

I · Numerical methods

AQA · A-Level · Mathematics

Scope and prerequisites

AQA A-level Mathematics 7357, section I (Numerical methods), version 1.3 (first assessment 2018). Original teaching notes.

Prerequisites: function evaluation and sign analysis from sections B and G.

These are original teaching examples, not official AQA material.

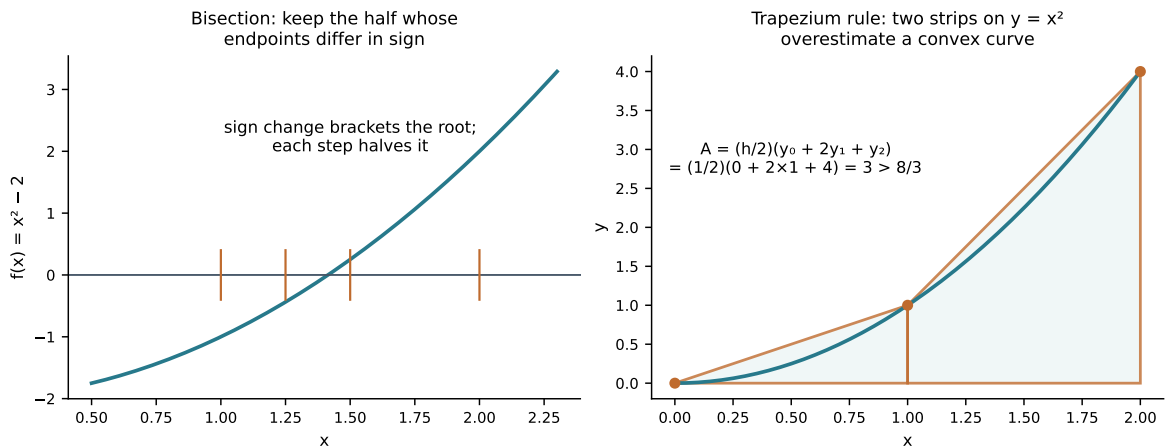
Numerical methods

■ The question

- The equation $x^3 - x - 1 = 0$ has no tidy algebraic solution, yet an engineer needs the root to four decimal places. How can an approximate answer be PROVED correct to a stated accuracy?
 - Before reading on, say what a sign change shows, and what could make a decimal answer trustworthy.
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■ The method

- A continuous function with opposite signs at two endpoints has a root between them. **Bisection** halves the interval, keeping the half whose endpoints still differ in sign; stop if the midpoint is itself a root.
- **Iteration** $x_{\text{next}} = g(x)$ refines an estimate from a rearrangement; a limit must satisfy $L = g(L)$ and be checked in the original equation. **Newton–Raphson** uses $x_{\text{next}} = x - f(x)/f'(x)$ with a nonzero derivative. The **trapezium rule** estimates a definite integral: $(h/2)(y_0 + 2y_1 + \dots + 2y_{n-1} + y_n)$.
- bisection: Halving a root bracket, keeping the half with a sign change; iteration: Repeating an update $x_{\text{next}} = g(x)$ from the previous estimate; Newton–Raphson: The root update $x_{\text{next}} = x - f(x)/f'(x)$; trapezium rule: Estimating an integral by adding trapezium strips.



Bisection keeps the half with a sign change; two trapezia give 3, above the true $8/3$, for this convex curve

■ Checking your work

- A sign change proves a root only where the function is CONTINUOUS: $1/x$ changes sign across $x = 0$ with no root there. Report a rounded root with its bracket, such as 1.3247 to four decimal places shown by a sign change of f on $[1.32465, 1.32475]$, so the accuracy is provable.
- Iteration can diverge or cycle: check the iterates, the domain and the original equation. The trapezium rule over- or under-estimates depending on curvature, not on whether the function merely increases.

■ A worked case

- Known: $f(x) = x^2 - 2$ and $x_0 = 1.5$ for Newton–Raphson. $f'(x) = 2x$, so $x_1 = x_0 - f(x_0)/f'(x_0) = 1.5 - (2.25 - 2)/3 = 1.5 - 0.0833... = 1.4166667$. A second step: $f(1.4166667) \approx 0.0069444$, $f' \approx 2.8333333$, so $x_2 \approx 1.4142157$, closing on $\sqrt{2}$.

Example:

Estimate $\int_0^2 x^2 dx$ with two equal trapezium strips and compare with the exact value, using the same sequence. (Answer: $h = 1$, heights 0, 1, 4, so $A = (1/2)(0 + 2 \times 1 + 4) = 3$. The exact value is $8/3 \approx 2.667$ – the convex curve makes the estimate an overestimate.)

■ Limits and traps

- Each method has conditions: bisection needs a continuous sign change, Newton needs a nonzero derivative and can fail from a poor start, and $|g'(L)| < 1$ is a sufficient local convergence condition when g' is continuous near the fixed point L and the starting value is close enough; it is not necessary. More strips improve a

trapezium estimate but never make it exact by assertion.

Warn:

Every sign change proves a root, including one across a discontinuity. This claim is false: the intermediate-value reasoning needs continuity on the whole interval – across an asymptote the function can jump from negative to positive with no root at all.

Key:

Numerical methods: bracket a root with a continuous sign change and halve with bisection; iterate with checked convergence; Newton–Raphson uses the tangent; the trapezium rule estimates integrals with curvature deciding the error direction.