sqrt{3}>3$: squaring both sides gives $5+2\sqrt{6}>9$, so $2\sqrt{6}>4$, which is true, so the claim is true. What is wrong?

A. Squaring an inequality is never allowed.

B. The argument starts from the conclusion and reaches a true statement.

C. $2\sqrt{6}$ is less than 4.

D. Nothing is wrong; it is a complete proof.

Evidence or proof?

Not a proof

Checking $n = 1$ to 100 for an all-integers claim.

Starting from the result and working to something true.

A proof about integers quoted for real numbers.

A proof

General variables: $2a + 1$, $(n - 1)n(n + 1)$.

A case list that provably covers everything.

A negation that leads to an impossibility.

Choose the method

Claim	**Method**
The sum of three **consecutive** integers is a multiple of 3.	deduction: $3(n+1)$
Every prime between 10 and 30 is one away from a multiple of 6.	exhaustion of 11, 13, ..., 29
$x^2 > x$ for every real x.	counterexample: $x = \frac{1}{2}$
There is no largest even number.	contradiction: $2N+2$

Check yourself

Question	**Answer**
Does $n = 0$ to 39 prove $n^2 + n + 41$ always prime?	No: $n = 40$ gives 41^2.
Write two independent even numbers.	$2a$ and $2b$
First line of a contradiction proof?	Assume the negation.
What must a counterexample satisfy?	the claim's condition
Is $x^2 = 4 \Rightarrow x = 2$ true?	No: $x = -2$.