

A · Proof · supported focus

AQA · A-Level · Mathematics

Scope and prerequisites

AQA A-level Mathematics 7357, section A (Proof), version 1.3 (first assessment 2018). Original teaching notes; objective-level mapping lives in `content/teaching/aqa_a_level_mathematics/objective-review.yaml`.

Prerequisites: algebraic manipulation from GCSE. Each lesson develops its own method before independent transfer.

These are original teaching examples, not official AQA material.

Proof

■ What would explain this observation?

- The formula $n^2 + n + 41$ gives prime numbers for $n = 1, 2, 3$, all the way to 39. A student concludes it is always prime. How can one number destroy forty confirmations?
 - Start with a prediction. State what kind of argument could settle a claim about EVERY case, and what could settle one about A SINGLE case.
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■ Build the model

- A **deductive proof** starts from stated definitions and follows logically: odd integers $2a+1$ and $2b+1$ sum to $2(a+b+1)$, which is even – proved for every pair.
- A **counterexample** refutes an all-values claim with one valid failing case: at $n = 41$, the value is $41 \times 43 = 1763$, not prime. A **proof by contradiction** assumes the negation and derives an impossibility: if $\sqrt{2} = p/q$ in lowest terms, then $p^2 = 2q^2$ forces p even, and writing $p = 2r$ forces q even too – contradicting lowest terms, so $\sqrt{2}$ is irrational.
- deduction: Proving a claim by logical steps from accepted facts; counterexample: A single valid case that disproves a universal claim; contradiction: Assuming the negation and deriving an impossibility.

Three ways to be sure: each has a required structure

Deduction	Counterexample	Contradiction
$2a+1$ and $2b+1$ are odd. Their sum is $2(a+b+1)$, which is even — for EVERY pair.	Claim: n^2+n+41 is always prime. $n=41$ gives $41 \times 43 = 1763$, not prime — claim dead in one case.	Suppose $\sqrt{2} = p/q$ in lowest terms. Then $p^2=2q^2$ forces p and q both even — impossible. So $\sqrt{2}$ is irrational.

Checking many cases is evidence, never a proof — one counterexample kills a universal claim

■ Choose evidence that can test it

- Checking $n = 0, 1, 2, \dots, 39$ is strong evidence but not a proof: the claim concerns EVERY positive integer. A proof must cover all cases at once – by general algebra, by exhaustion of a complete finite list, or by contradiction.
- Before accepting a proof, check its structure: does it start from assumptions rather than from the conclusion? Does the contradiction attack the stated assumption? Checking examples verifies arithmetic but cannot replace the general argument.

■ Work from known quantities

- State the known values and their units. Choose the relation because its assumptions fit this case, then rearrange before substitution.
- Known: the candidate counterexample $n = 41$. Substitution: $n^2 + n + 41 = 41^2 + 41 + 41 = 41(41 + 1 + 1) = 41 \times 43 = 1763$. The value has a factor of 41, so it is not prime – the universal claim fails at exactly one case.

Example:

Test whether the sum of two even numbers is always even using deduction: write the two numbers as $2a$ and $2b$ and show the structure. (Answer: $2a + 2b = 2(a + b)$, which is a multiple of 2, hence even for every pair – no examples needed.)

■ Check the conclusion and its limits

- Each method has a required shape: deduction needs general variables, exhaustion needs a genuinely complete case list, and contradiction needs the impossibility to follow from the negation itself. A proof about integers does not automatically cover real numbers, and an identity proved for all values differs from an equation true only at particular solutions.
- Return to the original observation. Explain what the result supports, which conditions it assumes, and one way to test a competing explanation.

Warn:

Checking a claim for the first hundred integers proves it for every positive integer. This claim is false: a finite sample is evidence, never a proof – $n^2 + n + 41$ survives 40 checks and fails at the 41st, so only a general argument or a counterexample settles a universal claim.

Key:

Proof: deduction works from definitions with general variables; a counterexample kills an all-values claim with one case; contradiction assumes the negation and derives an impossibility. Examples verify and disprove – they never prove a universal.