

Notes

AQA · A-Level · Mathematics

7357: course teaching notes

Version: Version 1.3, 31 January 2018; first A-level assessment 2018

These are original course-owned teaching notes. Objective-level exceptions are listed in the review, and lessons are tier labelled.

A · Deduction, contradiction and counterexamples

When does a pattern become a proof?

- Checking several integers can suggest a pattern. What turns the pattern into a proof for every integer?
 - This lesson studies **counterexample** : A single valid case that disproves a universal claim.
-

Choose the mathematical structure

- A deductive proof starts from stated definitions or assumptions. A counterexample refutes an all-values claim. For contradiction, assume the contrary and derive an impossibility. For exhaustion, check every case in a finite complete list.
 - State the allowed inputs and units before calculating. An equation should express the relationship, not just record a calculator entry.
-

Work through a checked case

$$(2a + 1) + (2b + 1) = 2(a + b + 1)$$

- Check the result against the starting quantities. Substitute into the original relation, or compare the graph and numerical answer where appropriate.

Worked example

Odd integers $2a+1$ and $2b+1$ sum to $2(a+b+1)$, hence even. To disprove that n^2+n+41 is always prime, use $n=41$: the value is $41 \times 43 = 1763$. In a contradiction proof that $\sqrt{2}$ is irrational, assume $\sqrt{2}=p/q$ in lowest terms; $p^2=2q^2$ forces both p and q even, contradicting lowest terms.

Odd + odd: $(2a + 1) + (2b + 1)$

Factor 2: $2(a + b + 1)$

The bracket is an integer $\rightarrow$ the sum is even

Test a tempting shortcut

- Do not assume the conclusion while proving it. In contradiction, identify the impossible consequence and reject the initial contrary assumption. Induction is not compulsory in AQA 7357; it belongs in its own appropriate further-mathematics scope.
- When a shortcut fails, identify the assumption it breaks. Keep an exact value until the requested final rounding.

Watch out

Checking the first ten integers proves a claim for every positive integer. This claim is false. Explain which definition or assumption it violates.

Interpret a new situation

- AQA proof includes deduction, exhaustion and contradiction. IAL P2 introduces exhaustion and counterexample, while P4 introduces contradiction. Keep the method matched to the named unit.
 - A complete solution gives the mathematical result and explains what it means. Check that it is possible in the stated context.
-

Use this in your course

- 7357 · A-level · A. Match the target tier and specification before assigning extensions.
- Give the method before the final answer, and use the paper's calculator and formula rules. Review a wrong answer by locating the first invalid step.

Key idea

A single valid case that disproves a universal claim. Choose the relationship, show the method, check its assumptions and interpret the result.

A · Complete cases and counterexamples

When do three cases cover every integer?

- Testing a thousand integers may miss the next counterexample. A complete list of possible remainders can instead settle every integer.
 - This lesson studies **proof by exhaustion** : A proof that checks every possibility in a finite, complete set of cases.
-

Choose the mathematical structure

- Write $n=3k$, $3k+1$ or $3k+2$ for integer k . These are all possible remainders on division by 3, including negative integers. Evaluate the expression in each case. One counterexample disproves a universal claim; examples supporting a claim do not prove it.
- State the allowed inputs and units before calculating. An equation should express the relationship, not just record a calculator entry.

Work through a checked case

$$n = 3k + r, \quad r \in \{0, 1, 2\}, \quad n^2 \not\equiv 2 \pmod{3}$$

- Check the result against the starting quantities. Substitute into the original relation, or compare the graph and numerical answer where appropriate.

Worked example

For $n=3k$, $n^2=9k^2$ is divisible by 3. For $n=3k+1$, $n^2=3(3k^2+2k)+1$. For $n=3k+2$, $n^2=3(3k^2+4k+1)+1$. Thus an integer square has remainder 0 or 1, never 2. The claim that n^2+n+41 is always prime fails at $n=41$: its value is $1763=41 \times 43$.

All integer squares: remainder 0 or 1 modulo 3

$$n = 3k$$

$$n^2 = 3(3k^2)$$

$$n = 3k + 1$$

$$n^2 = 3(3k^2 + 2k) + 1$$

$$n = 3k + 2$$

$$n^2 = 3(3k^2 + 4k + 1) + 1$$

Test a tempting shortcut

- A finite sample is not exhaustion unless it contains every possible case. Checking $n=0,1,2$ alone is not enough: the expressions involving arbitrary integer k justify all integers.
- When a shortcut fails, identify the assumption it breaks. Keep an exact value until the requested final rounding.

Watch out

If a claim works for the first hundred integers, it has been proved for all integers. This claim is false. Explain which definition or assumption it violates.

Interpret a new situation

- Choose a complete case split, such as parity or remainders. State why no case is missing. For a false universal claim, give a valid input and verify the failure directly.
 - A complete solution gives the mathematical result and explains what it means. Check that it is possible in the stated context.
-

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- Give the method before the final answer, and use the paper's calculator and formula rules. Review a wrong answer by locating the first invalid step.

Key idea

A proof that checks every possibility in a finite, complete set of cases. Choose the relationship, show the method, check its assumptions and interpret the result.

A · Irrational roots and infinitely many primes

Why cannot $\sqrt{2}$ be an exact fraction?

- A calculator prints finitely many digits of $\sqrt{2}$. Those digits cannot decide whether an exact ratio of integers exists.
 - This lesson studies **proof by contradiction** : A proof that assumes the negation of a claim and derives an impossibility.
-

Choose the mathematical structure

- To prove $\sqrt{2}$ irrational, assume $\sqrt{2}=p/q$ for coprime integers p,q with q nonzero. Then $p^2=2q^2$ forces p even; writing $p=2r$ forces q even too. This contradicts coprimality. To prove infinitely many primes, assume a complete finite prime list and construct a number with no listed prime divisor.
 - State the allowed inputs and units before calculating. An equation should express the relationship, not just record a calculator entry.
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Work through a checked case

$$p^2 = 2q^2, \quad p = 2r \implies q^2 = 2r^2$$

- Check the result against the starting quantities. Substitute into the original relation, or compare the graph and numerical answer where appropriate.

Worked example

From $p^2=2q^2$ and $p=2r$, obtain $q^2=2r^2$. Both p and q would have factor 2, contradicting lowest terms. For a supposed complete prime list $p_1, \dots, p_k$, set $N=p_1 \cdots p_k + 1$. $N > 1$ has a prime divisor, but every listed prime leaves remainder 1 when dividing N . That divisor is a new prime, contradicting completeness. N itself need not be prime: $2 \times 3 \times 5 \times 7 \times 11 \times 13 + 1 = 30031 = 59 \times 509$.

Assume a complete finite prime list: $p_1, \dots, p_k$

Construct $N = p_1 \times \dots \times p_k + 1$

Each listed prime leaves remainder 1

A prime divisor of N is missing $\rightarrow$ contradiction

Test a tempting shortcut

- The even-square step needs a reason: an odd integer $2r+1$ has odd square $4r(r+1)+1$. The prime proof needs a new prime divisor, not a claim that product-plus-one is always prime. A contradiction must attack the stated assumption.
- When a shortcut fails, identify the assumption it breaks. Keep an exact value until the requested final rounding.

Watch out

The product of any finite list of primes plus one is always prime. This claim is false. Explain which definition or assumption it violates.

Interpret a new situation

- For an unfamiliar proof, state the contrary assumption precisely, preserve its conditions and name the exact contradiction. Decimal approximations or a large finite list do not settle these two claims.
 - A complete solution gives the mathematical result and explains what it means. Check that it is possible in the stated context.
-

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- Give the method before the final answer, and use the paper's calculator and formula rules. Review a wrong answer by locating the first invalid step.

Key idea

A proof that assumes the negation of a claim and derives an impossibility. Choose the relationship, show the method, check its assumptions and interpret the result.

B · Indices, surds and standard form

How small is a microscopic length?

- A microscope records a length of 0.000072 metres. A compact representation must keep its size correct.
 - This lesson studies **index** : The power to which a base is raised.
-

Choose the mathematical structure

- For the same positive base, multiplication adds indices and division subtracts them. A negative index means reciprocal; a fractional index represents a root. Standard form has $1 \leq a < 10$.
 - State the allowed inputs and units before calculating. An equation should express the relationship, not just record a calculator entry.
-

Work through a checked case

$$a^m a^n = a^{m+n}, \quad a^{-n} = \frac{1}{a^n}, \quad a^{m/n} = \left(\sqrt[n]{a}\right)^m$$

- Check the result against the starting quantities. Substitute into the original relation, or compare the graph and numerical answer where appropriate.

Worked example

$0.000072 = 7.2 \times 10^{-5}$. Also $16^{3/4} = (16^{1/4})^3 = 2^3 = 8$. Simplify $\sqrt{72} = 6\sqrt{2}$, then rationalise $1/\sqrt{2} = \sqrt{2}/2$.

$$0.000072 = 7.2 \times 10^{-5}$$

7.2 is the coefficient; -5 is the power

The coefficient is between 1 and 10

Test a tempting shortcut

- Index laws do not turn a sum into a single power: $2^3 + 2^4 = 24$, not 2^7 . Do not round a surd when an exact answer is requested.
- When a shortcut fails, identify the assumption it breaks. Keep an exact value until the requested final rounding.

Watch out

For every positive a , $a^2 + a^3$ equals a^5 . This claim is false. Explain which definition or assumption it violates.

Interpret a new situation

- Check powers of ten against the original quantity. Use surds for exact geometry, and round only the final length when the question asks for a decimal.
- A complete solution gives the mathematical result and explains what it means. Check that it is possible in the stated context.

Use this in your course

- 7357 · A-level · B. Match the target tier and specification before assigning extensions.
- Give the method before the final answer, and use the paper's calculator and formula rules. Review a wrong answer by locating the first invalid step.

Key idea

The power to which a base is raised. Choose the relationship, show the method, check its assumptions and interpret the result.

B · Quadratics and inequalities

Which widths make enough space?

- A rectangular enclosure has area $x(10-x)$. What widths give at least 21 square metres?
 - This lesson studies **discriminant** : The quantity b^2-4ac that determines the real roots of $ax^2+bx+c=0$.
-

Choose the mathematical structure

- Factor where possible; otherwise complete the square or use the quadratic formula. A quadratic inequality needs the sign on intervals, not only the roots. The discriminant identifies repeated or missing real roots.
 - State the allowed inputs and units before calculating. An equation should express the relationship, not just record a calculator entry.
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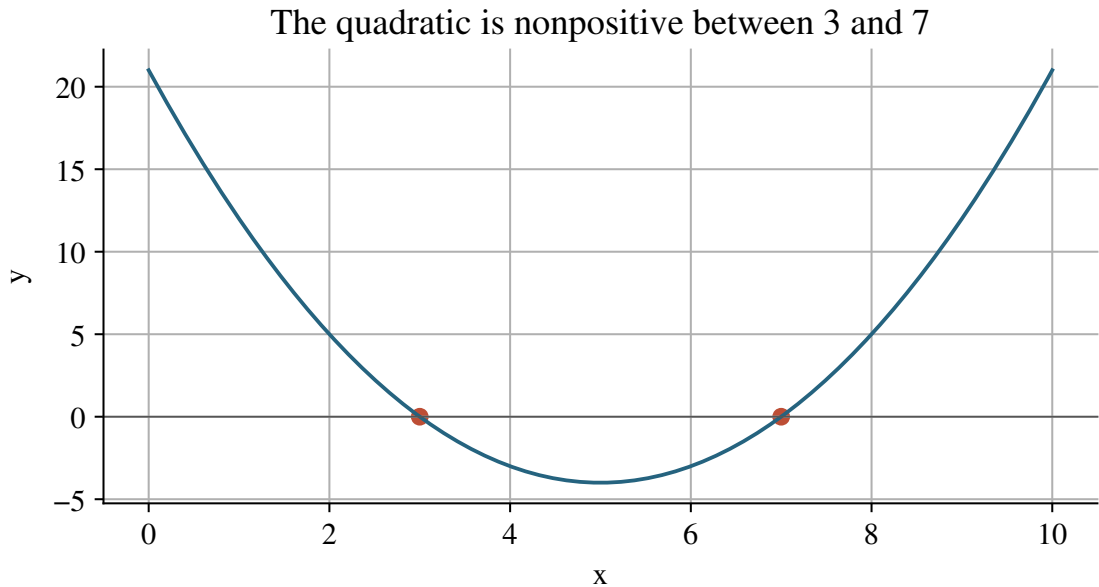
Work through a checked case

$$ax^2 + bx + c = 0, \quad \Delta = b^2 - 4ac$$

- Check the result against the starting quantities. Substitute into the original relation, or compare the graph and numerical answer where appropriate.

Worked example

The condition is $x(10-x) \geq 21$, so $x^2 - 10x + 21 \leq 0$. Factor $(x-3)(x-7) \leq 0$. The upward parabola is nonpositive between its roots, giving $3 \leq x \leq 7$. The maximum area is 25 at $x=5$.



Test a tempting shortcut

- Multiplying an inequality by a negative number reverses its direction. A sketch must show which side of each root satisfies the inequality. Geometry may restrict x further.
- When a shortcut fails, identify the assumption it breaks. Keep an exact value until the requested final rounding.

Watch out

A positive discriminant means that a quadratic has no real roots. This claim is false. Explain which definition or assumption it violates.

Interpret a new situation

- Use the vertex to interpret an optimum. Check an endpoint and a point between the roots; the algebra and graph should tell the same story.

- A complete solution gives the mathematical result and explains what it means. Check that it is possible in the stated context.
-

Use this in your course

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- Give the method before the final answer, and use the paper's calculator and formula rules. Review a wrong answer by locating the first invalid step.

Key idea

The quantity b^2-4ac that determines the real roots of $ax^2+bx+c=0$. Choose the relationship, show the method, check its assumptions and interpret the result.

B · Domains, inverses and composition

Which inputs are allowed?

- A square-root model returns a real output only for some inputs. Its formula alone does not specify a complete function.
 - This lesson studies **domain** : The set of allowed inputs to a function.
-

Choose the mathematical structure

- State the domain and range. For an inverse, first ensure the function is one-to-one on its domain. Composition fg means apply g first, then f ; the intermediate output must be an allowed input to f .
 - State the allowed inputs and units before calculating. An equation should express the relationship, not just record a calculator entry.
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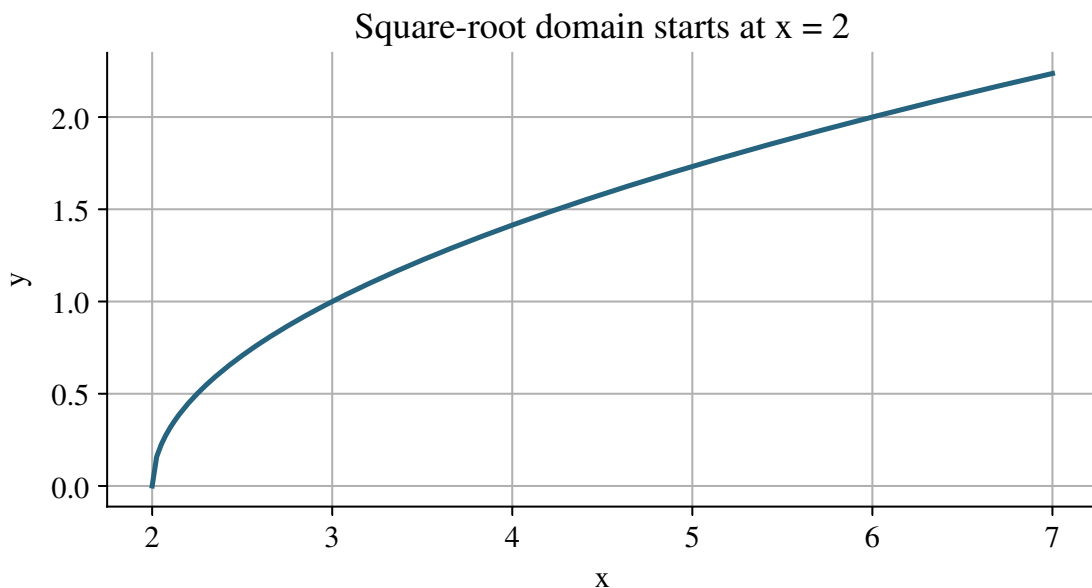
Work through a checked case

$$f(g(x)) = (f \circ g)(x), \quad f^{-1}(f(x)) = x$$

- Check the result against the starting quantities. Substitute into the original relation, or compare the graph and numerical answer where appropriate.

Worked example

For $f(x)=\sqrt{x-2}$, $x\geq 2$ and the range is $y\geq 0$. From $y=\sqrt{x-2}$, $x=y^2+2$. Thus $f^{-1}(x)=x^2+2$ with $x\geq 0$. For $g(x)=x+3$, $fg(1)=f(4)=\sqrt{2}$.



Test a tempting shortcut

- Squaring can introduce extraneous solutions. Restricting a parabola's domain is essential before claiming an inverse. A horizontal translation inside f has the opposite sign to the graph's movement.
- When a shortcut fails, identify the assumption it breaks. Keep an exact value until the requested final rounding.

Watch out

Every quadratic function on all real numbers has an inverse function. This claim is false. Explain which definition or assumption it violates.

Interpret a new situation

- Check $f(f^{-1}(x))=x$ on the inverse domain. Use a sketch to test whether a horizontal line meets the original graph more than once.
 - A complete solution gives the mathematical result and explains what it means. Check that it is possible in the stated context.
-

Use this in your course

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- Give the method before the final answer, and use the paper's calculator and formula rules. Review a wrong answer by locating the first invalid step.

Key idea

The set of allowed inputs to a function. Choose the relationship, show the method, check its assumptions and interpret the result.

B · Polynomial division, remainders and factors

How does one root unlock a cubic?

- A cubic graph crosses the axis at a known point. That root lets us reduce a cubic equation to a quadratic rather than guess the other roots.
 - This lesson studies **factor theorem** : The statement that $x-a$ is a factor of polynomial $P(x)$ exactly when $P(a)=0$.
-

Choose the mathematical structure

- Write $P(x)=(x-a)Q(x)+R$, with R constant. Substituting $x=a$ gives $R=P(a)$. Divide leading terms, multiply the whole divisor, subtract with brackets and repeat. Include zero coefficients for missing powers.
 - State the allowed inputs and units before calculating. An equation should express the relationship, not just record a calculator entry.
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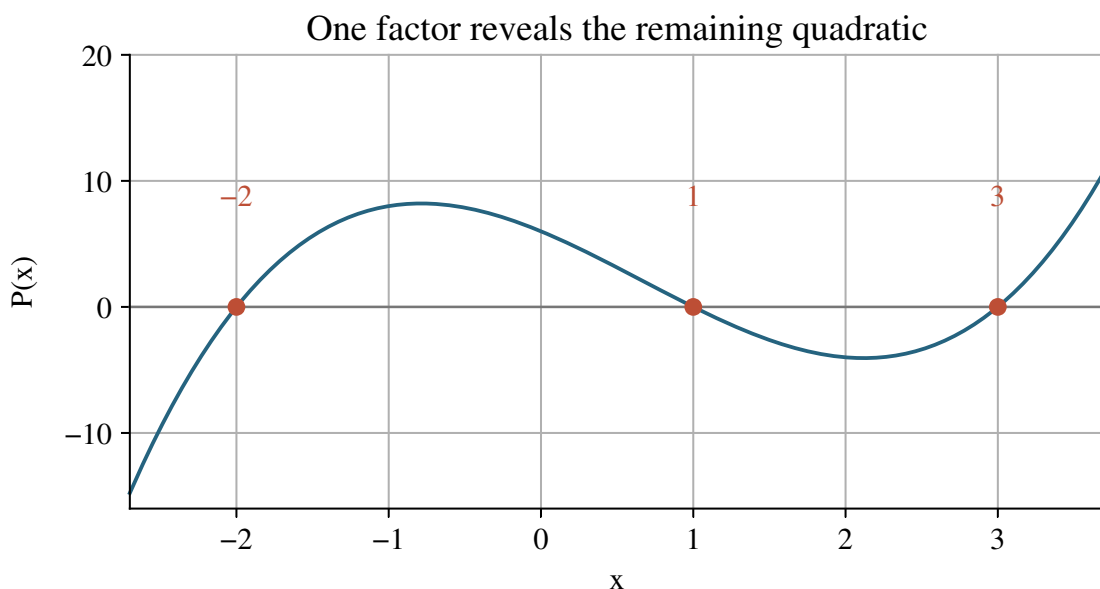
Work through a checked case

$$P(x) = (x - a)Q(x) + P(a)$$

- Check the result against the starting quantities. Substitute into the original relation, or compare the graph and numerical answer where appropriate.

Worked example

For $P(x)=x^3-2x^2-5x+6$, division by $x-1$ gives $Q(x)=x^2-x-6$ and remainder 0. Indeed $(x-1)(x^2-x-6)=P(x)$. Factor $Q=(x-3)(x+2)$, so the roots are 1, 3, -2. Dividing the same P by $x-2$ gives quotient x^2-5 and remainder -4 : $P=(x-2)(x^2-5)-4$. Hence $P/(x-2)=x^2-5-4/(x-2)$, for $x \neq 2$.



Test a tempting shortcut

- The sign in $x-a$ matters: for $x+2$ test $P(-2)$. A nonzero remainder does not disappear. Subtract the entire product at each division step; omitting brackets changes signs.
- When a shortcut fails, identify the assumption it breaks. Keep an exact value until the requested final rounding.

Watch out

If $P(a)$ is nonzero, $x-a$ is still a factor of P . This claim is false. Explain which definition or assumption it violates.

Interpret a new situation

- Check a quotient by expanding divisor times quotient plus remainder. Factorisation can solve $P(x)=0$, while division of a rational expression must retain every original excluded input.
 - A complete solution gives the mathematical result and explains what it means. Check that it is possible in the stated context.
-

Use this in your course

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- Give the method before the final answer, and use the paper's calculator and formula rules. Review a wrong answer by locating the first invalid step.

Key idea

The statement that $x-a$ is a factor of polynomial $P(x)$ exactly when $P(a)=0$. Choose the relationship, show the method, check its assumptions and interpret the result.

B · Algebraic fractions and excluded values

A cancelled fraction can look like a straight line and still have one missing point. Cancelling a factor does not put an undefined input back into the original formula.

- A cancelled fraction can look like a straight line and still have one missing point. Cancelling a factor does not put an undefined input back into the original formula.
 - This lesson studies **excluded value** : An input for which an original denominator or divisor is zero.
-

Choose the mathematical structure

- Factor every numerator and denominator before cancelling a common factor. A term joined by addition is not a cancellable factor. Record exclusions from the original expression first. For addition or subtraction use a common denominator, retaining brackets around the entire numerator. Multiply factored numerators and denominators. To divide, multiply by the reciprocal and exclude inputs making the divisor zero as well as inputs making any original fraction undefined.
 - State the allowed inputs and units before calculating. An equation should express the relationship, not just record a calculator entry.
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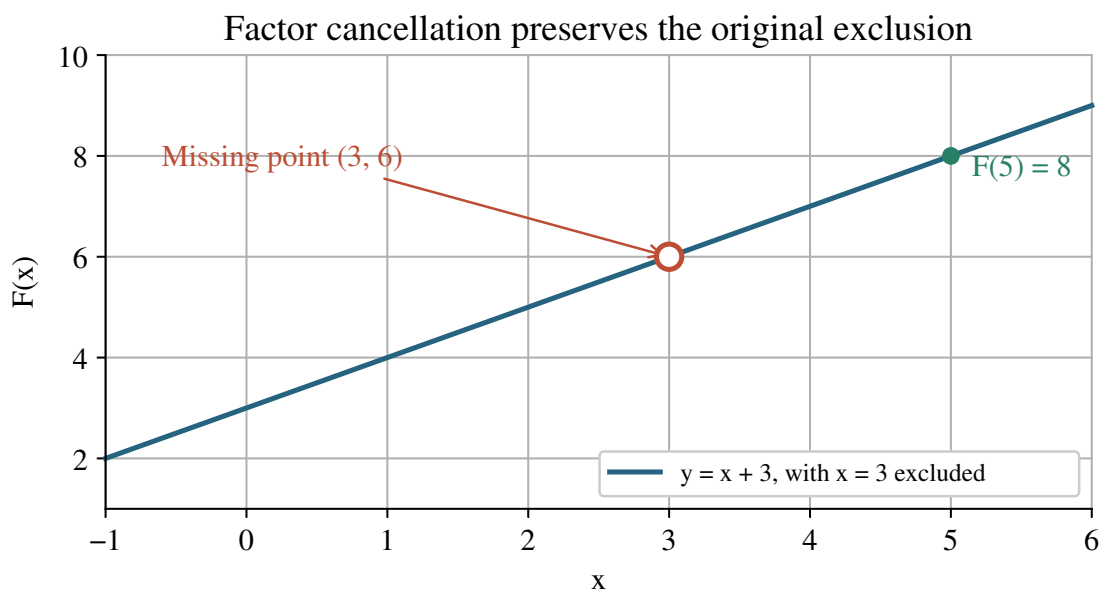
Work through a checked case

$$\frac{x^2 - 9}{x - 3} = x + 3 \quad (x \neq 3)$$

- Check the result against the starting quantities. Substitute into the original relation, or compare the graph and numerical answer where appropriate.

Worked example

For $F(x) = (x^2 - 9)/(x - 3)$, factor $x^2 - 9 = (x - 3)(x + 3)$. Thus $F(x) = x + 3$ for $x \neq 3$; $F(5) = 8$, but $F(3)$ is undefined, not 6. For addition, $1/(x - 1) + 2/(x + 1) = [(x + 1) + 2(x - 1)]/[(x - 1)(x + 1)] = (3x - 1)/(x^2 - 1)$, with $x \neq \pm 1$. At $x = 3$, both forms give 1. For subtraction, the numerator is $(x + 1) - 2(x - 1) = 3 - x$, so the minus sign acts on both terms. Multiplication $[(x - 1)/(x + 2)] \times [(x + 2)/(x + 1)]$ gives $(x - 1)/(x + 1)$, but the original excludes $x = -2$ and $x = -1$. Division $[(x^2 - 1)/(x^2 + 3x + 2)] \div [(x - 1)/(x + 2)]$ gives 1, yet $x = -2, -1, 1$ are excluded: $x = 1$ makes the divisor zero. Finally $F(x) = 6$ reduces to $x + 3 = 6$. Its only candidate $x = 3$ is forbidden, so this equation has no solution. Check candidates in the original equation before reporting them.



Test a tempting shortcut

- Never cancel the x in $(x+2)/x$. A simplified denominator does not show all original restrictions. In a division task, check when the divisor is zero. Cross-multiplication can produce an excluded candidate; a formal root is not automatically a solution.
- When a shortcut fails, identify the assumption it breaks. Keep an exact value until the requested final rounding.

Watch out

Cancelling a denominator factor always makes its excluded input valid. This claim is false. Explain which definition or assumption it violates.

Interpret a new situation

- 7357 B6 covers rational-expression manipulation, cancellation and algebraic division by linear expressions. State the original restrictions, including zero-divisor exclusions, and check every equation candidate in the original domain. Combine this lesson with polynomial division for an improper rational expression.
- A complete solution gives the mathematical result and explains what it means. Check that it is possible in the stated context.

Use this in your course

- 7357 · A-level · B. Match the target tier and specification before assigning extensions.
- Give the method before the final answer, and use the paper's calculator and formula rules. Review a wrong answer by locating the first invalid step.

Key idea

An input for which an original denominator or divisor is zero. Choose the relationship, show the method, check its assumptions and interpret the result.

B · Partial fractions with distinct linear factors

Can one fraction become a sum of simpler fractions?

- A product of two linear denominators can hide two simpler contributions. Separating them makes later integration and equation work easier.
- This lesson studies **partial fraction** : A simpler fraction in a sum that equals a rational expression on its original domain.

Choose the mathematical structure

- For proper rational expressions, use $A/(x+a)+B/(x+b)$ for distinct linear factors. Clear the denominator to obtain a polynomial identity, then substitute convenient roots or compare coefficients. Divide first if the numerator degree is not smaller.
- State the allowed inputs and units before calculating. An equation should express the relationship, not just record a calculator entry.

Work through a checked case

$$\frac{3x+5}{(x+1)(x+2)} = \frac{2}{x+1} + \frac{1}{x+2}$$

- Check the result against the starting quantities. Substitute into the original relation, or compare the graph and numerical answer where appropriate.

Worked example

For $(3x+5)/[(x+1)(x+2)]$, write $3x+5=A(x+2)+B(x+1)$. At $x=-1$, $A=2$; at $x=-2$, $B=1$. Thus the decomposition is $2/(x+1)+1/(x+2)$, with $x \neq -1, -2$. With three factors, $(3x+1)/[x(x+1)(x+2)]=1/(2x)+2/(x+1)-5/[2(x+2)]$. Clearing denominators and using $x=0, -1, -2$ gives $A=1/2$, $B=2$, $C=-5/2$.

$$(3x + 5)/[(x + 1)(x + 2)]$$

$$3x + 5 = A(x + 2) + B(x + 1)$$

$$A = 2, B = 1; \text{ exclude } x = -1, -2$$

Test a tempting shortcut

- The roots used after clearing denominators are allowed in the resulting polynomial identity, not in the original fraction. Retain all excluded inputs. A coefficient can be negative or zero.
- When a shortcut fails, identify the assumption it breaks. Keep an exact value until the requested final rounding.

Watch out

Once denominators are cleared, their excluded inputs become valid inputs of the original fraction. This claim is false. Explain which definition or assumption it violates.

Interpret a new situation

- Check by recombining all terms and comparing coefficients, not just one numerical input. In 7357, numerators here are constant or linear and the decomposition has no more than three terms.
 - A complete solution gives the mathematical result and explains what it means. Check that it is possible in the stated context.
-

Use this in your course

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- Give the method before the final answer, and use the paper's calculator and formula rules. Review a wrong answer by locating the first invalid step.

Key idea

A simpler fraction in a sum that equals a rational expression on its original domain. Choose the relationship, show the method, check its assumptions and interpret the result.

B · Repeated linear factors in partial fractions

Why does a squared factor need two terms?

- A squared factor needs two separate terms. Keeping only the squared term can leave too few coefficients to reproduce the numerator.
 - This lesson studies **repeated factor** : A denominator factor occurring more than once, requiring a term for each power.
-

Choose the mathematical structure

- For a repeated factor $(x+a)^2$, include $A/(x+a)+B/(x+a)^2$. With a distinct factor $x+b$, add $C/(x+b)$. Clear denominators, find convenient coefficients and use another input or coefficient comparison for the remaining unknown.
- State the allowed inputs and units before calculating. An equation should express the relationship, not just record a calculator entry.

Work through a checked case

$$\frac{2x+3}{(x+1)^2(x+2)} = \frac{1}{x+1} + \frac{1}{(x+1)^2} - \frac{1}{x+2}$$

- Check the result against the starting quantities. Substitute into the original relation, or compare the graph and numerical answer where appropriate.

Worked example

For $(2x+3)/[(x+1)^2(x+2)]$, the identity is $2x+3=A(x+1)(x+2)+B(x+2)+C(x+1)^2$. At $x=-1$, $B=1$. At $x=-2$, $C=-1$. The x^2 coefficient gives $A+C=0$, hence $A=1$. The result is $1/(x+1)+1/(x+1)^2-1/(x+2)$, with $x \neq -1, -2$. At $x=0$ both sides are $3/2$; recombination gives numerator $2x+3$ for every allowed x .

$$(2x+3)/[(x+1)^2(x+2)]$$

$$A/(x+1) + B/(x+1)^2 + C/(x+2)$$

$$B = 1, C = -1, A = 1$$

Both powers are needed; exclude $x = -1, -2$

Test a tempting shortcut

- Do not omit the first-power term or place $Ax+B$ over each linear factor. A numerical agreement at one point is only a check, not an identity proof. Keep the square on its own denominator.
- When a shortcut fails, identify the assumption it breaks. Keep an exact value until the requested final rounding.

Watch out

For every denominator $(x+1)^2(x+2)$, one term over $(x+1)^2$ and one over $x+2$ are always enough. This claim is false. Explain which definition or assumption it violates.

Interpret a new situation

- 7357 permits squared linear factors and at most three partial-fraction terms. Verify all numerator coefficients after recombination. Higher powers and irreducible quadratic factors are outside this lesson's specification scope.
 - A complete solution gives the mathematical result and explains what it means. Check that it is possible in the stated context.
-

Use this in your course

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- Give the method before the final answer, and use the paper's calculator and formula rules. Review a wrong answer by locating the first invalid step.

Key idea

A denominator factor occurring more than once, requiring a term for each power. Choose the relationship, show the method, check its assumptions and interpret the result.

B · Linear and quadratic simultaneous equations

How many places can a line meet a parabola?

- A line can cut a parabola twice, touch it once or miss it. An algebraic solution should give the same number of real intersection points as the sketch.
 - This lesson studies **simultaneous solution** : An ordered pair that satisfies every equation in a system at the same time.
-

Choose the mathematical structure

- For two linear equations, eliminate one unknown or substitute an expression for it. For one linear and one quadratic equation, express y from the line and substitute into the quadratic. Solve every resulting root, find its matching y and check both original equations.
 - State the allowed inputs and units before calculating. An equation should express the relationship, not just record a calculator entry.
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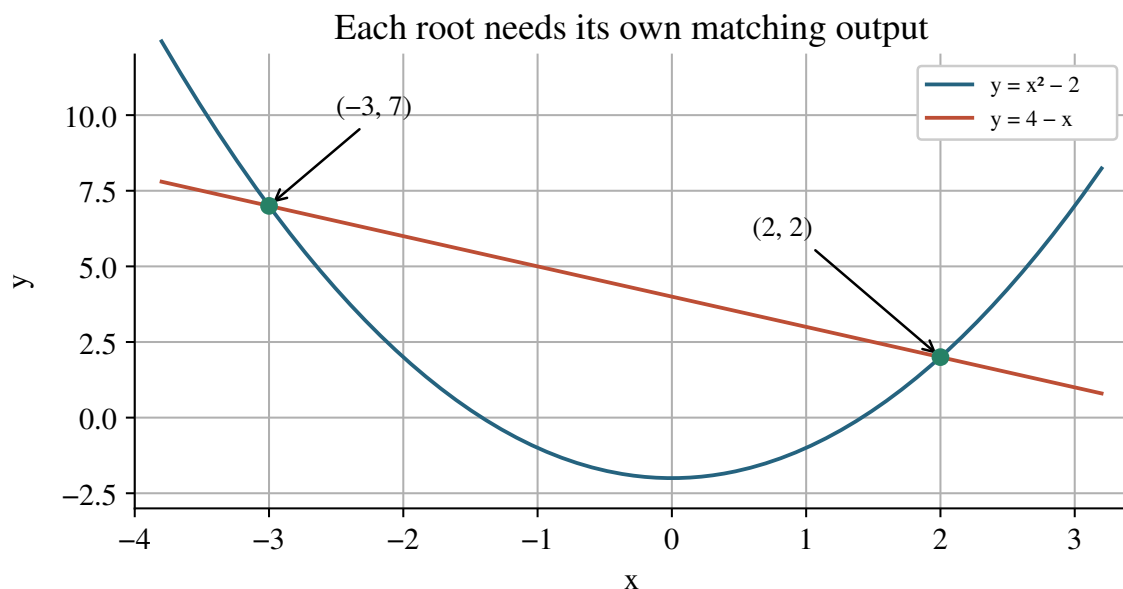
Work through a checked case

$$x + y = 4, \quad y = x^2 - 2 \implies (x - 2)(x + 3) = 0$$

- Check the result against the starting quantities. Substitute into the original relation, or compare the graph and numerical answer where appropriate.

Worked example

For $3x+2y=13$ and $x-y=1$, add twice the second equation to the first: $5x=15$, so $x=3$ and $y=2$. For $x+y=4$ and $y=x^2-2$, substitute $y=4-x$: $x^2+x-6=0=(x-2)(x+3)$. The solutions are $(2,2)$ and $(-3,7)$. The line $y=4x-6$ instead gives $(x-2)^2=0$, so only $(2,2)$: a tangent. The line $y=4x-7$ gives $x^2-4x+5=0$ with discriminant -4 , so no real intersection.



Test a tempting shortcut

- A root for x is not a complete simultaneous solution. Pair each x with its own y ; do not combine one root with another root's output. A repeated quadratic root gives one point, not two different points.
- When a shortcut fails, identify the assumption it breaks. Keep an exact value until the requested final rounding.

Watch out

Every system of two equations has exactly one real simultaneous solution. This claim is false. Explain which definition or assumption it violates.

Interpret a new situation

- Use the discriminant to distinguish two, one or no real intersections. The sketch helps predict the result, but substitutions into both original equations verify it. Preserve any restrictions if a more general system involves fractions.
 - A complete solution gives the mathematical result and explains what it means. Check that it is possible in the stated context.
-

Use this in your course

- 7357 · A-level · B. Match the target tier and specification before assigning extensions.
- Give the method before the final answer, and use the paper's calculator and formula rules. Review a wrong answer by locating the first invalid step.

Key idea

An ordered pair that satisfies every equation in a system at the same time. Choose the relationship, show the method, check its assumptions and interpret the result.

B · Rational indices and real domains

A cube grows to 27 times its original volume. Why does its edge grow by only a factor of 3?

- A cube grows to 27 times its original volume. Why does its edge grow by only a factor of 3?
 - This lesson studies **rational exponent** : An exponent written as a fraction of integers.
-

Choose the mathematical structure

- For a positive base a , $a^{p/q}$ is the q th root of a raised to p , with q positive. A negative exponent takes the reciprocal, so the base must be nonzero. Reduce p/q first. For a negative real base, a reduced odd denominator permits a real root; an even denominator does not.
 - State the allowed inputs and units before calculating. An equation should express the relationship, not just record a calculator entry.
-

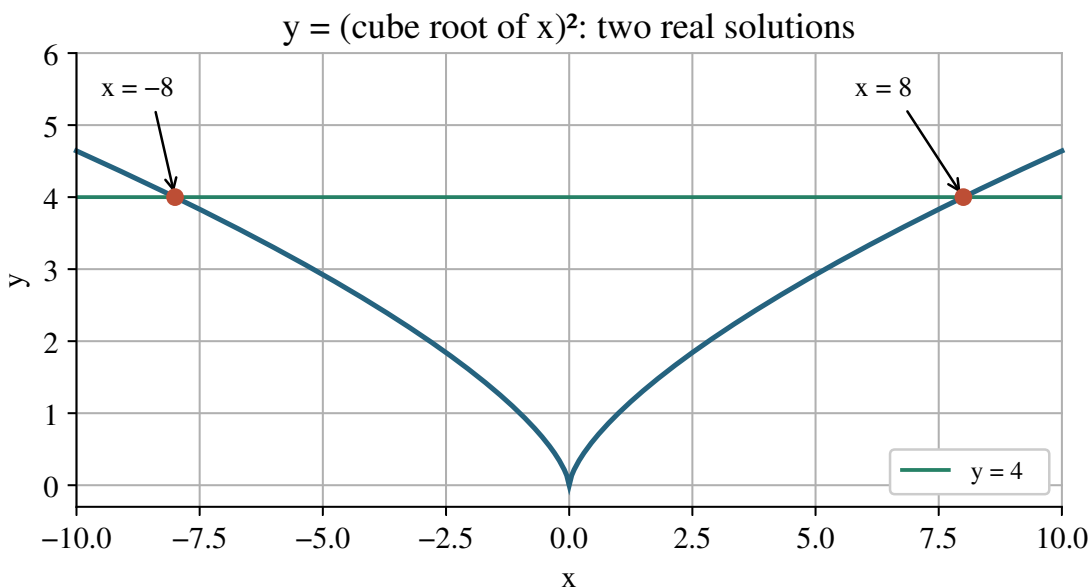
Work through a checked case

$$a^{-p/q} = \frac{1}{(\sqrt[q]{a})^p} \quad (a > 0)$$

- Check the result against the starting quantities. Substitute into the original relation, or compare the graph and numerical answer where appropriate.

Worked example

$27^{2/3} = (\text{cube root of } 27)^2 = 9$, while $16^{-3/4} = 1/(\text{fourth root of } 16)^3 = 1/8$. For $x^{2/3} = 4$ over the real numbers, let u be the cube root of x : $u^2 = 4$, so $u = \pm 2$ and $x = \pm 8$. Both inputs check. By contrast $x^{1/2} = -2$ has no real solution because the principal square root is nonnegative. For negative bases, $(-8)^{1/3} = -2$ and $(-8)^{2/3} = 4$.



Test a tempting shortcut

- Do not change a negative exponent into a negative answer. Do not discard a negative solution when an odd root is squared. General power-of-a-power manipulations can fail outside positive bases: $((-1)^2)^{(1/2)}=1$, whereas $(-1)^{(2 \times 1/2)}=-1$.
- When a shortcut fails, identify the assumption it breaks. Keep an exact value until the requested final rounding.

Watch out

The equation $x^{(2/3)}=4$ has only the solution $x=8$. This claim is false. Explain which definition or assumption it violates.

Interpret a new situation

- Use a positive scale factor for lengths and volumes: a volume ratio 27 gives edge ratio $27^{(1/3)}=3$ and area ratio $27^{(2/3)}=9$. In equations, state the real domain and substitute every candidate. Zero to a negative power is undefined; this lesson does not assign a value to 0^0 .
- A complete solution gives the mathematical result and explains what it means. Check that it is possible in the stated context.

Use this in your course

- 7357 · A-level · B. Match the target tier and specification before assigning extensions.
- Give the method before the final answer, and use the paper's calculator and formula rules. Review a wrong answer by locating the first invalid step.

Key idea

An exponent written as a fraction of integers. Choose the relationship, show the method, check its assumptions and interpret the result.

B · Surds and conjugate denominators

An exact length is $1/(\sqrt{5}-2)$. Can we remove the radical from its denominator without rounding?

- An exact length is $1/(\sqrt{5}-2)$. Can we remove the radical from its denominator without rounding?
- This lesson studies **conjugate** : A paired expression with the sign between two terms reversed.

Choose the mathematical structure

- For nonnegative a and b , $\sqrt{a} \times \sqrt{b} = \sqrt{ab}$, but $\sqrt{a+b}$ is generally not $\sqrt{a} + \sqrt{b}$. Extract square factors to simplify surds. For a two-term denominator, multiply numerator and denominator by its conjugate: $(a+b\sqrt{c})(a-b\sqrt{c}) = a^2 - b^2c$. The original denominator and the conjugate multiplier must both be nonzero.
- State the allowed inputs and units before calculating. An equation should express the relationship, not just record a calculator entry.

Work through a checked case

$$(a + b\sqrt{c})(a - b\sqrt{c}) = a^2 - b^2c$$

- Check the result against the starting quantities. Substitute into the original relation, or compare the graph and numerical answer where appropriate.

Worked example

$\sqrt{72} - \sqrt{8} = 6\sqrt{2} - 2\sqrt{2} = 4\sqrt{2}$. For $1/(\sqrt{5}-2)$, multiply by $(\sqrt{5}+2)/(\sqrt{5}+2)$: the denominator is $5-4=1$, giving $\sqrt{5}+2$. For $3/(2+\sqrt{3})$, multiply by $2-\sqrt{3}$ to get $3(2-\sqrt{3})/(4-3)=6-3\sqrt{3}$. Recombine with the original denominator to verify the exact result.

The numerator changes too; the exact value stays the same

Original exact fraction

$$\frac{1}{\sqrt{5}-2}$$

Multiply by the conjugate

$$\frac{1}{\sqrt{5}-2} \times \frac{\sqrt{5}+2}{\sqrt{5}+2}$$

Difference of squares

$$\frac{\sqrt{5}+2}{5-4} = \sqrt{5}+2$$

Test a tempting shortcut

- The conjugate changes one sign, not every sign. Multiply the numerator too. Squaring a sum produces a cross term: $(2+\sqrt{3})^2=7+4\sqrt{3}$. For real x , $\sqrt{(x^2)}=|x|$, so $\sqrt{((-3)^2)}=3$.
- When a shortcut fails, identify the assumption it breaks. Keep an exact value until the requested final rounding.

Watch out

For every real x , $\sqrt{(x^2)}=x$. This claim is false. Explain which definition or assumption it violates.

Interpret a new situation

- Keep exact surd expressions until a decimal is requested. Combine like surds only after simplification: $\sqrt{18}+\sqrt{8}=5\sqrt{2}$, but $\sqrt{2}+\sqrt{3}$ cannot be combined into one simple surd. If the conjugate is zero, this multiplier is invalid: simplify the original denominator directly, then check whether that denominator is zero. For example $1/(2+\sqrt{4})=1/4$ is valid even though its proposed conjugate is zero.
 - A complete solution gives the mathematical result and explains what it means. Check that it is possible in the stated context.
-

Use this in your course

- 7357 · A-level · B. Match the target tier and specification before assigning extensions.
- Give the method before the final answer, and use the paper's calculator and formula rules. Review a wrong answer by locating the first invalid step.

Key idea

A paired expression with the sign between two terms reversed. Choose the relationship, show the method, check its assumptions and interpret the result.

B · Completing the square and quadratic structure

A quadratic model reaches a lowest value. Which form tells us where it happens without drawing every point?

- A quadratic model reaches a lowest value. Which form tells us where it happens without drawing every point?
 - This lesson studies **completed square** : A quadratic written as a multiple of a squared bracket plus a constant.
-

Choose the mathematical structure

- For x^2+bx , add and subtract $(b/2)^2$ to obtain $(x+b/2)^2-(b/2)^2$. If the leading coefficient a is not 1, factor a from the x^2 and x terms first. The form $a(x-h)^2+k$ gives vertex (h,k) , axis $x=h$, and a minimum k if $a>0$ or a maximum k if $a<0$.
- State the allowed inputs and units before calculating. An equation should express the relationship, not just record a calculator entry.

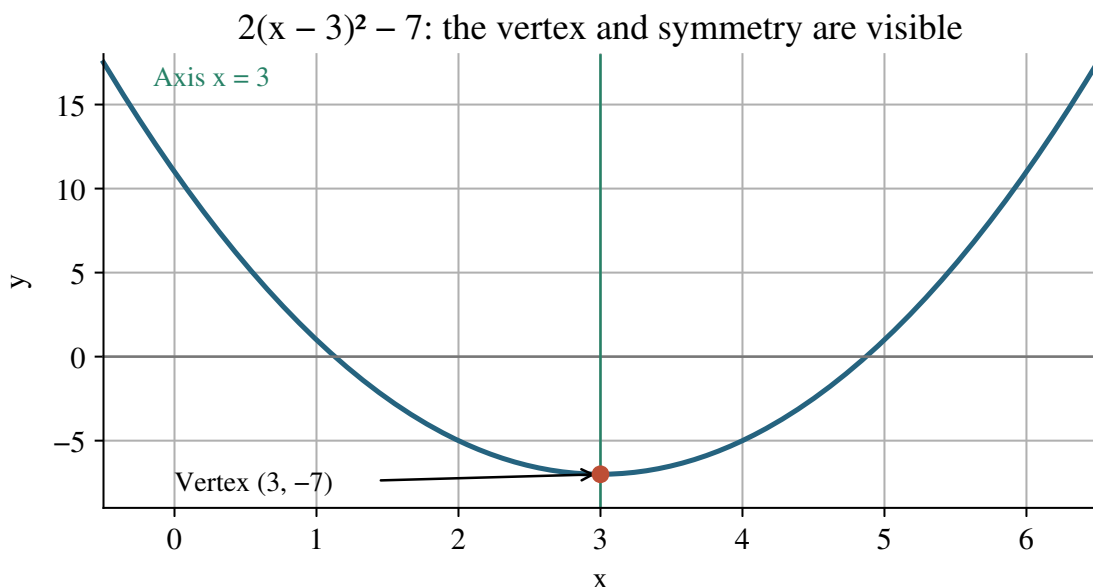
Work through a checked case

$$2x^2 - 12x + 11 = 2(x - 3)^2 - 7$$

- Check the result against the starting quantities. Substitute into the original relation, or compare the graph and numerical answer where appropriate.

Worked example

$2x^2-12x+11=2(x^2-6x)+11=2[(x-3)^2-9]+11=2(x-3)^2-7$. Its minimum is -7 at $x=3$. To solve $2x^2-12x+11=0$, obtain $(x-3)^2=7/2$, so $x=3\pm\sqrt{(7/2)}$. The roots lie equally far from $x=3$. In general $ax^2+bx+c=0$ becomes $(x+b/(2a))^2=(b^2-4ac)/(4a^2)$, for $a\neq 0$, leading to $x=(-b\pm\sqrt{(b^2-4ac)})/(2a)$. A negative discriminant means no real roots; zero gives one repeated root.



Test a tempting shortcut

- Half the coefficient inside the factored bracket, not the original coefficient of x . Keep the outside multiplier when subtracting the added square. A positive squared bracket does not imply the whole quadratic is positive: its constant can be negative.
- When a shortcut fails, identify the assumption it breaks. Keep an exact value until the requested final rounding.

Watch out

A completed-square expression always represents a quadratic with a minimum. This claim is false. Explain which definition or assumption it violates.

Interpret a new situation

- For the same model, $y \leq 1$ becomes $2(x-3)^2 \leq 8$, giving $1 \leq x \leq 5$. Check both end-points. A contextual restriction such as $x \geq 0$ must be combined with this interval. For $y = -2(x-3)^2 + 7$, the same vertex x -coordinate gives a maximum instead of a minimum.
 - A complete solution gives the mathematical result and explains what it means. Check that it is possible in the stated context.
-

Use this in your course

- 7357 · A-level · B. Match the target tier and specification before assigning extensions.
- Give the method before the final answer, and use the paper's calculator and formula rules. Review a wrong answer by locating the first invalid step.

Key idea

A quadratic written as a multiple of a squared bracket plus a constant. Choose the relationship, show the method, check its assumptions and interpret the result.

B · Equations quadratic in another expression

A fourth-degree equation can hide an ordinary quadratic. Solving for the new variable is only half the job.

- A fourth-degree equation can hide an ordinary quadratic. Solving for the new variable is only half the job.
 - This lesson studies **substitution** : Replacing a repeated expression with a new variable to reveal a simpler equation.
-

Choose the mathematical structure

- Identify a repeated expression and write u for it, including any restrictions. Solve the resulting quadratic in u . Then solve the original expression equal to each allowed u value. Substitute every final candidate into the original equation; a solution for u is not yet a solution for x .
 - State the allowed inputs and units before calculating. An equation should express the relationship, not just record a calculator entry.
-

Work through a checked case

$$x^4 - 5x^2 + 4 = 0 \implies u^2 - 5u + 4 = 0, \quad u = x^2 \geq 0$$

- Check the result against the starting quantities. Substitute into the original relation, or compare the graph and numerical answer where appropriate.

Worked example

For $x^4 - 5x^2 + 4 = 0$, set $u = x^2$ with $u \geq 0$. Then $u^2 - 5u + 4 = (u-1)(u-4) = 0$, so $x^2 = 1$ or 4 and $x = \pm 1, \pm 2$. For $x^4 + x^2 - 2 = 0$, the u -equation is $(u-1)(u+2) = 0$: $u = -2$ is impossible for real x , leaving $x = \pm 1$. For $(x+1)^2 - 5(x+1) + 6 = 0$, set $u = x+1$: $u = 2$ or 3 gives $x = 1$ or 2 . For $3\sqrt{x} - 4\sqrt{x} + 3 = 0$, set $u = \sqrt{x} > 0$: $u = 1$ or 3 gives $x = 0$ or 1 .

Two allowed u values give four distinct real x values

$$x^4 - 5x^2 + 4 = 0$$

$$u = x^2 \geq 0 : (u - 1)(u - 4) = 0$$

$$u = 1 \Rightarrow x = -1, 1$$

$$u = 4 \Rightarrow x = -2, 2$$

Test a tempting shortcut

- Do not report $u=1$ and $u=4$ as the x solutions of the quartic. Taking a square root needs both signs, but a principal square-root expression is nonnegative. A quadratic in u can have two real roots while one or both violate u 's domain.
- When a shortcut fails, identify the assumption it breaks. Keep an exact value until the requested final rounding.

Watch out

Two roots in the substituted variable always give exactly two real solutions in x . This claim is false. Explain which definition or assumption it violates.

Interpret a new situation

- Choose a substitution that repeats exactly: $3^{2x} = (3^x)^2$, while 3^{x^2} is a different expression. Count distinct real x solutions after returning to x , not before. Check by direct substitution: $x=-2$ gives $16-20+4=0$ in the first quartic.
 - A complete solution gives the mathematical result and explains what it means. Check that it is possible in the stated context.
-

Use this in your course

- 7357 · A-level · B. Match the target tier and specification before assigning extensions.
- Give the method before the final answer, and use the paper's calculator and formula rules. Review a wrong answer by locating the first invalid step.

Key idea

Replacing a repeated expression with a new variable to reveal a simpler equation. Choose the relationship, show the method, check its assumptions and interpret the result.

B · Inequalities, fractions and solution sets

A value must satisfy two conditions at once. How can we keep the allowed interval without losing a boundary?

- A value must satisfy two conditions at once. How can we keep the allowed interval without losing a boundary?
- This lesson studies **intersection** : The values common to two solution sets, corresponding to AND.

Choose the mathematical structure

- Adding the same expression preserves an inequality; multiplying or dividing by a negative number reverses it. Clear constant denominators using a positive common multiple. Factor a quadratic and test the sign on the intervals separated by its roots. AND takes the intersection of sets; OR takes their union. Distinguish included and excluded endpoints.
- State the allowed inputs and units before calculating. An equation should express the relationship, not just record a calculator entry.

Work through a checked case

$$(x-2)(x-5) < 0 \wedge x \geq 3 \implies x \in [3, 5)$$

- Check the result against the starting quantities. Substitute into the original relation, or compare the graph and numerical answer where appropriate.

Worked example

For $(2x-1)/3 - (x+2)/2 \geq 1/6$, multiply by 6: $4x-2-3x-6 \geq 1$, so $x \geq 9$. For $(x-2)(x-5) < 0$, the product is negative between its roots: $2 < x < 5$. Combining with $x \geq 3$ gives $3 \leq x < 5$, or $[3, 5)$. Taking OR with $x \leq 0$ gives $(-\infty, 0] \cup [3, 5)$. A denominator depending on x needs a separate sign/domain check: $(x-1)/(x+2) \geq 0$ has $x < -2$ or $x \geq 1$. The zero numerator at 1 is allowed; $x = -2$ is undefined.

AND keeps the overlap; open endpoints stay excluded

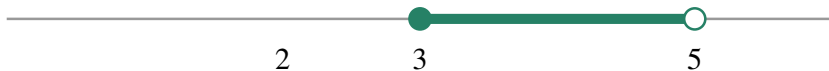
$$2 < x < 5$$



$$x \geq 3$$



$$\text{AND: } 3 \leq x < 5$$



Test a tempting shortcut

- Do not multiply by an expression of unknown sign as though it were positive. For $-2x < 6$, dividing by -2 gives $x > -3$. A root can be included only for a non-strict inequality and only where the expression is defined. Infinity is never a real endpoint to include.
- When a shortcut fails, identify the assumption it breaks. Keep an exact value until the requested final rounding.

Watch out

AND always means combine every value from both sets. This claim is false. Explain which definition or assumption it violates.

Interpret a new situation

- Use a sign table or test value on every interval, then write the whole set. The variable-denominator example extends the domain-checking method; B5's explicit core is linear/quadratic inequalities with brackets and fractions, and combined sets. Check $x=4$ and $x=6$ in the quadratic case to confirm which side is allowed.
 - A complete solution gives the mathematical result and explains what it means. Check that it is possible in the stated context.
-

Use this in your course

- 7357 · A-level · B. Match the target tier and specification before assigning extensions.
- Give the method before the final answer, and use the paper's calculator and formula rules. Review a wrong answer by locating the first invalid step.

Key idea

The values common to two solution sets, corresponding to AND. Choose the relationship, show the method, check its assumptions and interpret the result.

B · Regions between linear and quadratic boundaries

A point can lie above a parabola but still fail a second condition. Which points satisfy both boundaries?

- A point can lie above a parabola but still fail a second condition. Which points satisfy both boundaries?
 - This lesson studies **boundary** : The line or curve separating points that satisfy an inequality from those that do not.
-

Choose the mathematical structure

- Draw each equality boundary first. Use a solid line or curve for $\leq$ or $\geq$ and a dashed one for $<$ or $>$. For y greater than a function, shade above its graph; for y less than it, shade below. For AND, keep only the overlap. Test a point away from every boundary to confirm the chosen side.
- State the allowed inputs and units before calculating. An equation should express the relationship, not just record a calculator entry.

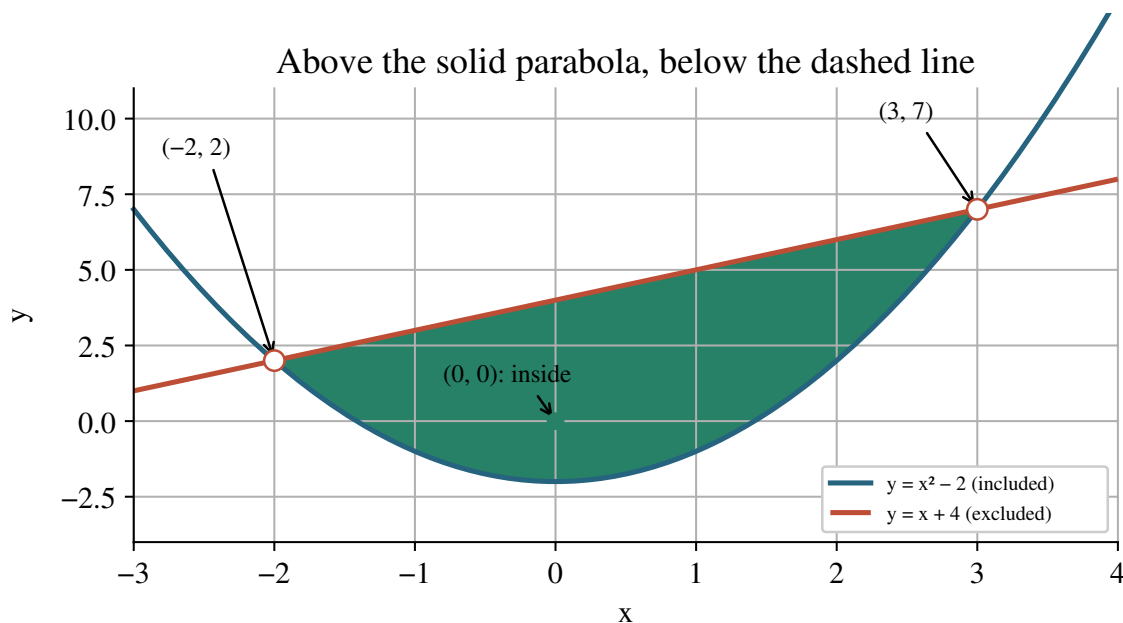
Work through a checked case

$$x^2 - 2 \leq y < x + 4, \quad -2 < x < 3$$

- Check the result against the starting quantities. Substitute into the original relation, or compare the graph and numerical answer where appropriate.

Worked example

For $y \geq x^2 - 2$ AND $y < x + 4$, draw the parabola solid and the line dashed. The boundaries meet when $x^2 - 2 = x + 4$: $(x - 3)(x + 2) = 0$, giving $(-2, 2)$ and $(3, 7)$. Between $-2 < x < 3$, the line is above the parabola. The allowed points satisfy $x^2 - 2 \leq y < x + 4$. At $x = 0$, the vertical interval is $-2 \leq y < 4$. The point $(0, 0)$ is included, $(0, 4)$ is excluded and $(0, -3)$ is excluded. The intersection points are excluded because the line inequality is strict.



Test a tempting shortcut

- A dashed boundary is not part of the region, even where it meets a solid boundary. Solving for intersection x-values gives the horizontal extent, not all allowed coordinates. A region for OR is the union, which can be much larger than the overlap.
- When a shortcut fails, identify the assumption it breaks. Keep an exact value until the requested final rounding.

Watch out

Every intersection point is included if either boundary is solid. This claim is false. Explain which definition or assumption it violates.

Interpret a new situation

- For a different inequality such as $y > x^2 - 2$, the permitted side is above the parabola and the curve itself is excluded. If a context also requires $x \geq 0$, clip the already combined region to that half-plane. Verify an interior point and a point outside each boundary separately.
 - A complete solution gives the mathematical result and explains what it means. Check that it is possible in the stated context.
-

Use this in your course

- 7357 · A-level · B. Match the target tier and specification before assigning extensions.
- Give the method before the final answer, and use the paper's calculator and formula rules. Review a wrong answer by locating the first invalid step.

Key idea

The line or curve separating points that satisfy an inequality from those that do not. Choose the relationship, show the method, check its assumptions and interpret the result.

B · Modulus graphs and piecewise solutions

A sensor measures distance from a target value. Values on either side give the same distance, but their algebraic signs differ.

- A sensor measures distance from a target value. Values on either side give the same distance, but their algebraic signs differ.
 - This lesson studies **modulus** : The nonnegative distance of a real number from zero.
-

Choose the mathematical structure

- For a real expression u , $|u|=u$ when $u \geq 0$ and $|u|=-u$ when $u < 0$. Find where a linear expression is zero to locate the corner of its V-shaped modulus graph. Solve equations by checking both branches and their domains; the right-hand side must be nonnegative.
 - State the allowed inputs and units before calculating. An equation should express the relationship, not just record a calculator entry.
-

Work through a checked case

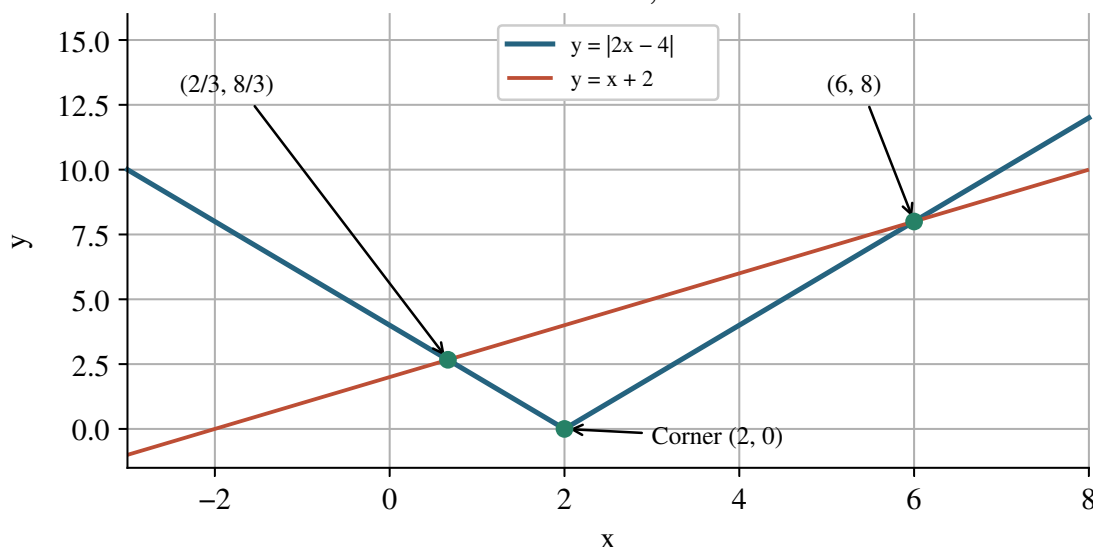
$$|2x - 4| = \begin{cases} 4 - 2x, & x < 2 \\ 2x - 4, & x \geq 2 \end{cases}$$

- Check the result against the starting quantities. Substitute into the original relation, or compare the graph and numerical answer where appropriate.

Worked example

For $y=|2x-4|$, the corner is $(2,0)$. The left branch is $y=4-2x$ for $x < 2$; the right is $y=2x-4$ for $x \geq 2$. To solve $|2x-4|=x+2$, require $x \geq -2$. The right branch gives $2x-4=x+2$, hence $x=6$. The left gives $4-2x=x+2$, hence $x=2/3$. Both satisfy their branch restrictions and the original equation. The graph meets the line at $(6,8)$ and $(2/3, 8/3)$.

Two checked intersections, one modulus corner



Test a tempting shortcut

- Do not replace $|u|$ by u on a negative branch. Squaring an equation can create candidates where its right-hand side is negative. The equation $|2x-4|=-1$ has no real solutions. The graph of $|2x-4|$ has one corner and its slopes change there.
- When a shortcut fails, identify the assumption it breaks. Keep an exact value until the requested final rounding.

Watch out

Every modulus equation has exactly two real solutions. This claim is false. Explain which definition or assumption it violates.

Interpret a new situation

- For $|2x-4|<6$, use $-6<2x-4<6$ to obtain $-1<x<5$. For $|2x-4|>6$, use $2x-4<-6$ OR $2x-4>6$: $x<-1$ or $x>5$. A bounded distance gives an inside interval; an excessive distance gives two outside intervals. Test both boundary and interior values.
- A complete solution gives the mathematical result and explains what it means. Check that it is possible in the stated context.

Use this in your course

- 7357 · A-level · B. Match the target tier and specification before assigning extensions.
- Give the method before the final answer, and use the paper's calculator and formula rules. Review a wrong answer by locating the first invalid step.

Key idea

The nonnegative distance of a real number from zero. Choose the relationship, show the method, check its assumptions and interpret the result.

B · Reciprocal graphs and asymptotes

An inverse model grows very large as its input approaches zero. Zero is a missing input, not a point to join across.

- An inverse model grows very large as its input approaches zero. Zero is a missing input, not a point to join across.
- This lesson studies **asymptote** : A line that a curve approaches in a limiting direction.

Choose the mathematical structure

- For nonzero a , $y=a/x$ has domain $x \neq 0$ and range $y \neq 0$, with asymptotes $x=0$ and $y=0$. Positive a gives branches in quadrants I and III; negative a gives II and IV. For $y=a/x^2$ the domain is still $x \neq 0$, but the graph is symmetric about the y -axis; both branches are positive if $a > 0$ and negative if $a < 0$.
- State the allowed inputs and units before calculating. An equation should express the relationship, not just record a calculator entry.

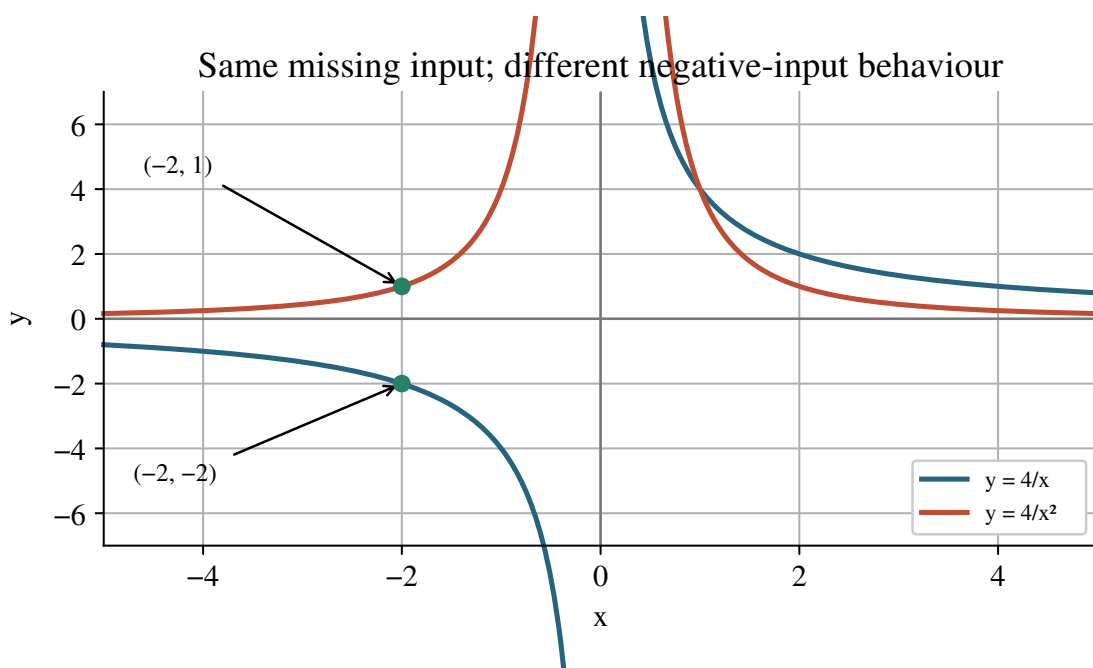
Work through a checked case

$$y = \frac{4}{x} \quad \text{or} \quad y = \frac{4}{x^2}, \quad x \neq 0$$

- Check the result against the starting quantities. Substitute into the original relation, or compare the graph and numerical answer where appropriate.

Worked example

For $y=4/x$, checked points are $(1,4)$, $(2,2)$, $(-1,-4)$, $(-2,-2)$. As x approaches zero from the right, y grows positively; from the left it grows negatively. As $|x|$ grows, y approaches zero. For $y=4/x^2$, $(1,4)$ and $(-1,4)$ have equal outputs, and both sides rise positively near zero. Intersecting $y=4/x$ with $y=x$ gives $x^2=4$, so $(2,2)$ and $(-2,-2)$. Intersecting $y=4/x^2$ with $y=1$ gives $x^2=4$, so $(2,1)$ and $(-2,1)$.



Test a tempting shortcut

- Never plot $x=0$ or draw a stroke joining the two branches through it. The reciprocal-square graph is not the reciprocal graph with the negative branch removed; the entire left branch is reflected above the x -axis for positive a . These particular curves do not meet $y=0$, but “no curve can ever cross any asymptote” is not a valid general rule.
- When a shortcut fails, identify the assumption it breaks. Keep an exact value until the requested final rounding.

Watch out

The graphs $y=4/x$ and $y=4/x^2$ have the same outputs for every negative x . This claim is false. Explain which definition or assumption it violates.

Interpret a new situation

- For positive physical inputs, $y=a/x$ can model inverse proportion and $y=a/x^2$ inverse-square proportion. If $y=4/x$ represents a time and speed relation, the negative branch is mathematically valid but outside that model's physical domain. Doubling a positive x halves $4/x$ and quarters $4/x^2$. Check graph intersections in the original relation after clearing a nonzero denominator.
 - A complete solution gives the mathematical result and explains what it means. Check that it is possible in the stated context.
-

Use this in your course

- 7357 · A-level · B. Match the target tier and specification before assigning extensions.
- Give the method before the final answer, and use the paper's calculator and formula rules. Review a wrong answer by locating the first invalid step.

Key idea

A line that a curve approaches in a limiting direction. Choose the relationship, show the method, check its assumptions and interpret the result.

B · Inverse graphs and restricted branches

A square model can give the same output from two inputs. Which branch must we choose before reversing it?

- A square model can give the same output from two inputs. Which branch must we choose before reversing it?
- This lesson studies **one-to-one** : A function for which different inputs always give different outputs.

Choose the mathematical structure

- An inverse reverses the input-output mapping, so the original function must be one-to-one on its stated domain. A horizontal line should meet its graph at most once. Swap x and y , solve for the new output and retain the correct branch. The inverse graph reflects the original in $y=x$: each point (a,b) becomes (b,a) . Original domain and range exchange roles.
 - State the allowed inputs and units before calculating. An equation should express the relationship, not just record a calculator entry.
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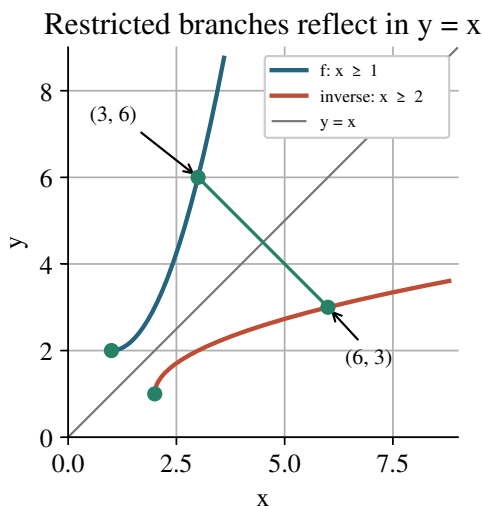
Work through a checked case

$$f(x) = (x-1)^2 + 2 \quad (x \geq 1), \quad f^{-1}(x) = 1 + \sqrt{x-2} \quad (x \geq 2)$$

- Check the result against the starting quantities. Substitute into the original relation, or compare the graph and numerical answer where appropriate.

Worked example

Let $f(x)=(x-1)^2+2$ for $x \geq 1$. Its range is $y \geq 2$. From $y=(x-1)^2+2$, the domain selects $x-1 \geq 0$, so $x=1+\sqrt{y-2}$. Thus f inverse $(x)=1+\sqrt{x-2}$, with domain $x \geq 2$ and range $y \geq 1$. The point $(3,6)$ reflects to $(6,3)$. Checking gives $f(f \text{ inverse}(x))=x$ for $x \geq 2$. In the other order, $f \text{ inverse}(f(x))=1+|x-1|=x$ for $x \geq 1$. Choosing the original domain $x \leq 1$ instead gives inverse $1-\sqrt{x-2}$, with range $y \leq 1$.



Test a tempting shortcut

- The same quadratic formula on all real inputs is not one-to-one. Do not choose the plus square root without using the original domain. An inverse function is not the reciprocal $1/f(x)$. A reflected graph needs its domain and range labels as well as its formula.
- When a shortcut fails, identify the assumption it breaks. Keep an exact value until the requested final rounding.

Watch out

Every quadratic on all real inputs has an inverse function. This claim is false. Explain which definition or assumption it violates.

Interpret a new situation

- On the branch $x \leq 1$, output 6 reverses to input -1 , while on $x \geq 1$ it reverses to 3. Both are correct for their own original domains. Check both compositions on their actual domains; applying the inverse identity to an input outside the original domain can fail.
 - A complete solution gives the mathematical result and explains what it means. Check that it is possible in the stated context.
-

Use this in your course

- 7357 · A-level · B. Match the target tier and specification before assigning extensions.
- Give the method before the final answer, and use the paper's calculator and formula rules. Review a wrong answer by locating the first invalid step.

Key idea

A function for which different inputs always give different outputs. Choose the relationship, show the method, check its assumptions and interpret the result.

B · Composite functions and intermediate domains

One machine's output becomes another machine's input. An allowed starting input can still produce an invalid intermediate value.

- One machine's output becomes another machine's input. An allowed starting input can still produce an invalid intermediate value.
 - This lesson studies **composition** : Applying one function to the output of another, in a specified order.
-

Choose the mathematical structure

- Here fg means f composed with g : $fg(x)=f(g(x))$, not multiplication. Apply g first. A composite input must belong to the inner function's domain, and its output must belong to the outer function's domain. In general fg and gf have different formulas and different domains.
 - State the allowed inputs and units before calculating. An equation should express the relationship, not just record a calculator entry.
-

Work through a checked case

$$(f \circ g)(x) = \sqrt{\frac{1}{x-1} - 2}, \quad 1 < x \leq \frac{3}{2}$$

- Check the result against the starting quantities. Substitute into the original relation, or compare the graph and numerical answer where appropriate.

Worked example

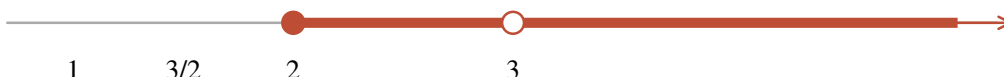
Let $f(x)=\sqrt{x-2}$, with $x \geq 2$, and $g(x)=1/(x-1)$, with $x \neq 1$. Then $fg(x)=\sqrt{1/(x-1)-2}$. It requires $1/(x-1) \geq 2$. A negative denominator cannot meet this positive lower bound; for $x > 1$, multiplying by $x-1$ preserves order and gives $x \leq 3/2$. Thus the domain is $1 < x \leq 3/2$. In the reverse order, $gf(x)=1/(\sqrt{x-2}-1)$. It requires $x \geq 2$ and $\sqrt{x-2} \neq 1$, so $x \neq 3$. Its domain is $[2,3) \cup (3,\infty)$. At $x=6$, $gf(6)=1$, while $fg(6)$ is not real.

The intermediate output changes the allowed inputs

fg: $1 < x \leq 3/2$



gf: $x \geq 2$, with $x = 3$ excluded



Test a tempting shortcut

- Checking only the inner domain is insufficient. Do not multiply the inequality by $x-1$ before deciding its sign. An excluded intermediate denominator remains excluded even if another form appears simpler. A formula and domain together define the composite function.
- When a shortcut fails, identify the assumption it breaks. Keep an exact value until the requested final rounding.

Watch out

A composite is defined wherever its inner function is defined. This claim is false. Explain which definition or assumption it violates.

Interpret a new situation

- Trace the intermediate value: $gf(6)$ uses $f(6)=2$ then $g(2)=1$. For $fg(5/4)$, $g(5/4)=4$ then $f(4)=\sqrt{2}$. At $x=3$, $f(3)=1$ is valid, but $g(1)$ is undefined, so $gf(3)$ is undefined. Mark every retained or excluded endpoint in interval notation.
- A complete solution gives the mathematical result and explains what it means. Check that it is possible in the stated context.

Use this in your course

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Key idea

Applying one function to the output of another, in a specified order. Choose the relationship, show the method, check its assumptions and interpret the result.

B · Combined graph transformations and order

A compressed curve is moved to the right. Does moving it first give the same final graph?

- A compressed curve is moved to the right. Does moving it first give the same final graph?
- This lesson studies **horizontal scale factor** : The factor multiplying the original horizontal coordinates when a graph is stretched or compressed.

Choose the mathematical structure

- For $g(x) = a f(b(x-h)) + k$ with $b \neq 0$, an original point (u, v) maps to $(u/b + h, av + k)$. Horizontal scale is $1/|b|$, with reflection in the y-axis if $b < 0$. Vertical scale is $|a|$, with reflection in the x-axis if $a < 0$. Read the factored inner expression before naming a horizontal translation. The transformed domain follows $u = b(x-h)$.
- State the allowed inputs and units before calculating. An equation should express the relationship, not just record a calculator entry.

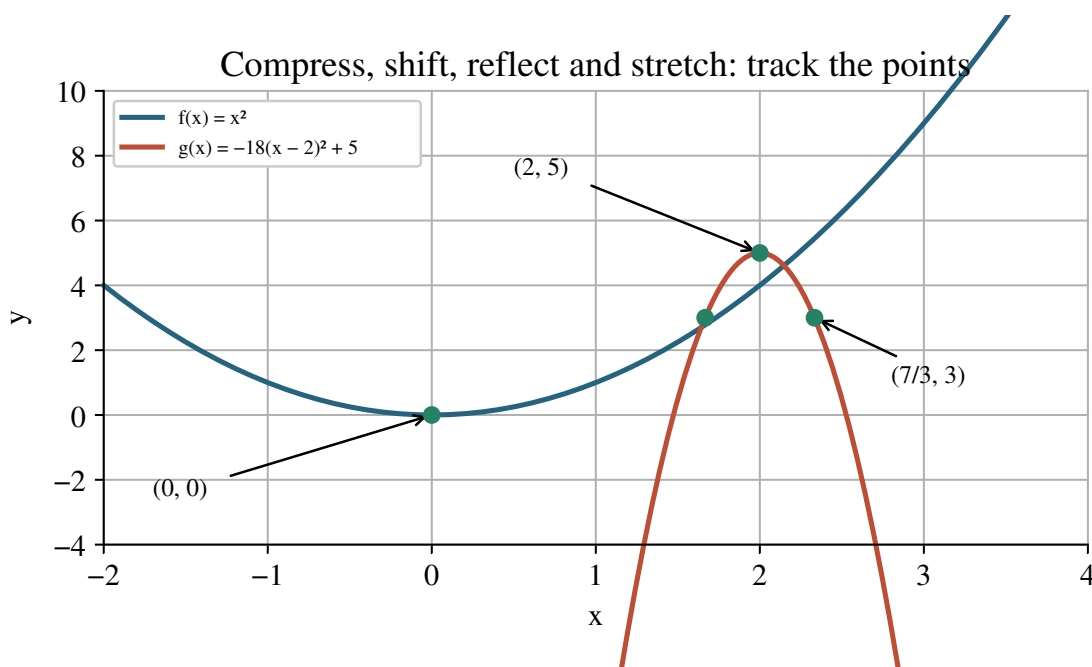
Work through a checked case

$$g(x) = af(b(x - h)) + k : (u, v) \mapsto (u/b + h, av + k)$$

- Check the result against the starting quantities. Substitute into the original relation, or compare the graph and numerical answer where appropriate.

Worked example

With $f(x)=x^2$, $g(x)=-2f(3x-6)+5=-2f(3(x-2))+5=-18(x-2)^2+5$. Compress horizontally by factor $1/3$, then translate right 2; reflect vertically and stretch by factor 2, then translate up 5. The points $(-1,1),(0,0),(1,1)$ become $(5/3,3),(2,5),(7/3,3)$. Thus the vertex is $(2,5)$ and the range is $y \leq 5$. Compressing first and then shifting right 2 gives $f(3(x-2))$; shifting first and then compressing gives $f(3x-2)$, whose vertex is at $x=2/3$ instead.



Test a tempting shortcut

- The coefficient 3 inside f gives horizontal factor $1/3$, not 3. In $f(3x-6)$, the translation is 2, not 6. Translation and scaling generally do not commute. A symmetric example can hide a reflection: $f(-x)=f(x)$ for x^2 , but not for every f .
- When a shortcut fails, identify the assumption it breaks. Keep an exact value until the requested final rounding.

Watch out

A horizontal translation and horizontal scaling can always be applied in either order. This claim is false. Explain which definition or assumption it violates.

Interpret a new situation

- Use point mapping to check every combined transformation, and transfer any restricted domain. If the original f is defined only for $0 \leq u \leq 3$, the inner condition $0 \leq 3(x-2) \leq 3$ gives $2 \leq x \leq 3$. Changing b to a negative value reverses endpoint order when solving the domain inequality. Plot the transformed reference points before joining the curve.
 - A complete solution gives the mathematical result and explains what it means. Check that it is possible in the stated context.
-

Use this in your course

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- Give the method before the final answer, and use the paper's calculator and formula rules. Review a wrong answer by locating the first invalid step.

Key idea

The factor multiplying the original horizontal coordinates when a graph is stretched or compressed. Choose the relationship, show the method, check its assumptions and interpret the result.

B · Fitting and refining a function model

Two different curve models fit the same three measurements. Which extra observation can help choose between them?

- Two different curve models fit the same three measurements. Which extra observation can help choose between them?
- This lesson studies **residual** : An observed output minus the output predicted by a model.

Choose the mathematical structure

- State inputs, outputs, units and the modelling domain. Fit parameters using the stated observations, then compare predictions with independent measurements. An exact fit to the calibration points does not prove the chosen function is correct everywhere. A residual records observed minus predicted output; model refinement needs evidence and a reasoned change.
 - State the allowed inputs and units before calculating. An equation should express the relationship, not just record a calculator entry.
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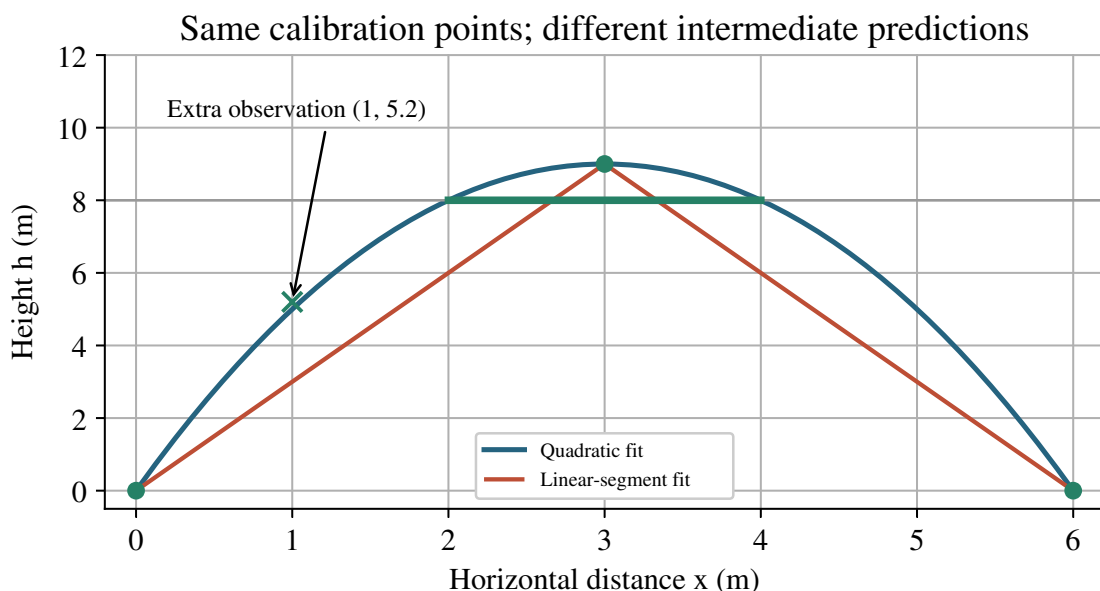
Work through a checked case

$$h(x) = -x^2 + 6x = 9 - (x - 3)^2, \quad 0 \leq x \leq 6$$

- Check the result against the starting quantities. Substitute into the original relation, or compare the graph and numerical answer where appropriate.

Worked example

For a proposed arch, x is horizontal distance and h its height, both in metres. Use the illustrative points $(0,0), (3,9), (6,0)$, with $0 \leq x \leq 6$. Let $h = ax^2 + bx + c$. Then $c = 0$, $9a + 3b = 9$ and $36a + 6b = 0$ give $a = -1, b = 6$. Thus $h = -x^2 + 6x = 9 - (x - 3)^2$, with maximum 9 m at $x = 3$. A piecewise-linear alternative $h = 3x$ for $0 \leq x \leq 3$ and $h = 18 - 3x$ for $3 \leq x \leq 6$ fits the same three points. At $x = 1$, an illustrative extra measurement 5.2 m gives residual 0.2 m for the quadratic but 2.2 m for the linear alternative. This favours the quadratic at that point, not a universal verdict.



Test a tempting shortcut

- Do not use a fitted calibration point as independent validation. Outside $0 \leq x \leq 6$, the quadratic can give negative height and no longer describes the proposed arch. A smaller residual at one point does not establish accuracy everywhere. The simpler model can be easier to interpret but still miss curvature.
- When a shortcut fails, identify the assumption it breaks. Keep an exact value until the requested final rounding.

Watch out

A function that fits three calibration points is proven correct at every input. This claim is false. Explain which definition or assumption it violates.

Interpret a new situation

- For height at least 8 m, $9 - (x-3)^2 \geq 8$ gives $2 \leq x \leq 4$: modelled horizontal width 2 m. Collect more intermediate measurements and estimate measurement uncertainty before relying on this clearance. Thickness and changes in shape can require a refined domain or formula; explain which observation motivates the refinement.
- A complete solution gives the mathematical result and explains what it means. Check that it is possible in the stated context.

Use this in your course

- 7357 · A-level · B. Match the target tier and specification before assigning extensions.
- Give the method before the final answer, and use the paper's calculator and formula rules. Review a wrong answer by locating the first invalid step.

Key idea

An observed output minus the output predicted by a model. Choose the relationship, show the method, check its assumptions and interpret the result.

B · Polynomial sketches, repeated roots and signs

Two roots can appear in a factorised formula but make very different marks on its graph. Which one crosses the axis?

- Two roots can appear in a factorised formula but make very different marks on its graph. Which one crosses the axis?
- This lesson studies **multiplicity** : The number of times a root's linear factor appears in a polynomial.

Choose the mathematical structure

- Find real roots and their multiplicities, the y-intercept and the leading term. A root of odd multiplicity changes the sign and crosses the x-axis; a root of even multiplicity keeps the sign and touches the axis. For a positive leading coefficient, an odd-degree curve goes down on the far left and up on the far right; an even-degree curve goes up at both ends. A negative leading coefficient reverses these directions.
 - State the allowed inputs and units before calculating. An equation should express the relationship, not just record a calculator entry.
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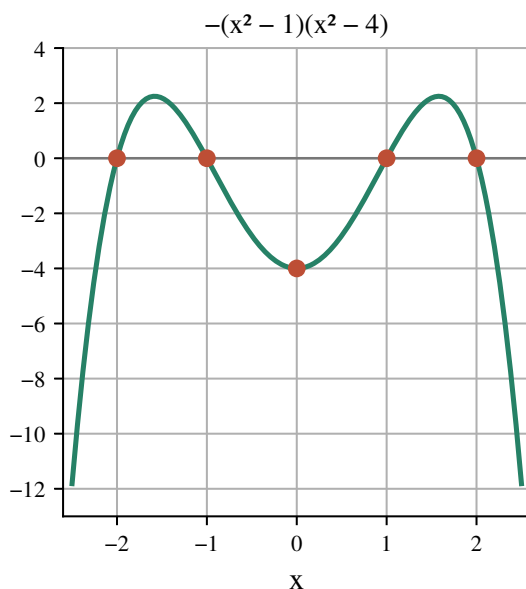
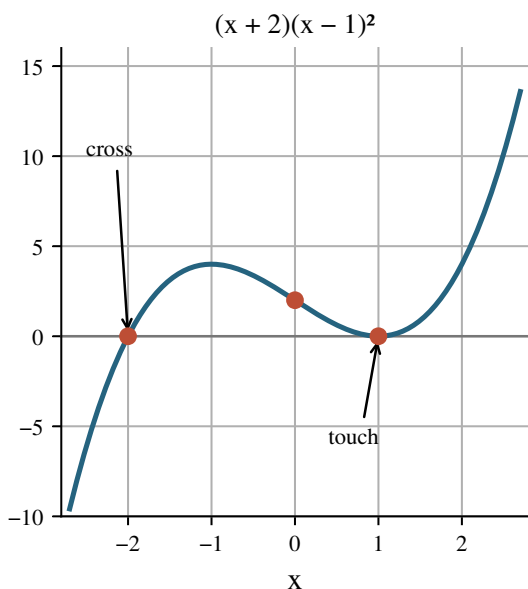
Work through a checked case

$$P(x) = (x + 2)(x - 1)^2, \quad Q(x) = -(x^2 - 1)(x^2 - 4)$$

- Check the result against the starting quantities. Substitute into the original relation, or compare the graph and numerical answer where appropriate.

Worked example

For $P(x) = (x+2)(x-1)^2$, roots are -2 with multiplicity 1 and 1 with multiplicity 2. The y-intercept is $P(0)=2$. The leading term is x^3 , so the left tail falls and the right tail rises. P crosses at -2 and touches at 1 . It is negative for $x < -2$, positive for $-2 < x < 1$ and $x > 1$, and zero at both roots. For $Q(x) = -(x^2-1)(x^2-4)$, the four simple roots are $-2, -1, 1, 2$, the y-intercept is -4 and the leading term is $-x^4$. Both tails fall. Q is positive on $(-2, -1)$ and $(1, 2)$, and negative on the other three root-separated intervals.



Test a tempting shortcut

- The number of distinct roots need not equal the degree. A repeated root is one x-intercept even though its factor occurs more than once. Do not alternate the sign at an even-multiplicity root. A sketch shows structure; exact turning-point coordinates require further analysis, not guessed readings from the picture.
- When a shortcut fails, identify the assumption it breaks. Keep an exact value until the requested final rounding.

Watch out

Every real root of a polynomial must change the sign of its values. This claim is false. Explain which definition or assumption it violates.

Interpret a new situation

- Graphical solutions of $P(x)=k$ are intersections of the curve with $y=k$. For $k=0$ the factored roots are exact; for another k a graph may only estimate solutions. Use the sign intervals to solve $P(x)\leq 0$: $x\leq -2$ or $x=1$. That isolated root matters even though its neighbouring values are positive.
 - A complete solution gives the mathematical result and explains what it means. Check that it is possible in the stated context.
-

Use this in your course

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- Give the method before the final answer, and use the paper's calculator and formula rules. Review a wrong answer by locating the first invalid step.

Key idea

The number of times a root's linear factor appears in a polynomial. Choose the relationship, show the method, check its assumptions and interpret the result.

C · Coordinate geometry and tangents

How does a path's slope become an equation?

- A path rises 6 metres over a horizontal distance of 3 metres. Its gradient connects a diagram to an equation.
 - This lesson studies **gradient** : The change in y divided by the corresponding change in x .
-

Choose the mathematical structure

- A line through (x_1, y_1) with gradient m has $y - y_1 = m(x - x_1)$. Parallel lines have equal gradients. Finite perpendicular gradients multiply to -1 . A circle has $(x - a)^2 + (y - b)^2 = r^2$.
 - State the allowed inputs and units before calculating. An equation should express the relationship, not just record a calculator entry.
-

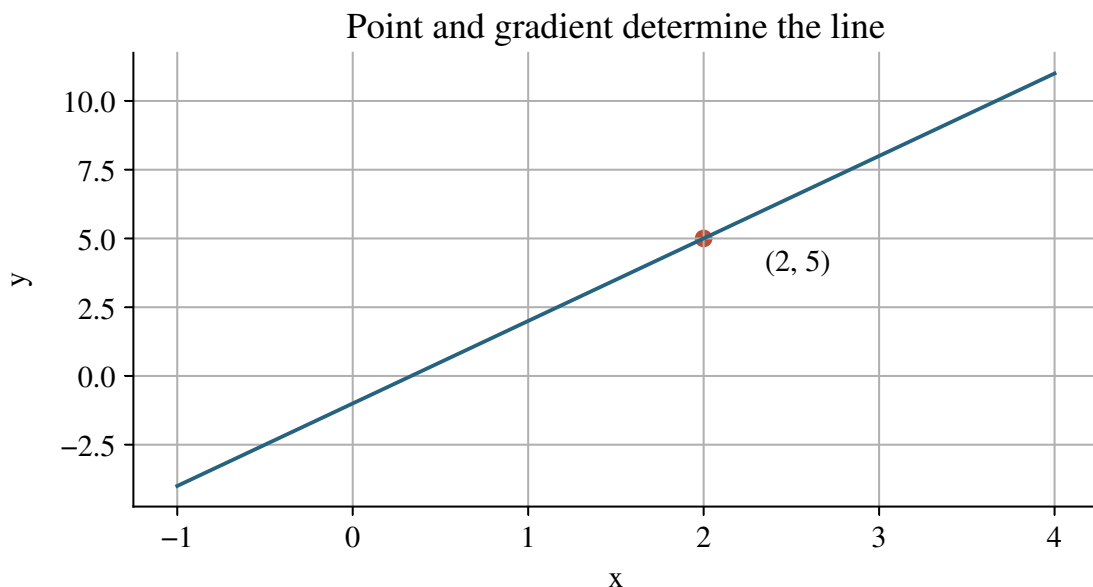
Work through a checked case

$$y - y_1 = m(x - x_1), \quad (x - a)^2 + (y - b)^2 = r^2$$

- Check the result against the starting quantities. Substitute into the original relation, or compare the graph and numerical answer where appropriate.

Worked example

Through $(2, 5)$ with gradient 3 , $y - 5 = 3(x - 2)$, so $y = 3x - 1$. A perpendicular through the same point has $y - 5 = -(x - 2)/3$. The circle $(x - 2)^2 + (y + 1)^2 = 25$ has centre $(2, -1)$ and radius 5 .



Test a tempting shortcut

- A vertical line has no finite gradient; do not force it into $y=mx+c$. Read the signs of a circle's centre carefully. The radius to a tangent is perpendicular to the tangent.
- When a shortcut fails, identify the assumption it breaks. Keep an exact value until the requested final rounding.

Watch out

Perpendicular nonvertical lines always have equal gradients. This claim is false. Explain which definition or assumption it violates.

Interpret a new situation

- Before solving a line-circle intersection, predict whether there are zero, one or two intersections. Substitution produces a quadratic whose discriminant checks the prediction.
 - A complete solution gives the mathematical result and explains what it means. Check that it is possible in the stated context.
-

Use this in your course

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- Give the method before the final answer, and use the paper's calculator and formula rules. Review a wrong answer by locating the first invalid step.

Key idea

The change in y divided by the corresponding change in x . Choose the relationship, show the method, check its assumptions and interpret the result.

C · Parametric curves, domains and direction

What information disappears when time is eliminated?

- Two coordinates can change together as a third variable changes. Eliminating that variable can hide which part of the curve was traced.
 - This lesson studies **parameter** : A variable that determines both coordinates of a point.
-

Choose the mathematical structure

- For $x=f(t)$, $y=g(t)$, express t from one equation if possible and substitute into the other. Transfer the parameter interval to the Cartesian curve and mark the direction of increasing t . With trigonometric parameters, use identities and keep sign or branch restrictions.
 - State the allowed inputs and units before calculating. An equation should express the relationship, not just record a calculator entry.
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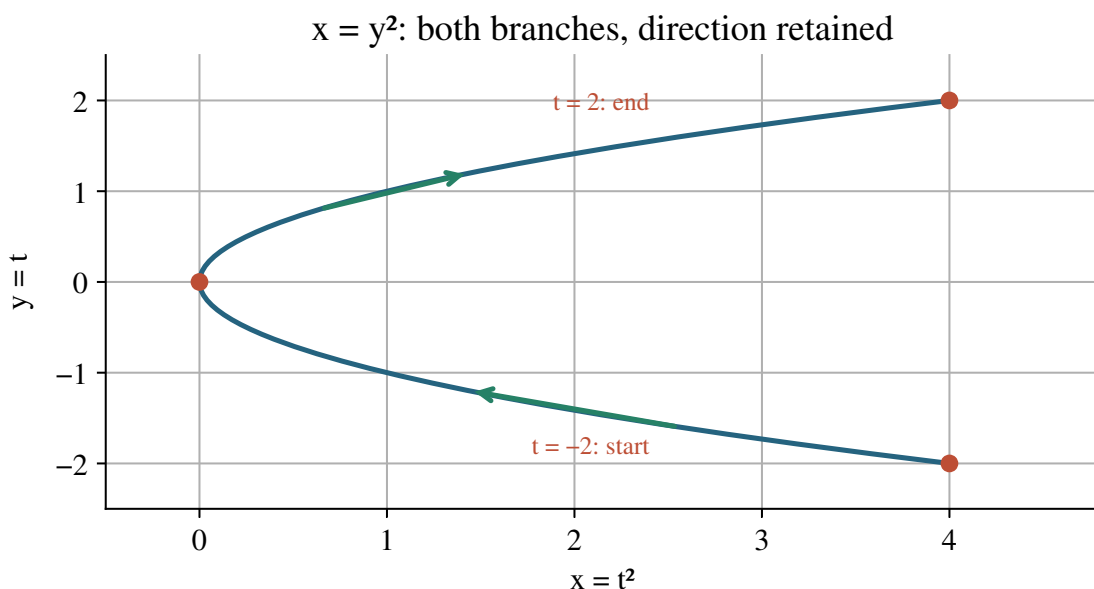
Work through a checked case

$$x = 2t + 1, \quad y = t^2 - 2 \implies y = \frac{(x-1)^2}{4} - 2, \quad 1 \leq x \leq 7$$

- Check the result against the starting quantities. Substitute into the original relation, or compare the graph and numerical answer where appropriate.

Worked example

For $x=2t+1$, $y=t^2-2$ and $0 \leq t \leq 3$, $t=(x-1)/2$ gives $y=(x-1)^2/4-2$ with $1 \leq x \leq 7$. The point moves from $(1,-2)$ to $(7,7)$. For $x=t^2$, $y=t$ and $-2 \leq t \leq 2$, the relation is $x=y^2$ with $-2 \leq y \leq 2$: both signs of y occur. For $x=3\cos\theta$, $y=2\sin\theta$ and $0 \leq \theta \leq \pi$, $x^2/9+y^2/4=1$ with $y \geq 0$; increasing θ traces the upper half from $(3,0)$ to $(-3,0)$. Conversely, $y=x^2$ with $-1 \leq x \leq 2$ can be parametrised by $x=t$, $y=t^2$ and $-1 \leq t \leq 2$.



Test a tempting shortcut

- Writing $y = \sqrt{x}$ from $x = t^2, y = t$ loses the part where t is negative. A Cartesian equation alone does not record direction or the original parameter interval. Squaring can erase a sign restriction.
- When a shortcut fails, identify the assumption it breaks. Keep an exact value until the requested final rounding.

Watch out

Eliminating a parameter always preserves the original domain and direction automatically. This claim is false. Explain which definition or assumption it violates.

Interpret a new situation

- Use a table of parameter values to sketch endpoints and intermediate points. A parameter can revisit an x value or a whole point. Check the original equations and interval when converting back to parametric form.
 - A complete solution gives the mathematical result and explains what it means. Check that it is possible in the stated context.
-

Use this in your course

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- Give the method before the final answer, and use the paper's calculator and formula rules. Review a wrong answer by locating the first invalid step.

Key idea

A variable that determines both coordinates of a point. Choose the relationship, show the method, check its assumptions and interpret the result.

C · Parametric modelling in motion and design

When does a mathematical path stop describing a real flight?

- A ball's height and horizontal position depend on the same time. A Cartesian path cannot tell us when it is reached without the time model.
- This lesson studies **parametric model** : A model expressing two quantities using a common variable and stated assumptions.

Choose the mathematical structure

- Name the parameter, its units and allowed interval. Evaluate both model quantities at the same parameter value. Eliminate the parameter to relate the quantities, then interpret endpoints and limitations. A mathematical curve beyond the allowed interval need not describe the physical situation.
- State the allowed inputs and units before calculating. An equation should express the relationship, not just record a calculator entry.

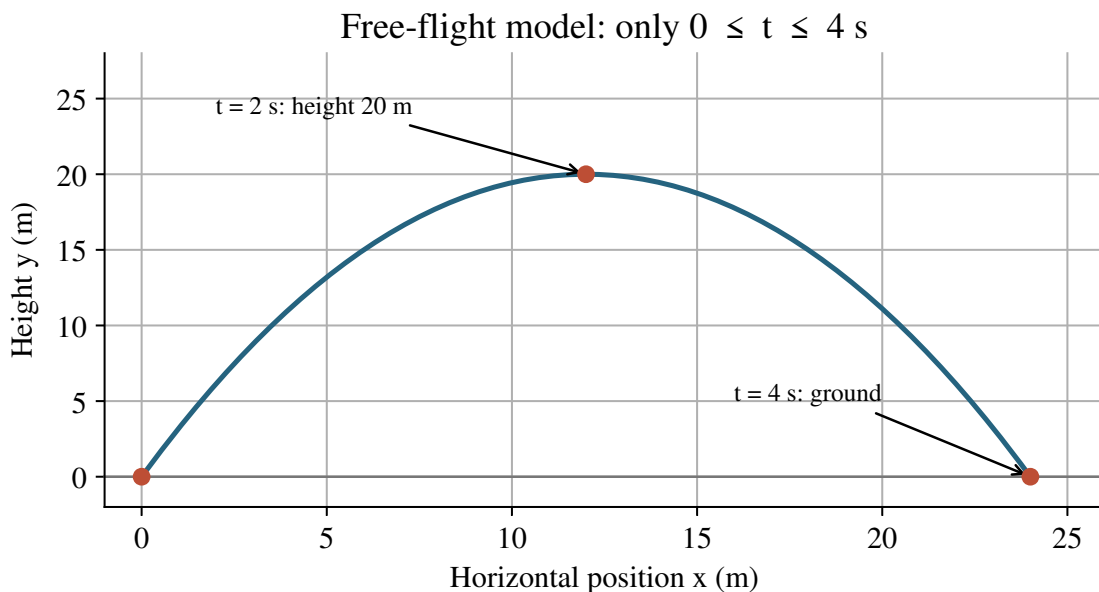
Work through a checked case

$$x = 6t, \quad y = 20t - 5t^2, \quad 0 \leq t \leq 4$$

- Check the result against the starting quantities. Substitute into the original relation, or compare the graph and numerical answer where appropriate.

Worked example

A simplified flight model uses $x=6t$ and $y=20t-5t^2$ in metres, with t in seconds and $0 \leq t \leq 4$. It assumes constant horizontal speed and downward acceleration 10 metres per second squared, ignoring air resistance. The noninitial ground time is $t=4$, giving $x=24$ m. Completing the square gives $y=20-5(t-2)^2$, so maximum height is 20 m at $t=2$. Eliminating t gives $y=10x/3-5x^2/36$ for $0 \leq x \leq 24$. For a rectangle of width t and length $t+2$ metres, $P=4t+4$ and $A=t(t+2)$. With $0 < t \leq 5$, $A=(P^2-16)/16$ and $4 < P \leq 24$.



Test a tempting shortcut

- The flight formula giving negative height after $t=4$ does not describe continued free flight below the ground. Do not use t measured in minutes with coefficients defined for seconds. In the rectangle model, $t=0$ is excluded because the width must be positive.
- When a shortcut fails, identify the assumption it breaks. Keep an exact value until the requested final rounding.

Watch out

The Cartesian flight curve is a physically valid model for every real x . This claim is false. Explain which definition or assumption it violates.

Interpret a new situation

- Check a model against observed data and its assumptions. Wind or air resistance could require refinement of the flight model; a different length–width relation changes the rectangle model. The acceleration 10 here is a stated simplification, not a replacement for a paper’s instructed value of g .
 - A complete solution gives the mathematical result and explains what it means. Check that it is possible in the stated context.
-

Use this in your course

- 7357 · A-level · C. Match the target tier and specification before assigning extensions.
- Give the method before the final answer, and use the paper’s calculator and formula rules. Review a wrong answer by locating the first invalid step.

Key idea

A model expressing two quantities using a common variable and stated assumptions. Choose the relationship, show the method, check its assumptions and interpret the result.

C · General-form lines and contextual models

Which part of a delivery charge changes with distance?

- A straight-line cost model has a fixed charge and a rate per kilometre. Its intercept and gradient have different meanings and units.
 - This lesson studies **general form** : The form $ax+by+c=0$ for a straight line, with a and b not both zero.
-

Choose the mathematical structure

- For a nonvertical line through two points, $m=(y_2-y_1)/(x_2-x_1)$ and $y-y_1=m(x-x_1)$. Rearrange into $ax+by+c=0$. Parallel nonvertical lines have

equal gradients; perpendicular finite gradients multiply to -1 . Treat vertical and horizontal lines separately.

- State the allowed inputs and units before calculating. An equation should express the relationship, not just record a calculator entry.

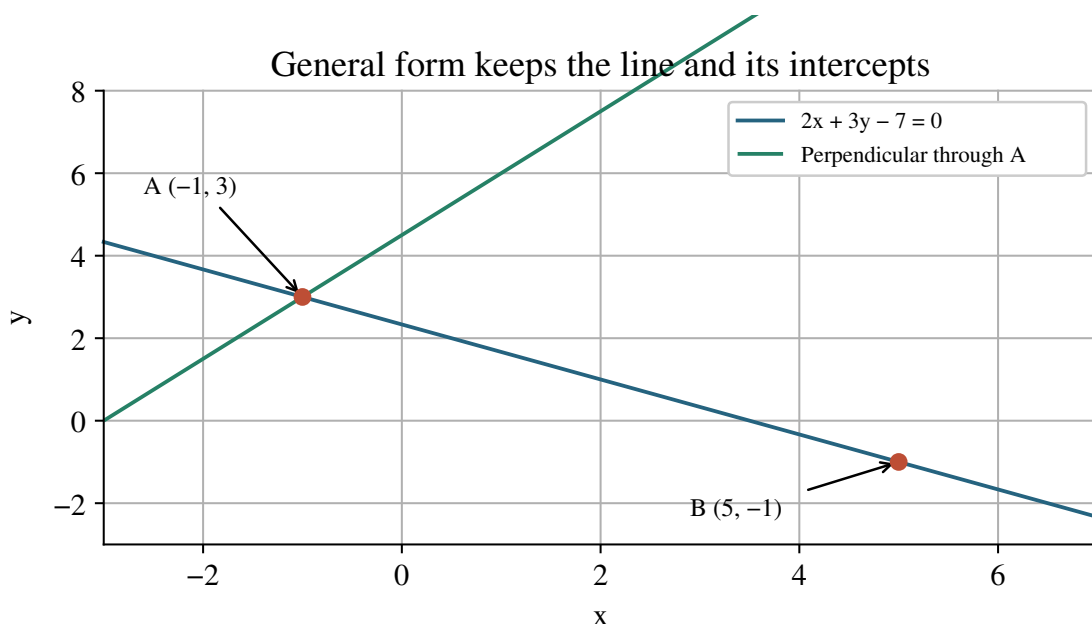
Work through a checked case

$$y - y_1 = m(x - x_1) \implies ax + by + c = 0$$

- Check the result against the starting quantities. Substitute into the original relation, or compare the graph and numerical answer where appropriate.

Worked example

Through $A(-1,3)$ and $B(5,-1)$, $m = -4/6 = -2/3$. Thus $y - 3 = -2(x + 1)/3$, giving $2x + 3y - 7 = 0$. The perpendicular through A has gradient $3/2$ and equation $3x - 2y + 9 = 0$. A vertical line $x = 4$ is perpendicular to a horizontal line $y = 3$ without having a finite gradient. Delivery quotes $C = 8$ at $d = 0$ km and $C = 38$ at $d = 12$ km give rate $(38 - 8)/12 = 2.5$ currency units per kilometre and $C = 8 + 2.5d$. At $d = 8$, the predicted cost is 28.



Test a tempting shortcut

- For $ax+by+c=0$, the gradient is $-a/b$ only if $b \neq 0$. Do not confuse the fixed charge with the price per kilometre. A line fitted to two quoted distances does not prove the same tariff applies at all distances.
- When a shortcut fails, identify the assumption it breaks. Keep an exact value until the requested final rounding.

Watch out

Every straight line has a finite gradient and can be written as $y=mx+c$. This claim is false. Explain which definition or assumption it violates.

Interpret a new situation

- Interpret the cost intercept, gradient and nonnegative distance domain. Within $0 \leq d \leq 12$, interpolation uses the observed quote range; beyond it, extrapolation needs a tariff assumption. Check both original points after rearranging a geometric line.
 - A complete solution gives the mathematical result and explains what it means. Check that it is possible in the stated context.
-

Use this in your course

- 7357 · A-level · C. Match the target tier and specification before assigning extensions.
- Give the method before the final answer, and use the paper's calculator and formula rules. Review a wrong answer by locating the first invalid step.

Key idea

The form $ax+by+c=0$ for a straight line, with a and b not both zero. Choose the relationship, show the method, check its assumptions and interpret the result.

C · Shifted circles, chords and tangents

Where is the centre hidden in an expanded circle equation?

- An expanded circle equation can hide its centre and radius. Completing the square exposes the geometry needed for a chord or tangent.
 - This lesson studies **centre** : The point at equal distance from every point on a circle.
-

Choose the mathematical structure

- Rewrite a circle as $(x-a)^2+(y-b)^2=r^2$ by completing both squares. A perpendicular from the centre bisects a chord, so half-chord length follows from a right triangle. A tangent is perpendicular to the radius at the contact point. An angle subtended by a diameter at the circumference is 90° .
 - State the allowed inputs and units before calculating. An equation should express the relationship, not just record a calculator entry.
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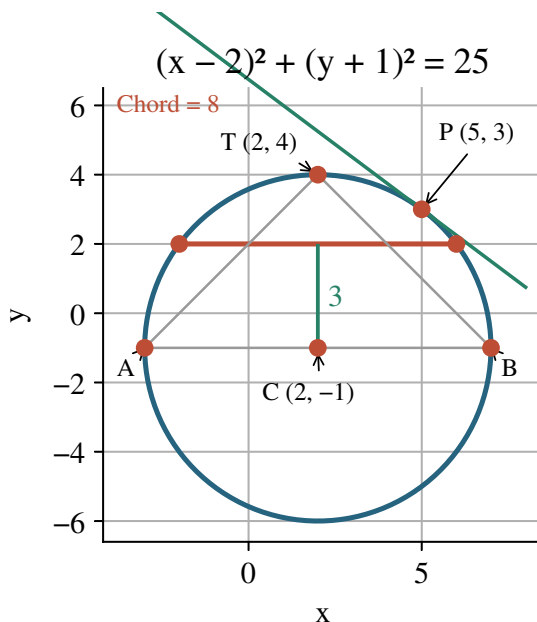
Work through a checked case

$$x^2 + y^2 - 4x + 2y - 20 = 0 \implies (x - 2)^2 + (y + 1)^2 = 25$$

- Check the result against the starting quantities. Substitute into the original relation, or compare the graph and numerical answer where appropriate.

Worked example

For $x^2+y^2-4x+2y-20=0$, completing squares gives $(x-2)^2+(y+1)^2=25$: centre $(2,-1)$, radius 5. The chord $y=2$ is 3 units above the centre; its half-length is $\sqrt{(25-9)}=4$, so endpoints are $(-2,2),(6,2)$ and length 8. At $P(5,3)$, the radius gradient is $4/3$, giving tangent $y-3=-3(x-5)/4$ or $3x+4y=27$. For diameter endpoints $A(-3,-1)$, $B(7,-1)$ and circumference point $T(2,4)$, gradients $AT=1$ and $BT=-1$ verify angle $ATB=90^\circ$.



Test a tempting shortcut

- Completing a square adds a constant: adjust the other side too. The circle's centre has the opposite signs to the bracket constants. A proposed tangent point must lie on the circle; perpendicularity alone at an unrelated point is insufficient.
- When a shortcut fails, identify the assumption it breaks. Keep an exact value until the requested final rounding.

Watch out

A line perpendicular to any radius is automatically a tangent, wherever it crosses the plane. This claim is false. Explain which definition or assumption it violates.

Interpret a new situation

- Check radius distance for each contact point and both chord endpoints. A negative completed r^2 gives no real circle, while $r^2=0$ gives a single point rather than a nondegenerate circle. Handle a horizontal or vertical radius without dividing by zero.
- A complete solution gives the mathematical result and explains what it means. Check that it is possible in the stated context.

Use this in your course

- 7357 · A-level · C. Match the target tier and specification before assigning extensions.
- Give the method before the final answer, and use the paper's calculator and formula rules. Review a wrong answer by locating the first invalid step.

Key idea

The point at equal distance from every point on a circle. Choose the relationship, show the method, check its assumptions and interpret the result.

D · Sequences, series and recurrence

Does the change add or multiply?

- A saving plan adds ¥30 more each week; a population model grows by 5% each year. Equal differences and equal ratios need different models.
- This lesson studies **common ratio** : The constant multiplier between consecutive terms of a geometric sequence.

Choose the mathematical structure

- For an arithmetic progression, $u_n = a + (n-1)d$ and $S_n = n(2a + (n-1)d)/2$. For a geometric progression, $u_n = ar^{n-1}$ and $S_n = a(1-r^n)/(1-r)$. An infinite geometric sum exists only if $|r| < 1$.
- State the allowed inputs and units before calculating. An equation should express the relationship, not just record a calculator entry.

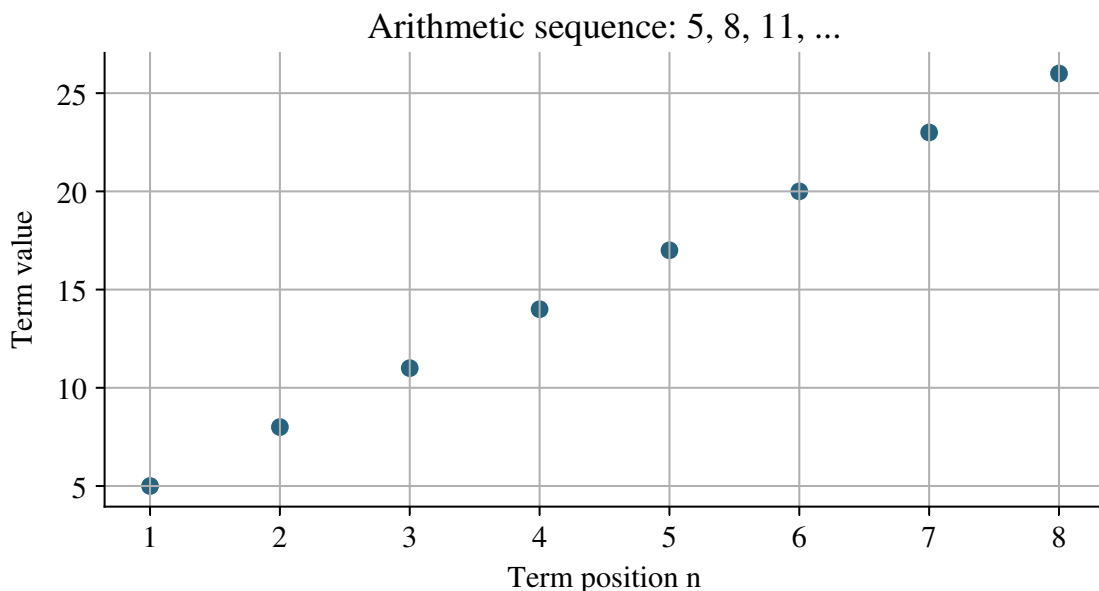
Work through a checked case

$$u_n = a + (n-1)d, \quad S_n = \frac{n}{2} [2a + (n-1)d]$$

- Check the result against the starting quantities. Substitute into the original relation, or compare the graph and numerical answer where appropriate.

Worked example

For $a=5, d=3, n=8$, $u_8=5+7\times 3=26$ and $S_8=8(10+21)/2=124$. For $a=12, r=1/2$, $S_{\infty}=12/(1-1/2)=24$. For $u_{(n+1)}=2u_n+1$ with $u_1=1$, the next terms are 3, 7, 15.



Test a tempting shortcut

- The first term has index 1, so the exponent is $n-1$. A sequence is a list; a series is a sum. A geometric sequence can alternate in sign and still converge.
- When a shortcut fails, identify the assumption it breaks. Keep an exact value until the requested final rounding.

Watch out

Every geometric series has a finite sum to infinity. This claim is false. Explain which definition or assumption it violates.

Interpret a new situation

- Explain whether the context justifies additive or multiplicative change. In finance, distinguish a single deposit from a stream of deposits before choosing a sum formula.

- A complete solution gives the mathematical result and explains what it means. Check that it is possible in the stated context.
-

Use this in your course

- 7357 · A-level · D. Match the target tier and specification before assigning extensions.
- Give the method before the final answer, and use the paper's calculator and formula rules. Review a wrong answer by locating the first invalid step.

Key idea

The constant multiplier between consecutive terms of a geometric sequence. Choose the relationship, show the method, check its assumptions and interpret the result.

D · Binomial expansion and valid approximations

How does a small change affect a power?

- A small measurement change affects a power of a quantity. An expansion can show the size of first and second effects.
 - This lesson studies **binomial coefficient** : The number of ways to choose a specified number of objects from a set.
-

Choose the mathematical structure

- For positive integer n , $(a+b)^n$ is a finite binomial expansion. For noninteger n , expand $(1+x)^n$ as $1+nx+n(n-1)x^2/2+\dots$ with $|x|<1$. Factor out constants before using this form.
 - State the allowed inputs and units before calculating. An equation should express the relationship, not just record a calculator entry.
-

Work through a checked case

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2}x^2 + \dots$$

- Check the result against the starting quantities. Substitute into the original relation, or compare the graph and numerical answer where appropriate.

Worked example

$(1+2x)^5 = 1 + 10x + 40x^2 + 80x^3 + 80x^4 + 32x^5$. For $(1+x)^{-1}$, the first three terms are $1 - x + x^2$; at $x=0.1$ this gives 0.91 versus the exact $1/1.1 \approx 0.909091$.

$$(1 + 2x)^5$$

Coefficients: 1, 10, 40, 80, 80, 32

The x^2 coefficient includes $(2x)^2$

Test a tempting shortcut

- An expansion in $2x$ requires $|2x| < 1$ for the infinite series, not merely $|x| < 1$. Positive-integer expansions are finite and do not have that convergence restriction.
- When a shortcut fails, identify the assumption it breaks. Keep an exact value until the requested final rounding.

Watch out

Every binomial expansion is a finite polynomial, including powers that are not positive integers. This claim is false. Explain which definition or assumption it violates.

Interpret a new situation

- State the range of validity alongside an approximation. Retain enough terms to support the requested accuracy, and distinguish a coefficient from the whole term containing x .
 - A complete solution gives the mathematical result and explains what it means. Check that it is possible in the stated context.
-

Use this in your course

- 7357 · A-level · D. Match the target tier and specification before assigning extensions.
- Give the method before the final answer, and use the paper's calculator and formula rules. Review a wrong answer by locating the first invalid step.

Key idea

The number of ways to choose a specified number of objects from a set. Choose the relationship, show the method, check its assumptions and interpret the result.

D · Factorials, combinations and binomial coefficients

A binomial coefficient counts which factors contribute an x term. Why is the x^2 coefficient not just the middle number in Pascal's triangle?

- A binomial coefficient counts which factors contribute an x term. Why is the x^2 coefficient not just the middle number in Pascal's triangle?
 - This lesson studies **combination** : A selection in which order does not distinguish different choices.
-

Choose the mathematical structure

- For a nonnegative integer n , $n! = n(n-1)\dots 1$ and $0! = 1$. For $0 \leq r \leq n$, ${}^nC_r = \frac{n!}{r!(n-r)!}$ counts unordered selections. In $(a+bx)^n$ with positive integer

n, the x^r coefficient is $nCr \times a^{n-r} \times b^r$. The term number is $r+1$ when the expansion is written in ascending powers of x .

- State the allowed inputs and units before calculating. An equation should express the relationship, not just record a calculator entry.

Work through a checked case

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}, \quad [x^r](a+bx)^n = \binom{n}{r} a^{n-r} b^r$$

- Check the result against the starting quantities. Substitute into the original relation, or compare the graph and numerical answer where appropriate.

Worked example

$4!=24$ and $4C2=24/(2 \times 2)=6$. The coefficients for $n=4$ are 1,4,6,4,1. In $(2+3x)^4$, the x^2 term is $4C2 \times 2^2 \times (3x)^2 = 216x^2$; the constant is $2^4=16$ and the x term is $4 \times 2^3 \times 3x = 96x$. For four independent trials each with success probability $1/3$, exactly two successes have probability $4C2 \times (1/3)^2 \times (2/3)^2 = 8/27$. The coefficient 6 counts the possible success positions; each arrangement has the same probability.

Choose positions, then include every factor's power

$$\begin{array}{ccccccc}
 & & & & 1 & & \\
 & & & & & & \\
 & & & & 1 & & 1 \\
 & & & & & & \\
 & & & 1 & & 2 & & 1 \\
 & & & & & & & \\
 & & 1 & & 3 & & 3 & & 1 \\
 & & & & & & & & \\
 & 1 & & 4 & & 6 & & 4 & & 1
 \end{array}$$

x^2 coefficient: $6 \times 2^2 \times 3^2 = 216$

Test a tempting shortcut

- Do not omit the powers of a or b when using a combination coefficient. The x^2 term is the third term, not the second. The probability expression needs a fixed number of trials, independent binary outcomes and the same success probability; a changing probability invalidates this simple model.
- When a shortcut fails, identify the assumption it breaks. Keep an exact value until the requested final rounding.

Watch out

The x^2 coefficient of $(a+bx)^n$ is always just $nC2$. This claim is false. Explain which definition or assumption it violates.

Interpret a new situation

- Use symmetry $nCr = nC(n-r)$ and cancel factorial factors before calculating. To find the x^3 coefficient in $(2+3x)^4$, use $4C3 \times 2 \times 3^3 = 216$. The finite polynomial formula here is for positive integer n ; a rational exponent instead gives the separate general binomial series with its stated validity interval.
 - A complete solution gives the mathematical result and explains what it means. Check that it is possible in the stated context.
-

Use this in your course

- 7357 · A-level · D. Match the target tier and specification before assigning extensions.
- Give the method before the final answer, and use the paper's calculator and formula rules. Review a wrong answer by locating the first invalid step.

Key idea

A selection in which order does not distinguish different choices. Choose the relationship, show the method, check its assumptions and interpret the result.

D · Increasing, decreasing and periodic recurrences

The same update rule can increase one starting value and decrease another. What decides its behaviour?

- The same update rule can increase one starting value and decrease another. What decides its behaviour?
 - This lesson studies **periodic sequence** : A sequence whose terms repeat after a fixed positive number of steps.
-

Choose the mathematical structure

- A recurrence needs an initial term as well as a rule. Compute each new term from the previous one, preserving exact values where useful. Compare $u_{n+1} - u_n$ and check bounds before claiming increasing or decreasing behaviour. A periodic sequence repeats after a fixed number of steps; its smallest positive repeating step is its period.
 - State the allowed inputs and units before calculating. An equation should express the relationship, not just record a calculator entry.
-

Work through a checked case

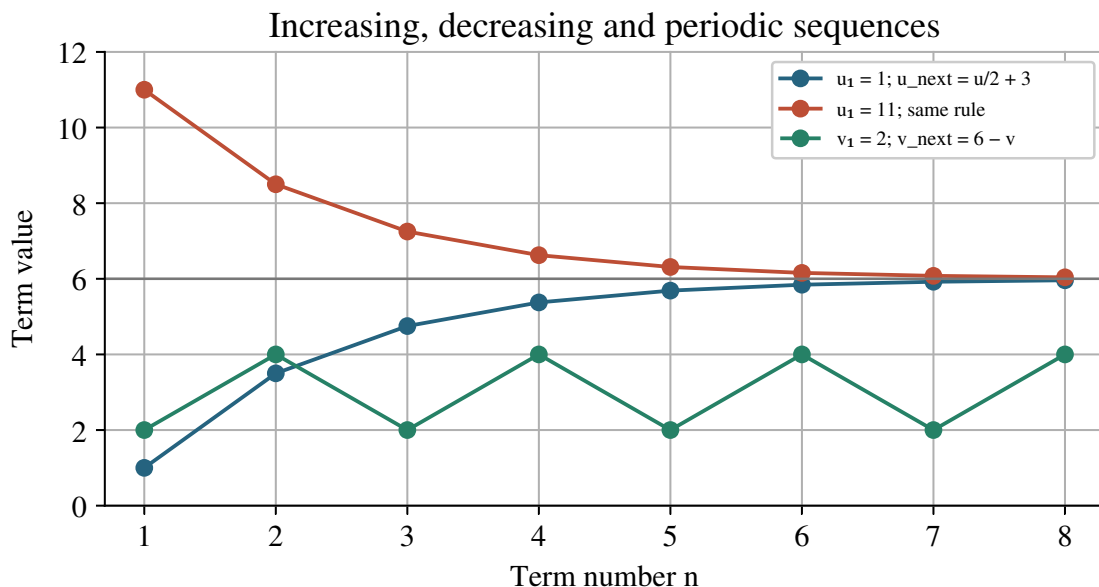
$$u_{n+1} = \frac{1}{2}u_n + 3, \quad u_n = 6 + \frac{u_1 - 6}{2^{n-1}}$$

- Check the result against the starting quantities. Substitute into the original relation, or compare the graph and numerical answer where appropriate.

Worked example

For $u_1=1$ and $u_{n+1}=u_n/2+3$, the first terms are $1, 7/2, 19/4, 43/8$. The fixed value solves $L=L/2+3$, giving $L=6$. If $u_n < 6$, the next term remains below 6 and $u_{n+1} - u_n = (6 - u_n)/2 > 0$. Thus this sequence increases toward 6. Starting instead at 11 gives $11, 17/2, 29/4, \dots$, decreasing while staying above 6. The exact form is $u_n = 6 + (u_1 - 6)/2^{n-1}$. Separately, $v_1=2$ and $v_{n+1}=6-v_n$ gives $2, 4, 2, 4, \dots$

with period 2 because applying the rule twice returns the original term. Starting that second rule at 3 gives a constant sequence with smallest period 1.



Test a tempting shortcut

- Early numerical terms suggest behaviour but do not prove it for every n . A fixed-point equation alone does not prove convergence: the update must keep approaching it. Alternating values need not be periodic; $(-2)^n$ alternates in sign while its magnitude grows. State whether “increasing” means strictly increasing or merely nondecreasing.
- When a shortcut fails, identify the assumption it breaks. Keep an exact value until the requested final rounding.

Watch out

Every recurrence with a fixed value must converge to it from every starting value. This claim is false. Explain which definition or assumption it violates.

Interpret a new situation

- Verify the explicit form by substituting it into the recurrence and checking the initial term. The distance from 6 halves at every update, explaining convergence in this example. A different initial term changes the direction or gives a constant sequence, even though the recurrence formula stays the same.
 - A complete solution gives the mathematical result and explains what it means. Check that it is possible in the stated context.
-

Use this in your course

- 7357 · A-level · D. Match the target tier and specification before assigning extensions.
- Give the method before the final answer, and use the paper's calculator and formula rules. Review a wrong answer by locating the first invalid step.

Key idea

A sequence whose terms repeat after a fixed positive number of steps. Choose the relationship, show the method, check its assumptions and interpret the result.

D · Sigma notation, indices and changed limits

A sum starts at the third term. How many terms does it contain, and what changes if we rename or shift its index?

- A sum starts at the third term. How many terms does it contain, and what changes if we rename or shift its index?
 - This lesson studies **summation index** : The variable that takes each integer value between a sum's stated limits.
-

Choose the mathematical structure

- In a sum from $k=a$ to b , substitute each integer $a, a+1, \dots, b$ into the summand and add the results. There are $b-a+1$ terms when $a \leq b$. The summation index is

a dummy variable, but other parameters keep their values. Renaming an index preserves the sum only when its bound occurrences are renamed consistently.

- State the allowed inputs and units before calculating. An equation should express the relationship, not just record a calculator entry.

Work through a checked case

$$\sum_{k=3}^7 (2k - 1) = \sum_{j=1}^5 (2j + 3) = 45$$

- Check the result against the starting quantities. Substitute into the original relation, or compare the graph and numerical answer where appropriate.

Worked example

The sum from $k=3$ to 7 of $(2k-1)$ is $5+7+9+11+13=45$, with five terms. It also equals the sum from $k=1$ to 7 minus the sum from $k=1$ to 2: $49-4=45$. With $j=k-2$, the lower limit becomes 1, the upper becomes 5 and $2k-1$ becomes $2j+3$, again giving 45. By linearity, the original sum is 2 times the sum of k from 3 to 7, minus five copies of 1: $2 \times 25 - 5 = 45$.

Five included indices; the shifted sum is unchanged

$$\sum_{k=3}^7 (2k - 1)$$

k: 3 4 5 6 7

term: 5 7 9 11 13

$$j = k - 2 : \quad \sum_{j=1}^5 (2j + 3) = 45$$

Test a tempting shortcut

- An inclusive sum from 3 to 7 has five terms, not four or seven. Subtract the prefix through $a-1$, not through a , when changing the lower limit. A constant is added once per index value: the sum of c from a to b is $c(b-a+1)$. Shifting the index changes both bounds and the summand.
- When a shortcut fails, identify the assumption it breaks. Keep an exact value until the requested final rounding.

Watch out

Renaming or shifting an index never requires changing the summand. This claim is false. Explain which definition or assumption it violates.

Interpret a new situation

- For the sum of $(2k-1)$ from $k=1$ to n , the index k changes while n is the fixed upper bound; the result is n^2 for positive integer n . Thus a tail from $k=m$ to n is $n^2-(m-1)^2$ for $1 \leq m \leq n$. Expand a few terms to check a symbolic manipulation before applying a sum formula.
 - A complete solution gives the mathematical result and explains what it means. Check that it is possible in the stated context.
-

Use this in your course

- 7357 · A-level · D. Match the target tier and specification before assigning extensions.
- Give the method before the final answer, and use the paper's calculator and formula rules. Review a wrong answer by locating the first invalid step.

Key idea

The variable that takes each integer value between a sum's stated limits. Choose the relationship, show the method, check its assumptions and interpret the result.

D · Arithmetic series and inverse problems

Two known terms reveal an arithmetic sequence.
Can a stated total tell us exactly how many terms were added?

- Two known terms reveal an arithmetic sequence. Can a stated total tell us exactly how many terms were added?
 - This lesson studies **common difference** : The fixed amount added to each term to obtain the next.
-

Choose the mathematical structure

- For first term a and common difference d , $u_n = a + (n-1)d$. The sum of n terms is $S_n = n[2a + (n-1)d]/2 = n(a + u_n)/2$. Pairing the first and last terms gives the same pair total throughout. An inverse problem may require solving simultaneous equations for a, d or a quadratic for n ; a term count must be a positive integer.
 - State the allowed inputs and units before calculating. An equation should express the relationship, not just record a calculator entry.
-

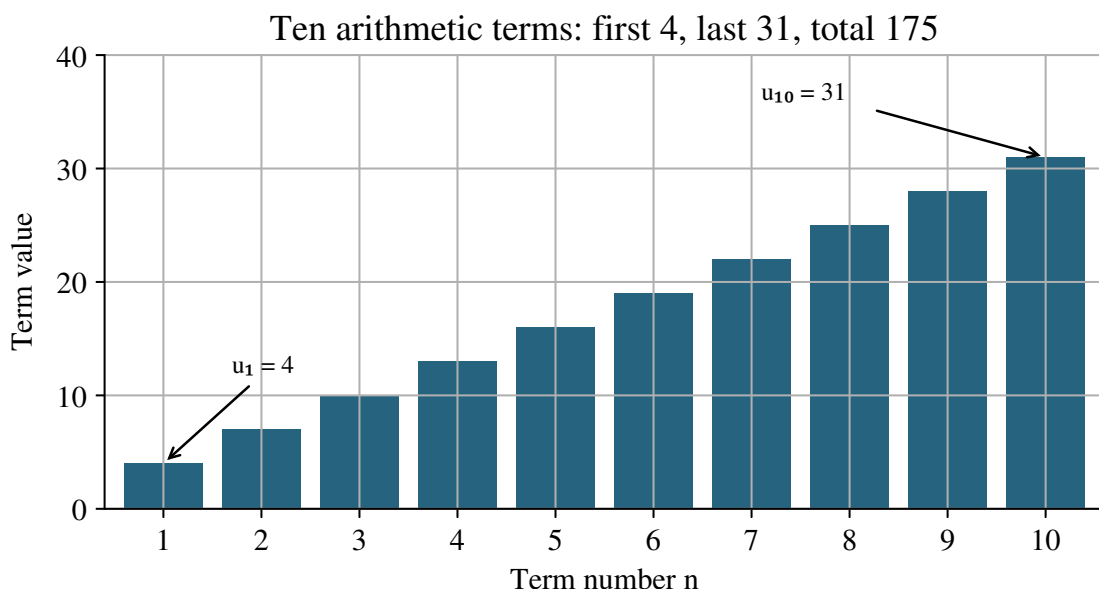
Work through a checked case

$$u_n = a + (n-1)d, \quad S_n = \frac{n}{2}(2a + (n-1)d)$$

- Check the result against the starting quantities. Substitute into the original relation, or compare the graph and numerical answer where appropriate.

Worked example

Given $u_3=10$ and $u_7=22$, write $a+2d=10$ and $a+6d=22$. Subtraction gives $d=3$ and $a=4$, so $u_n=3n+1$ and $S_n=n(3n+5)/2$. To find a total of 175, solve $3n^2+5n-350=0=(n-10)(3n+35)$. The roots are 10 and $-35/3$; only $n=10$ is a positive integer. Check $u_{10}=31$ and $S_{10}=10(4+31)/2=175$. A total of exactly 200 has no integer n : $S_{10}=175$ and $S_{11}=209$, and these positive terms make the sums strictly increasing.



Test a tempting shortcut

- The n th term and sum of n terms are different quantities. The n th term contains $n-1$ differences, not n . Do not round a noninteger solution for an exact number of terms. If the question asks for the first total at least a target, check the neighbouring integer totals. Not every arithmetic series has increasing partial sums.
- When a shortcut fails, identify the assumption it breaks. Keep an exact value until the requested final rounding.

Watch out

A noninteger solution for an exact term count can always be rounded to the nearest integer. This claim is false. Explain which definition or assumption it violates.

Interpret a new situation

- For this sequence, the least n giving $S_n \geq 200$ is 11, since the previous total is 175. State whether a problem asks for an exact total or a threshold. The sum formula still works for negative common differences, but any positivity or monotonicity assumption must be checked from the actual terms.
- A complete solution gives the mathematical result and explains what it means. Check that it is possible in the stated context.

Use this in your course

- 7357 · A-level · D. Match the target tier and specification before assigning extensions.
- Give the method before the final answer, and use the paper's calculator and formula rules. Review a wrong answer by locating the first invalid step.

Key idea

The fixed amount added to each term to obtain the next. Choose the relationship, show the method, check its assumptions and interpret the result.

D · Geometric sums, alternating ratios and convergence

Alternating payments can have a stable accumulated total. Why does a negative ratio not automatically prevent convergence?

- Alternating payments can have a stable accumulated total. Why does a negative ratio not automatically prevent convergence?
- This lesson studies **common ratio** : The fixed factor multiplying each term to obtain the next.

Choose the mathematical structure

- For first term a and nonzero ratio r , $u_n = ar^{n-1}$. If $r=0$, the terms are $a, 0, 0, \dots$, avoiding a 0^0 expression at the first term. For $r \neq 1$, the finite sum is $S_n = a(1-r^n)/(1-r)$; for $r=1$ it is na . For nonzero a , an infinite geometric sum exists when $|r| < 1$ and equals $a/(1-r)$. A negative ratio alternates signs; convergence depends on its magnitude, not its sign alone.
- State the allowed inputs and units before calculating. An equation should express the relationship, not just record a calculator entry.

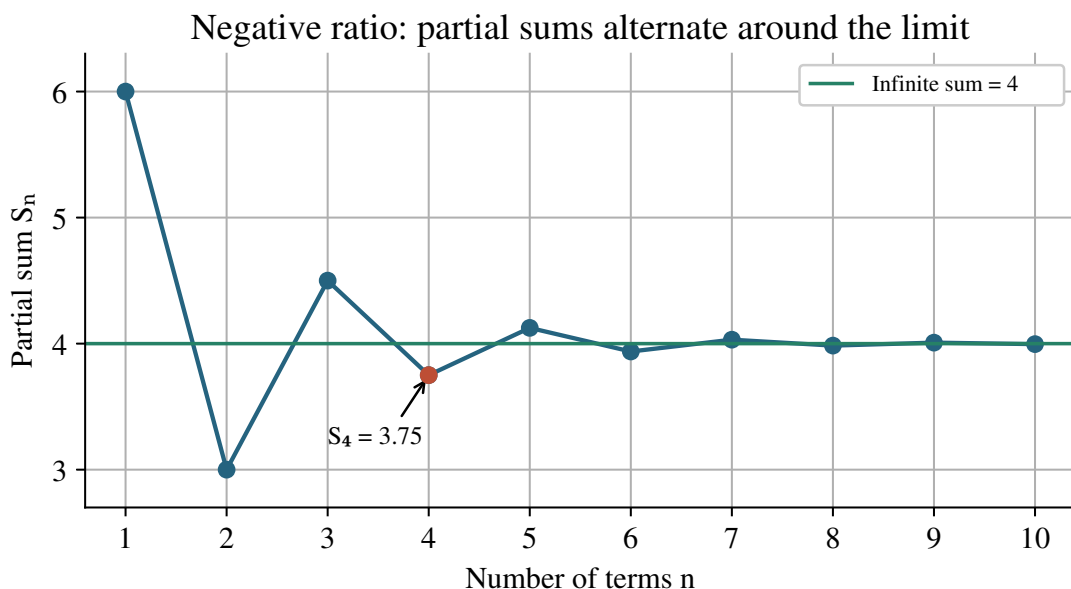
Work through a checked case

$$S_n = \frac{a(1 - r^n)}{1 - r}, \quad S_\infty = \frac{a}{1 - r} \quad (|r| < 1)$$

- Check the result against the starting quantities. Substitute into the original relation, or compare the graph and numerical answer where appropriate.

Worked example

For $a=6$ and $r=-1/2$, the terms are $6, -3, 3/2, -3/4, \dots$. The first four sum to $15/4=3.75$. The finite formula gives $S_4=6[1-(-1/2)^4]/(1+1/2)=15/4$. Since $|r|=1/2<1$, $S_\infty=6/(1+1/2)=4$. The signed remainder is $4-S_n=4(-1/2)^n$, so odd partial sums are above 4 and even partial sums below 4. For error below 0.01, require $4/2^n < 0.01$. At $n=8$ the error is $1/64 > 0.01$; at $n=9$ it is $1/128 < 0.01$, so 9 terms first meet the condition.



Test a tempting shortcut

- Do not replace $r=-1/2$ by $r=1/2$ in the finite formula: the parity of n matters. The infinite sum formula is not justified at $r=1$ or $r=-1$ for nonzero a ; their terms do not tend to zero. A finite sum can exist even when its infinite continuation diverges.
- When a shortcut fails, identify the assumption it breaks. Keep an exact value until the requested final rounding.

Watch out

Every geometric series with a negative ratio diverges. This claim is false. Explain which definition or assumption it violates.

Interpret a new situation

- Check the first few terms and partial sums directly before using a formula. If $|r| > 1$ and a is nonzero, term magnitudes grow instead of vanishing. At $r = -1$, partial sums alternate between a and 0 rather than approaching one value. The zero sequence $a = 0$ is a degenerate exception; identify it rather than applying a nonzero-first-term convergence claim.
 - A complete solution gives the mathematical result and explains what it means. Check that it is possible in the stated context.
-

Use this in your course

- 7357 · A-level · D. Match the target tier and specification before assigning extensions.
- Give the method before the final answer, and use the paper's calculator and formula rules. Review a wrong answer by locating the first invalid step.

Key idea

The fixed factor multiplying each term to obtain the next. Choose the relationship, show the method, check its assumptions and interpret the result.

D · Repeated-deposit sequence models and timing

Two equal payments can earn different amounts of interest if one is paid earlier. Which timeline does the sum describe?

- Two equal payments can earn different amounts of interest if one is paid earlier. Which timeline does the sum describe?
 - This lesson studies **deposit** : An amount added to an account at a stated time.
-

Choose the mathematical structure

- Define the balance immediately after a stated event and give the initial balance. With growth factor r and an end-of-period deposit d , $B_0=0$ and $B_n=rB_{n-1}+d$. The n th balance is $d(1+r+\dots+r^{n-1})=d(r^n-1)/(r-1)$ for $r \neq 1$. For $r=1$, use $B_n=nd$. The period count n is a nonnegative integer.
- State the allowed inputs and units before calculating. An equation should express the relationship, not just record a calculator entry.

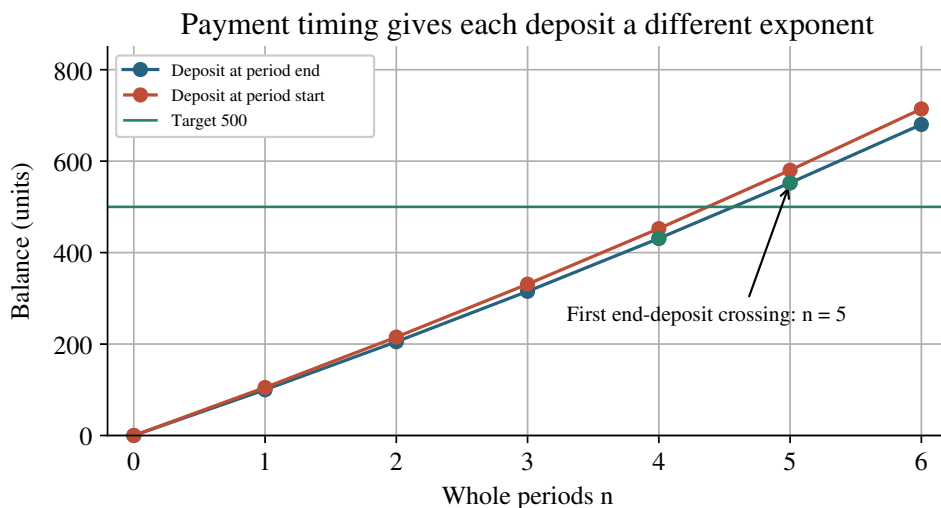
Work through a checked case

$$B_0 = 0, \quad B_n = rB_{n-1} + d, \quad B_n = d \frac{r^n - 1}{r - 1} \quad (r \neq 1)$$

- Check the result against the starting quantities. Substitute into the original relation, or compare the graph and numerical answer where appropriate.

Worked example

Use an illustrative growth rate of 5% per period and deposit 100 units at each period end. $B_1=100$, $B_2=1.05 \times 100 + 100 = 205$ and $B_3=1.05 \times 205 + 100 = 315.25$. The first deposit earns interest for $n-1$ periods, while the final one earns none at the time B_n is measured. Thus $B_n = 100(1.05^n - 1)/0.05$. For a target of at least 500, $B_4 = 431.0125 < 500$ and $B_5 = 552.563125 \geq 500$, so 5 periods first suffice. The balance exceeds the five deposits totalling 500 by 52.563125 units. If the same deposits are made at each period start and the balance is measured at the period end, $C_n = 1.05(B_n)$; hence $C_3 = 331.0125$, not 315.25.



Test a tempting shortcut

- Do not give the final end-of-period deposit an extra interest period. A balance includes both deposits and growth; it is not the interest alone. An exact target and an at-least target need different integer checks. A quoted annual rate cannot simply be used as a per-month factor without an appropriate conversion.
- When a shortcut fails, identify the assumption it breaks. Keep an exact value until the requested final rounding.

Watch out

Beginning- and end-of-period deposits always give the same final balance. This claim is false. Explain which definition or assumption it violates.

Interpret a new situation

- The illustration assumes a fixed per-period rate, equal deposits, no fees or withdrawals and the stated compounding/payment times. If these change, refine the recurrence: $B_n = (1+i_n)B_{n-1} + d_n - w_n$ for end-of-period deposits/withdrawals. Keep a separate ledger of payments to identify growth. Compare adjacent period balances before claiming the first target crossing; here all balances are nonnegative and $B_n - B_{n-1} = 0.05B_{n-1} + 100 > 0$.
 - A complete solution gives the mathematical result and explains what it means. Check that it is possible in the stated context.
-

Use this in your course

- 7357 · A-level · D. Match the target tier and specification before assigning extensions.
- Give the method before the final answer, and use the paper's calculator and formula rules. Review a wrong answer by locating the first invalid step.

Key idea

An amount added to an account at a stated time. Choose the relationship, show the method, check its assumptions and interpret the result.

E · Right triangles and non-right triangles

Which side does the ladder need?

- A ladder reaches a height of 4 m while its foot is 3 m from a wall. Which sides are known, and which angle do we need?
 - This lesson studies **hypotenuse** : The side opposite the right angle in a right-angled triangle.
-

Choose the mathematical structure

- In a right triangle $a^2 + b^2 = c^2$; $\sin\theta = \text{opposite/hypotenuse}$, $\cos\theta = \text{adjacent/hypotenuse}$ and $\tan\theta = \text{opposite/adjacent}$. For other triangles, use the sine or cosine rule, or $\text{area} = ab \sin C/2$.
 - State the allowed inputs and units before calculating. An equation should express the relationship, not just record a calculator entry.
-

Work through a checked case

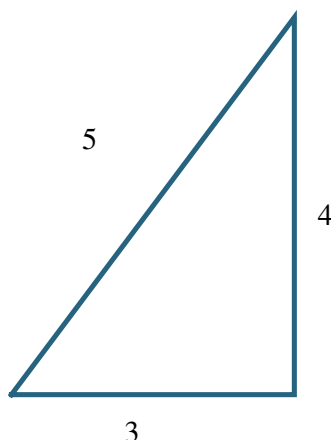
$$a^2 + b^2 = c^2, \quad \tan\theta = \frac{\text{opposite}}{\text{adjacent}}$$

- Check the result against the starting quantities. Substitute into the original relation, or compare the graph and numerical answer where appropriate.

Worked example

The ladder length is $c = \sqrt{3^2 + 4^2} = 5$ m. Its angle to the ground satisfies $\tan\theta = 4/3$, so $\theta \approx 53.1^\circ$. With two sides 6 and 8 enclosing 60° , $c^2 = 6^2 + 8^2 - 2 \times 6 \times 8 \cos 60^\circ = 52$.

Label the known sides before choosing a rule



Test a tempting shortcut

- Label sides relative to the chosen angle. Pythagoras needs a right angle. A calculator angle mode error can produce a plausible but wrong result. Keep unrounded values for later steps.
- When a shortcut fails, identify the assumption it breaks. Keep an exact value until the requested final rounding.

Watch out

Pythagoras applies to every triangle, including triangles without a right angle. This claim is false. Explain which definition or assumption it violates.

Interpret a new situation

- Use a plan or elevation for a three-dimensional problem before applying a triangle rule. Explain why the chosen triangle contains the required length or angle.
 - A complete solution gives the mathematical result and explains what it means. Check that it is possible in the stated context.
-

Use this in your course

- 7357 · A-level · E. Match the target tier and specification before assigning extensions.
- Give the method before the final answer, and use the paper's calculator and formula rules. Review a wrong answer by locating the first invalid step.

Key idea

The side opposite the right angle in a right-angled triangle. Choose the relationship, show the method, check its assumptions and interpret the result.

E · Radians, identities and trigonometric equations

How far does the rim travel?

- A rotating wheel sweeps an arc. Radians let its angle connect directly to the distance travelled along the rim.
- This lesson studies **radian** : The angle subtended by an arc whose length equals the radius.

Choose the mathematical structure

- For θ in radians, arc length $s=r\theta$ and sector area $A=\frac{1}{2}r^2\theta$. Use $\sin^2\theta+\cos^2\theta=1$ and angle identities to simplify or solve. List every solution in the stated interval.
- State the allowed inputs and units before calculating. An equation should express the relationship, not just record a calculator entry.

Work through a checked case

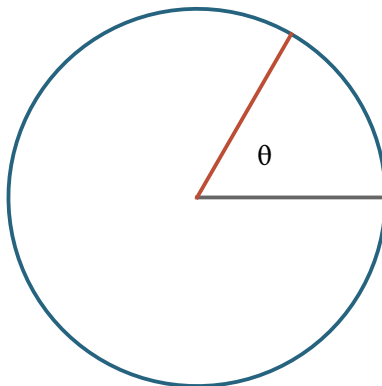
$$s = r\theta, \quad A = \frac{1}{2}r^2\theta, \quad \sin^2\theta + \cos^2\theta = 1$$

- Check the result against the starting quantities. Substitute into the original relation, or compare the graph and numerical answer where appropriate.

Worked example

For $r=4$ and $\theta=\pi/3$, $s=4\pi/3$ and $A=8\pi/3$. Solving $\sin x=1/2$ on $0\leq x<2\pi$ gives $x=\pi/6$ and $5\pi/6$. For $\sin 2x=1/2$, solve first on $0\leq 2x<4\pi$ before dividing by 2.

A radius fixes distance from the centre



radius $\times$ angle gives arc length

Test a tempting shortcut

- A calculator's inverse sine gives one principal value. It does not give every solution. Dividing by $\sin x$ can discard solutions where $\sin x=0$; check that case first.
- When a shortcut fails, identify the assumption it breaks. Keep an exact value until the requested final rounding.

Watch out

The principal inverse sine value is always the only solution in a full turn. This claim is false. Explain which definition or assumption it violates.

Interpret a new situation

- Choose an identity that reduces the equation to one trigonometric function. Keep exact familiar angles where possible, and check each solution against the original equation.
- A complete solution gives the mathematical result and explains what it means. Check that it is possible in the stated context.

Use this in your course

- 7357 · A-level · E. Match the target tier and specification before assigning extensions.
- Give the method before the final answer, and use the paper's calculator and formula rules. Review a wrong answer by locating the first invalid step.

Key idea

The angle subtended by an arc whose length equals the radius. Choose the relationship, show the method, check its assumptions and interpret the result.

E · Radian measure, arcs, sectors and segments

A curved window has a straight lower edge. How can we find the glass area without counting the triangle below the arc?

- A curved window has a straight lower edge. How can we find the glass area without counting the triangle below the arc?
- This lesson studies **sector** : The region bounded by two radii and their circular arc.

Choose the mathematical structure

- One radian is the central angle with arc length equal to the radius. A full turn is 2π radians= 360° , so radians=degrees $\times\pi/180$. For $r>0$ and a nonnegative swept angle θ in radians, arc length $s=r\theta$ and sector area $A=r^2\theta/2$. A sector perimeter is $2r+s$, not s alone. Specify minor or major region.
 - State the allowed inputs and units before calculating. An equation should express the relationship, not just record a calculator entry.
-

Work through a checked case

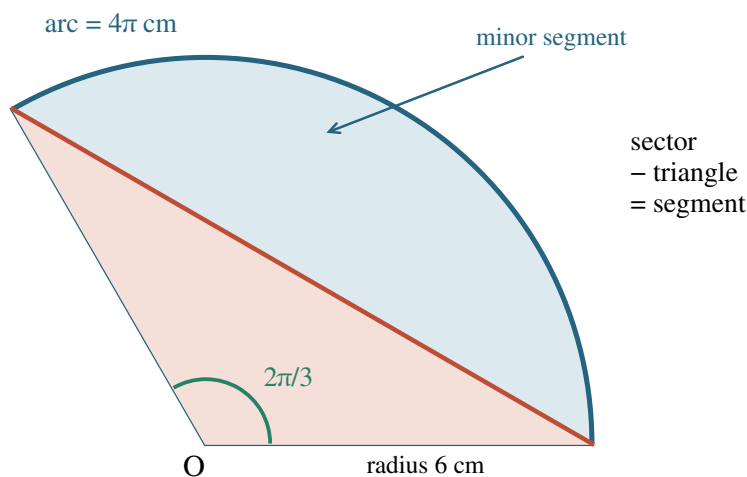
$$s = r\theta, \quad A_{\text{sector}} = \frac{1}{2}r^2\theta, \quad A_{\text{minor segment}} = \frac{1}{2}r^2(\theta - \sin \theta)$$

- Check the result against the starting quantities. Substitute into the original relation, or compare the graph and numerical answer where appropriate.

Worked example

With $r=6$ cm and $\theta=120^\circ=2\pi/3$, $s=4\pi$ cm and sector area $=12\pi$ cm². Bisect the isosceles triangle: chord $=2r \sin(\theta/2)=12 \sin(\pi/3)=6\sqrt{3}$ cm. The triangle area is $r^2 \sin\theta/2=9\sqrt{3}$ cm². The minor segment above the chord has area $12\pi-9\sqrt{3}$ cm². Its boundary length is arc+chord $=4\pi+6\sqrt{3}$ cm. The complementary major segment has area $36\pi-(12\pi-9\sqrt{3})=24\pi+9\sqrt{3}$ cm². In an inverse task, radius 5 cm and arc 8 cm give $\theta=8/5=1.6$ radians and sector area $=5\times 8/2=20$ cm².

A chord separates the triangle from the minor segment



Test a tempting shortcut

- Never put 120 directly into $r\theta$: first convert degrees to radians. A chord is straight and generally shorter than its minor arc. Sector, segment and triangle have different boundaries. The minor-segment subtraction shown uses $0 \leq \theta \leq \pi$; for a major segment subtract the complementary minor region from the disk.
- When a shortcut fails, identify the assumption it breaks. Keep an exact value until the requested final rounding.

Watch out

The perimeter of a sector consists only of its circular arc. This claim is false. Explain which definition or assumption it violates.

Interpret a new situation

- Draw the radii, chord and required boundary first. Recover θ from s/r or $2A/r^2$ when needed. Keep π and surds exact; attach length or area units only after choosing the correct formula. In the inverse example the sector perimeter is $10+8=18$ cm.
 - A complete solution gives the mathematical result and explains what it means. Check that it is possible in the stated context.
-

Use this in your course

- 7357 · A-level · E. Match the target tier and specification before assigning extensions.
- Give the method before the final answer, and use the paper's calculator and formula rules. Review a wrong answer by locating the first invalid step.

Key idea

The region bounded by two radii and their circular arc. Choose the relationship, show the method, check its assumptions and interpret the result.

E · Small-angle approximations and their limits

A small turn moves a pointer almost sideways.

When is replacing a trigonometric function by the angle itself accurate enough?

- A small turn moves a pointer almost sideways. When is replacing a trigonometric function by the angle itself accurate enough?
 - This lesson studies **approximation** : A nearby simpler value used with a stated accuracy limit.
-

Choose the mathematical structure

- When θ is close to zero and measured in radians, $\sin\theta\approx\theta$, $\tan\theta\approx\theta$ and $\cos\theta\approx 1-\theta^2/2$. These are approximations, not identities. The discarded sine/tangent terms begin at order θ^3 ; the discarded cosine term begins at order θ^4 . A smaller absolute angle usually improves these local approximations.
 - State the allowed inputs and units before calculating. An equation should express the relationship, not just record a calculator entry.
-

Work through a checked case

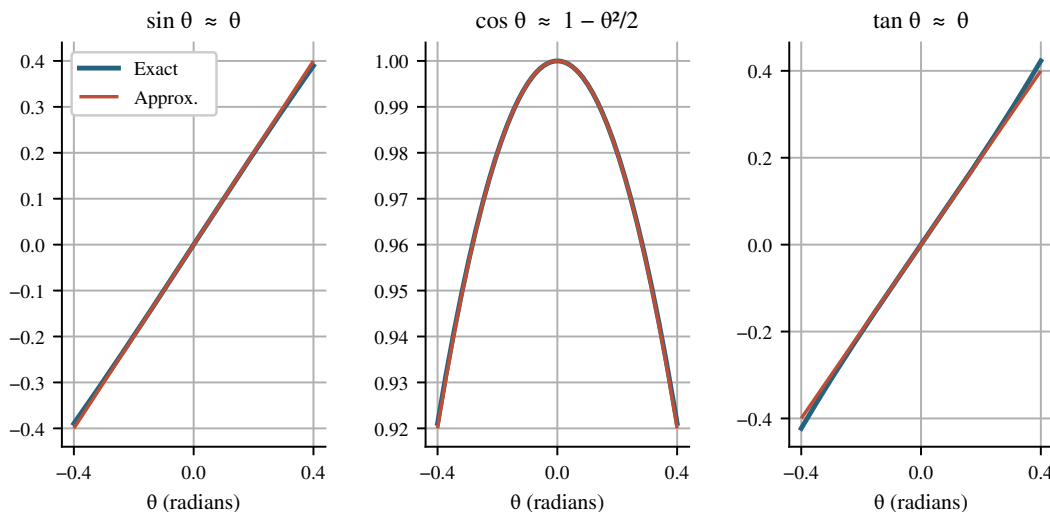
$$\sin\theta\approx\theta, \quad \tan\theta\approx\theta, \quad \cos\theta\approx 1-\frac{\theta^2}{2} \quad (\theta \text{ in radians})$$

- Check the result against the starting quantities. Substitute into the original relation, or compare the graph and numerical answer where appropriate.

Worked example

At $\theta=0.1$ radians, $\sin\theta\approx 0.1$ (actual 0.0998334), $\tan\theta\approx 0.1$ (actual 0.100335) and $\cos\theta\approx 0.995$ (actual 0.995004). Absolute errors are about 0.000167, 0.000335 and 0.00000417. For $6(1-\cos x)=0.03$, approximate $1-\cos x$ by $x^2/2$: $3x^2\approx 0.03$, so $x\approx\pm 0.1$ radians. Check the small-angle assumption after solving; substitution at $x=0.1$ gives 0.0299750 rather than exactly 0.03. Also $\sin(2x)/x\approx 2$ for small nonzero x , while $1-\cos(2x)\approx 2x^2$.

Local agreement improves as the angle approaches zero



Test a tempting shortcut

- Degrees do not work in these formulas: $\sin(1^\circ) \approx \pi/180$, not 1. Write $\approx$ rather than $=$. Subtraction can make relative error more important; preserve the θ^2 term in $1 - \cos\theta$. A ratio at $\theta=0$ may be undefined even when its nearby limit exists.
- When a shortcut fails, identify the assumption it breaks. Keep an exact value until the requested final rounding.

Watch out

The formula $\sin\theta \approx \theta$ is equally valid for an angle entered in degrees. This claim is false. Explain which definition or assumption it violates.

Interpret a new situation

- Convert to radians first and identify the argument of each trig function: $\sin(3x) \approx 3x$, not x . For a model, compare the approximation error with the precision required. If a solution is not small, use the original trigonometric equation; these formulas have no universal accuracy cutoff.
 - A complete solution gives the mathematical result and explains what it means. Check that it is possible in the stated context.
-

Use this in your course

- 7357 · A-level · E. Match the target tier and specification before assigning extensions.
- Give the method before the final answer, and use the paper's calculator and formula rules. Review a wrong answer by locating the first invalid step.

Key idea

A nearby simpler value used with a stated accuracy limit. Choose the relationship, show the method, check its assumptions and interpret the result.

E · All-angle definitions, exact values and periodic graphs

A wheel turns beyond a right angle. Which coordinate gives its vertical position after more than half a turn?

- A wheel turns beyond a right angle. Which coordinate gives its vertical position after more than half a turn?
 - This lesson studies **unit circle** : The circle of radius one centred at the coordinate origin.
-

Choose the mathematical structure

- For a signed angle θ from the positive x-axis, the unit-circle point is $(\cos\theta, \sin\theta)$; $\tan\theta = \sin\theta/\cos\theta$ where $\cos\theta \neq 0$. Sine and cosine have period 2π and range $[-1, 1]$. Tangent has period π , every real output and vertical asymptotes $\theta = \pi/2 + k\pi$ for integer k . Sine/tangent are odd functions; cosine is even.
 - State the allowed inputs and units before calculating. An equation should express the relationship, not just record a calculator entry.
-

Work through a checked case

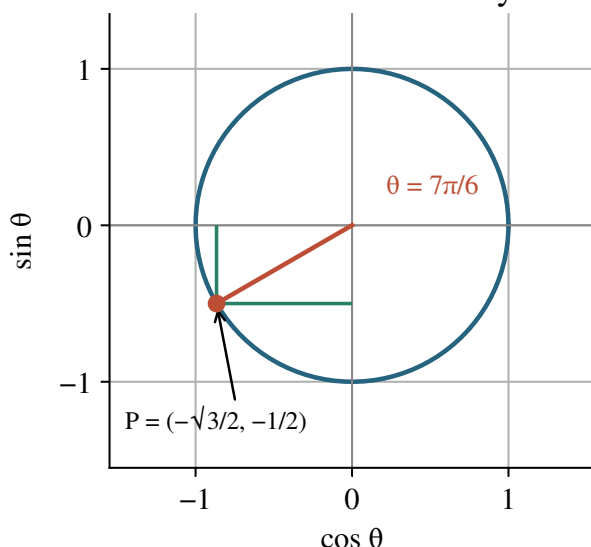
$$(x, y) = (\cos \theta, \sin \theta), \quad \tan \theta = \frac{\sin \theta}{\cos \theta}$$

- Check the result against the starting quantities. Substitute into the original relation, or compare the graph and numerical answer where appropriate.

Worked example

Special triangles give $(\sin\theta, \cos\theta)$: at 0 use (0,1); $\pi/6$ gives $(1/2, \sqrt{3}/2)$; $\pi/4$ gives $(\sqrt{2}/2, \sqrt{2}/2)$; $\pi/3$ gives $(\sqrt{3}/2, 1/2)$; $\pi/2$ gives (1,0); π gives (0,-1). Divide to obtain tangent, excluding $\pi/2$. Reference angles and quadrant signs give $\sin(7\pi/6) = -1/2$, $\cos(4\pi/3) = -1/2$ and $\tan(3\pi/4) = -1$. Subtract complete turns before finding the quadrant: $\sin(19\pi/6) = \sin(7\pi/6)$. For $y = 2 \sin(3x - \pi) + 1$, amplitude = 2, period = $2\pi/3$, centre line $y = 1$ and range $[-1, 3]$. Writing $3x - \pi = 3(x - \pi/3)$ gives a right shift $\pi/3$.

Coordinates define sine and cosine beyond acute angles



Test a tempting shortcut

- A negative angle is clockwise; do not automatically make its sine positive. $\tan(\pi/2)$ is undefined rather than a very large finite number. Multiply the input frequency to shorten the period; adding a vertical shift does not change the period. The sine graph crosses its centre line rather than always crossing $y=0$.
- When a shortcut fails, identify the assumption it breaks. Keep an exact value until the requested final rounding.

Watch out

Sine and cosine repeat after π radians for every input angle. This claim is false. Explain which definition or assumption it violates.

Interpret a new situation

- Build a graph from a full cycle of key points, its symmetry and period. For $\cos \theta$ use reflection symmetry about $\theta=0$; for $\sin \theta / \tan \theta$ use origin symmetry. Keep exact reference values, then use a calculator only when an unfamiliar angle needs a decimal. State angle mode and the domain interval.
- A complete solution gives the mathematical result and explains what it means. Check that it is possible in the stated context.

Use this in your course

- 7357 · A-level · E. Match the target tier and specification before assigning extensions.
- Give the method before the final answer, and use the paper's calculator and formula rules. Review a wrong answer by locating the first invalid step.

Key idea

The circle of radius one centred at the coordinate origin. Choose the relationship, show the method, check its assumptions and interpret the result.

E · Reciprocal trigonometric functions and excluded angles

A formula divides by a rotating point's horizontal coordinate. What happens as that coordinate approaches zero?

- A formula divides by a rotating point's horizontal coordinate. What happens as that coordinate approaches zero?
- This lesson studies **secant** : The reciprocal of cosine wherever cosine is nonzero.

Choose the mathematical structure

- $\sec\theta=1/\cos\theta$ and $\operatorname{cosec}\theta=1/\sin\theta$. Define $\cot\theta=\cos\theta/\sin\theta$, so it exists when $\sin\theta\neq0$. Secant excludes $\theta=\pi/2+k\pi$; cosecant and cotangent exclude $\theta=k\pi$, for integer k . Secant/cosecant have period 2π and range $(-\infty,-1]\cup[1,\infty)$. Cotangent has period π and every real output. Secant is even; cosecant/cotangent are odd.
- State the allowed inputs and units before calculating. An equation should express the relationship, not just record a calculator entry.

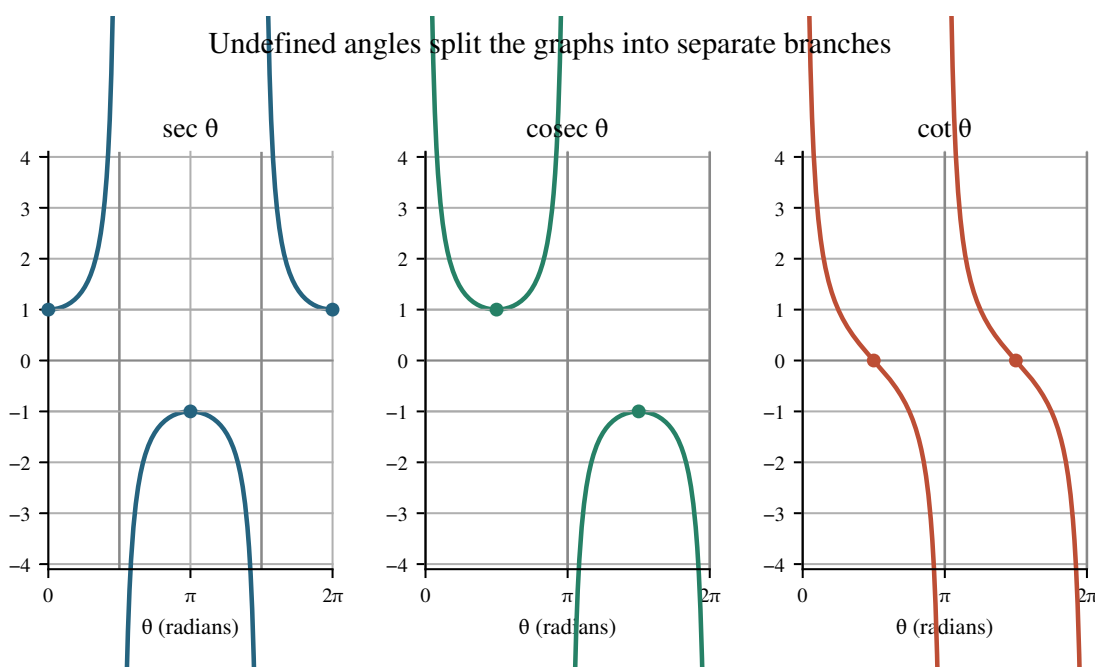
Work through a checked case

$$\sec \theta = \frac{1}{\cos \theta}, \quad \operatorname{cosec} \theta = \frac{1}{\sin \theta}, \quad \cot \theta = \frac{\cos \theta}{\sin \theta}$$

- Check the result against the starting quantities. Substitute into the original relation, or compare the graph and numerical answer where appropriate.

Worked example

At $\theta=2\pi/3$, $\cos\theta=-1/2$ and $\sin\theta=\sqrt{3}/2$, so $\sec\theta=-2$, $\operatorname{cosec}\theta=2\sqrt{3}/3$ and $\cot\theta=-\sqrt{3}/3$. At $\theta=\pi/2$, $\cot\theta=0/1=0$ and $\operatorname{cosec}\theta=1$, but $\sec\theta$ is undefined. Draw secant branches outside the strip $-1 < y < 1$: $y=1$ at 0 , $y=-1$ at π , with vertical asymptotes at $\pi/2$ and $3\pi/2$. Cosecant reaches 1 at $\pi/2$ and -1 at $3\pi/2$, with asymptotes at 0 , π and 2π . Cotangent decreases from positive to negative infinity on $(0, \pi)$, crossing zero at $\pi/2$; repeat after π .



Test a tempting shortcut

- $\cot\theta=1/\tan\theta$ is usable only where both expressions exist and $\tan\theta \neq 0$. It cannot define $\cot(\pi/2)$, because $\tan(\pi/2)$ is undefined while $\cot(\pi/2)=0$. The exponent -1 in $\sin^{-1}$ often denotes inverse sine, not the reciprocal; write $\operatorname{cosec}\theta$ when you mean $1/\sin\theta$.
- When a shortcut fails, identify the assumption it breaks. Keep an exact value until the requested final rounding.

Watch out

Cotangent is undefined at every angle where tangent is undefined. This claim is false. Explain which definition or assumption it violates.

Interpret a new situation

- Sketch the base sine/cosine zeros first: these locate the reciprocal asymptotes. Values 1 and -1 are unchanged by taking a reciprocal. Split the domain at each excluded angle and never join branches through an asymptote. State the function's domain before rearranging an equation involving its denominator.
 - A complete solution gives the mathematical result and explains what it means. Check that it is possible in the stated context.
-

Use this in your course

- 7357 · A-level · E. Match the target tier and specification before assigning extensions.
- Give the method before the final answer, and use the paper's calculator and formula rules. Review a wrong answer by locating the first invalid step.

Key idea

The reciprocal of cosine wherever cosine is nonzero. Choose the relationship, show the method, check its assumptions and interpret the result.

E · Inverse trigonometric branches and principal values

**A height ratio gives many possible wheel angles.
How does a calculator choose just one of them?**

- A height ratio gives many possible wheel angles. How does a calculator choose just one of them?
 - This lesson studies **principal value** : The single inverse output chosen from a stated restricted interval.
-

Choose the mathematical structure

- Restrict sine to $[-\pi/2, \pi/2]$ to define $\arcsin$ with domain $[-1, 1]$ and range $[-\pi/2, \pi/2]$. Restrict cosine to $[0, \pi]$ for $\arccos$, with domain $[-1, 1]$ and range $[0, \pi]$. Restrict tangent to $(-\pi/2, \pi/2)$ for $\arctan$, with every real input and range $(-\pi/2, \pi/2)$. These inverse graphs reflect the restricted base graphs in $y=x$.
 - State the allowed inputs and units before calculating. An equation should express the relationship, not just record a calculator entry.
-

Work through a checked case

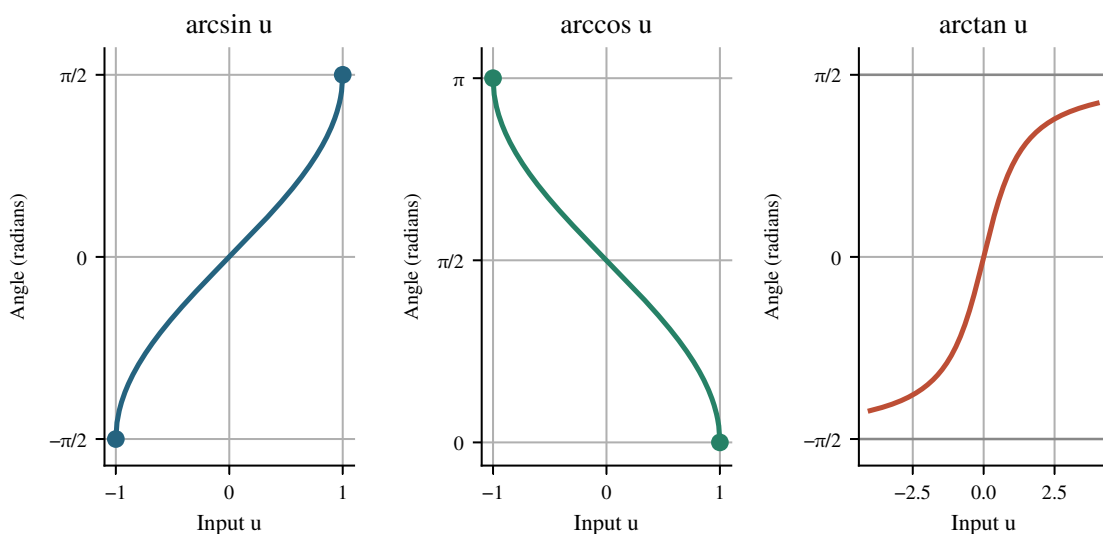
$$\arcsin : [-1, 1] \rightarrow [-\pi/2, \pi/2], \quad \arccos : [-1, 1] \rightarrow [0, \pi]$$

- Check the result against the starting quantities. Substitute into the original relation, or compare the graph and numerical answer where appropriate.

Worked example

$\arcsin(1/2) = \pi/6$, $\arccos(-1/2) = 2\pi/3$ and $\arctan(-1) = -\pi/4$. Although $\sin(5\pi/6) = 1/2$, $\arcsin(\sin(5\pi/6)) = \pi/6$, not $5\pi/6$. Similarly $\arccos(\cos(4\pi/3)) = 2\pi/3$ and $\arctan(\tan(3\pi/4)) = -\pi/4$. For $\sin x = 1/2$ on $[0, 2\pi)$, the principal value $\alpha = \pi/6$ generates both α and $\pi - \alpha = 5\pi/6$. For $\cos x = -1/2$ the solutions are $2\pi/3$ and $4\pi/3$; tangent repeats its principal value after π .

Principal inverse branches have different output intervals



Test a tempting shortcut

- $\sin(\arcsin u)=u$ only for $-1 \leq u \leq 1$; $\arcsin(\sin x)=x$ only on the restricted sine branch. Arccos is decreasing, so $\arccos(-1)=\pi$ and $\arccos(1)=0$. Arctan approaches $\pm\pi/2$ as input grows in magnitude but never attains those limits. A principal value alone is not a full interval solution set.
- When a shortcut fails, identify the assumption it breaks. Keep an exact value until the requested final rounding.

Watch out

Applying inverse sine after sine returns the original angle for every real angle. This claim is false. Explain which definition or assumption it violates.

Interpret a new situation

- Mark domain endpoints and output intervals before composing functions. On the inverse graphs $\arcsin$ runs from $(-1, -\pi/2)$ to $(1, \pi/2)$, $\arccos$ from $(-1, \pi)$ to $(1, 0)$, and $\arctan$ passes through $(0, 0)$ between horizontal asymptotes. Keep answers in radians unless degrees are requested, and check all equation candidates in the original interval.
 - A complete solution gives the mathematical result and explains what it means. Check that it is possible in the stated context.
-

Use this in your course

- 7357 · A-level · E. Match the target tier and specification before assigning extensions.
- Give the method before the final answer, and use the paper's calculator and formula rules. Review a wrong answer by locating the first invalid step.

Key idea

The single inverse output chosen from a stated restricted interval. Choose the relationship, show the method, check its assumptions and interpret the result.

E · Reciprocal identities and one-sided proof chains

Two formulas agree at one angle. Does that prove they agree for every angle where they are defined?

- Two formulas agree at one angle. Does that prove they agree for every angle where they are defined?
 - This lesson studies **identity** : An equality true for every input in its stated common domain.
-

Choose the mathematical structure

- The unit-circle equation gives $\sin^2\theta + \cos^2\theta = 1$. Divide by $\cos^2\theta$ when $\cos\theta \neq 0$ to obtain $\tan^2\theta + 1 = \sec^2\theta$. Divide by $\sin^2\theta$ when $\sin\theta \neq 0$ to obtain $1 + \cot^2\theta = \operatorname{cosec}^2\theta$. Record exclusions before cancelling or dividing. A proof starts from one side and reaches the other through valid equalities; numerical checks alone are not a proof.
 - State the allowed inputs and units before calculating. An equation should express the relationship, not just record a calculator entry.
-

Work through a checked case

$$\sec^2\theta = 1 + \tan^2\theta, \quad \operatorname{cosec}^2\theta = 1 + \cot^2\theta$$

- Check the result against the starting quantities. Substitute into the original relation, or compare the graph and numerical answer where appropriate.

Worked example

Prove $\tan^2\theta - \sin^2\theta = \tan^2\theta \sin^2\theta$ for $\cos\theta \neq 0$. The left side is $\sin^2\theta(1/\cos^2\theta - 1) = \sin^2\theta(1 - \cos^2\theta)/\cos^2\theta = \sin^4\theta/\cos^2\theta = \tan^2\theta \sin^2\theta$. No division by $\sin\theta$ is needed, so $\sin\theta = 0$ remains allowed. For $\sin\theta \neq 0$, $(1 - \cos\theta)/\sin\theta$ becomes $(1 - \cos\theta)(1 + \cos\theta)/[\sin\theta(1 + \cos\theta)] = \sin^2\theta/[\sin\theta(1 + \cos\theta)] = \sin\theta/(1 + \cos\theta)$. The stated exclusion also makes $1 + \cos\theta$ nonzero. If $\sec\theta = 2$ with θ acute, $\tan^2\theta = 4 - 1 = 3$, $\cos\theta = 1/2$, $\sin^2\theta = 3/4$, $\operatorname{cosec}^2\theta = 4/3$ and $\cot^2\theta = 1/3$.

A proof chain keeps its common domain

$$\tan^2 \theta - \sin^2 \theta$$

$$= \sin^2 \theta \left(\frac{1}{\cos^2 \theta} - 1 \right)$$

$$= \frac{\sin^2 \theta (1 - \cos^2 \theta)}{\cos^2 \theta}$$

$\cos \theta \neq 0$
 $\sin \theta = 0$ allowed

$$= \frac{\sin^4 \theta}{\cos^2 \theta} = \tan^2 \theta \sin^2 \theta$$

Test a tempting shortcut

- Proving an identity does not mean assuming both sides equal and manipulating that assumption without justification. A squared identity does not fix a function's sign: the quadrant does. The last expression $\sin\theta/(1+\cos\theta)$ is defined at $\theta=0$, but the original $(1-\cos\theta)/\sin\theta$ is not; simplification does not repair the original domain.
- When a shortcut fails, identify the assumption it breaks. Keep an exact value until the requested final rounding.

Watch out

Checking a proposed identity at one angle proves it for all allowed angles. This claim is false. Explain which definition or assumption it violates.

Interpret a new situation

- Choose the more complicated side, replace reciprocal functions by sine/cosine, use a common denominator and apply $\sin^2 + \cos^2 = 1$. Annotate every cancelled factor. To disprove a claimed identity, one valid counterexample suffices; to prove it, explain why the chain works for all angles in the common domain.
- A complete solution gives the mathematical result and explains what it means. Check that it is possible in the stated context.

Use this in your course

- 7357 · A-level · E. Match the target tier and specification before assigning extensions.
- Give the method before the final answer, and use the paper's calculator and formula rules. Review a wrong answer by locating the first invalid step.

Key idea

An equality true for every input in its stated common domain. Choose the relationship, show the method, check its assumptions and interpret the result.

E · Compound-angle formulae and geometric proofs

Two unit-radius points define a chord. Can its length explain why cosine of a difference is not the difference of two cosines?

- Two unit-radius points define a chord. Can its length explain why cosine of a difference is not the difference of two cosines?
- This lesson studies **compound angle** : An angle expressed as the sum or difference of two angles.

Choose the mathematical structure

- $\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$. $\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$: the cosine sign changes. $\tan(A \pm B) = (\tan A \pm \tan B) / (1 \mp \tan A \tan B)$ where the tangents and denominator are defined. Derive tangent by dividing the corresponding sine and cosine formulas; do not extend a quotient through a zero denominator.
 - State the allowed inputs and units before calculating. An equation should express the relationship, not just record a calculator entry.
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Work through a checked case

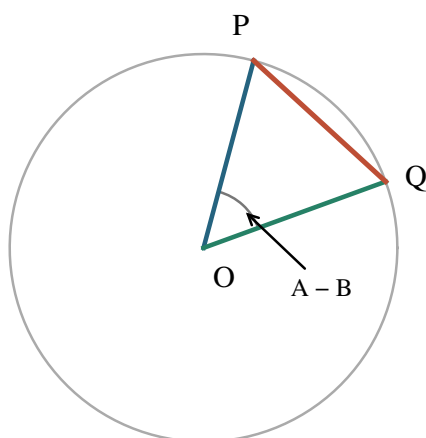
$$\cos(A - B) = \cos A \cos B + \sin A \sin B, \quad \sin(A + B) = \sin A \cos B + \cos A \sin B$$

- Check the result against the starting quantities. Substitute into the original relation, or compare the graph and numerical answer where appropriate.

Worked example

Let $P = (\cos A, \sin A)$ and $Q = (\cos B, \sin B)$ on the unit circle. Coordinate distance gives $PQ^2 = 2 - 2(\cos A \cos B + \sin A \sin B)$. The cosine rule on the two unit radii gives $PQ^2 = 2 - 2\cos(A - B)$, hence the cosine difference formula. Replace B by $-B$, using cosine even/sine odd, to obtain cosine addition. Apply cosine difference to $(\pi/2 - A)$ and B to obtain $\sin(A + B)$; then change B 's sign for sine subtraction. Exact evaluations: $\sin 75^\circ = \sin(45^\circ + 30^\circ) = (\sqrt{6} + \sqrt{2})/4$; $\cos 75^\circ = (\sqrt{6} - \sqrt{2})/4$; $\sin 15^\circ = (\sqrt{6} - \sqrt{2})/4$, $\cos 15^\circ = (\sqrt{6} + \sqrt{2})/4$ and $\tan 15^\circ = 2 - \sqrt{3}$.

Measure the same chord in two ways



Coordinate distance

$$PQ^2 = 2 - 2(\cos A \cos B + \sin A \sin B)$$

Cosine rule, radii both 1

$$PQ^2 = 2 - 2\cos(A - B)$$

Test a tempting shortcut

- $\sin(A + B)$ is not $\sin A + \sin B$. The cosine rule chord uses the smaller included angle when necessary, but its cosine is still $\cos(A - B)$. Coordinate distance and the resulting identities hold for all signed angles; the initial diagram is one illustrative case. $\tan(45^\circ + 45^\circ)$ is undefined because $1 - \tan 45^\circ \tan 45^\circ = 0$.

- When a shortcut fails, identify the assumption it breaks. Keep an exact value until the requested final rounding.

Watch out

Cosine of a sum always equals the sum of the two cosine values. This claim is false. Explain which definition or assumption it violates.

Interpret a new situation

- Choose familiar angles whose sum or difference is the target. Keep exact roots and rationalise a tangent quotient if needed. In a geometric proof, state both expressions for the same chord rather than assuming the desired identity. Check the denominator before applying a tangent addition/subtraction formula.
 - A complete solution gives the mathematical result and explains what it means. Check that it is possible in the stated context.
-

Use this in your course

- 7357 · A-level · E. Match the target tier and specification before assigning extensions.
- Give the method before the final answer, and use the paper's calculator and formula rules. Review a wrong answer by locating the first invalid step.

Key idea

An angle expressed as the sum or difference of two angles. Choose the relationship, show the method, check its assumptions and interpret the result.

E · Double-angle identities, quadrant signs and squared forms

Doubling a wheel angle can move its point into another quadrant. Which signs survive the calculation?

- Doubling a wheel angle can move its point into another quadrant. Which signs survive the calculation?

- This lesson studies **double angle** : An angle equal to twice the original angle.

Choose the mathematical structure

- Set $B=A=\theta$ in the compound formulas: $\sin 2\theta = 2\sin\theta \cos\theta$; $\cos 2\theta = \cos^2\theta - \sin^2\theta = 1 - 2\sin^2\theta = 2\cos^2\theta - 1$. Where $\tan\theta$ and $\tan 2\theta$ exist, $\tan 2\theta = 2\tan\theta/(1 - \tan^2\theta)$. Rearranging gives $\sin^2\theta = (1 - \cos 2\theta)/2$ and $\cos^2\theta = (1 + \cos 2\theta)/2$. These identities depend on the full angle 2θ , not on squaring the function.
- State the allowed inputs and units before calculating. An equation should express the relationship, not just record a calculator entry.

Work through a checked case

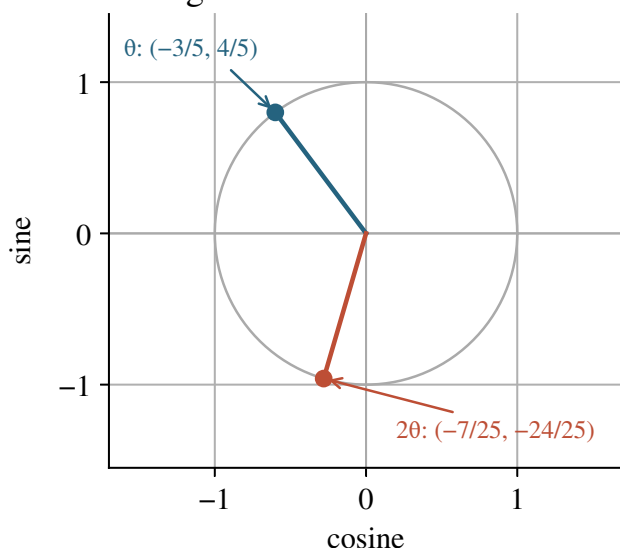
$$\sin 2\theta = 2 \sin \theta \cos \theta, \quad \cos 2\theta = 1 - 2 \sin^2 \theta = 2 \cos^2 \theta - 1$$

- Check the result against the starting quantities. Substitute into the original relation, or compare the graph and numerical answer where appropriate.

Worked example

If $\cos\theta = -3/5$ and θ is in quadrant II, $\sin\theta = +4/5$. Then $\sin 2\theta = 2(4/5)(-3/5) = -24/25$ and $\cos 2\theta = 9/25 - 16/25 = -7/25$. Both are negative, so the doubled angle lies in quadrant III modulo a turn; $\tan 2\theta = 24/7$. The tangent formula gives the same answer from $\tan\theta = -4/3$. Recover $\sin^2\theta$ from $\cos 2\theta$: $[1 - (-7/25)]/2 = 16/25$. At $\theta = \pi/4$, the sine/cosine double-angle formulas still work, but $\tan 2\theta$ is undefined. At $\theta = \pi/2$, $\tan\theta$ is undefined while $\tan 2\theta = 0$; use sine/cosine rather than the tangent quotient.

A doubled angle can move into a different quadrant



Test a tempting shortcut

- $\sin 2\theta$ means $\sin(2\theta)$; $\sin^2\theta$ means $(\sin\theta)^2$. A squared value gives a magnitude but not a sign; use the stated quadrant before taking a root. Never infer the doubled angle's quadrant from the original one without checking signs. Simplified identities do not remove the original tangent exclusions.
- When a shortcut fails, identify the assumption it breaks. Keep an exact value until the requested final rounding.

Watch out

The functions $\sin 2\theta$ and $\sin^2\theta$ have the same value for every angle. This claim is false. Explain which definition or assumption it violates.

Interpret a new situation

- Choose the cosine form that matches the known quantity: $2\cos^2\theta - 1$ when cosine is given, or $1 - 2\sin^2\theta$ when sine is given. Use the squared forms to replace a trig square with a constant and a double-angle term. Verify a derived sign against the unit circle and retain exact fractions until a decimal is requested.
- A complete solution gives the mathematical result and explains what it means. Check that it is possible in the stated context.

Use this in your course

- 7357 · A-level · E. Match the target tier and specification before assigning extensions.
- Give the method before the final answer, and use the paper's calculator and formula rules. Review a wrong answer by locating the first invalid step.

Key idea

An angle equal to twice the original angle. Choose the relationship, show the method, check its assumptions and interpret the result.

E · Auxiliary-angle forms, ranges and interval solutions

Two periodic effects act together. Can one shifted curve show their combined maximum and minimum?

- Two periodic effects act together. Can one shifted curve show their combined maximum and minimum?
- This lesson studies **amplitude** : The nonnegative size of an oscillation about its centre line.

Choose the mathematical structure

- For a and b not both zero, write $a \cos \theta + b \sin \theta = R \cos(\theta - \alpha)$, where $R = \sqrt{a^2 + b^2} > 0$, $\cos \alpha = a/R$ and $\sin \alpha = b/R$. Choose α in the correct quadrant. The equivalent sine form $R \sin(\theta + \beta)$ has $\sin \beta = a/R$ and $\cos \beta = b/R$. A constant c shifts the range to $[c - R, c + R]$. If $a = b = 0$, the expression is zero and no unique phase is defined.
- State the allowed inputs and units before calculating. An equation should express the relationship, not just record a calculator entry.

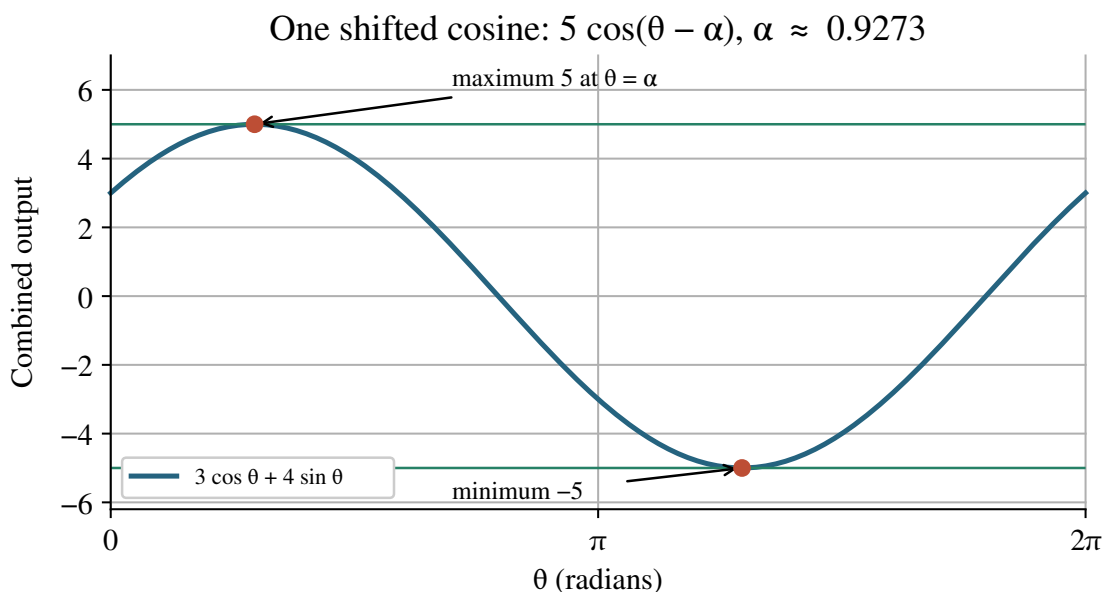
Work through a checked case

$$a \cos \theta + b \sin \theta = R \cos(\theta - \alpha), \quad R = \sqrt{a^2 + b^2}$$

- Check the result against the starting quantities. Substitute into the original relation, or compare the graph and numerical answer where appropriate.

Worked example

$3\cos\theta + 4\sin\theta = 5\cos(\theta - \alpha)$, with $\cos\alpha = 3/5$, $\sin\alpha = 4/5$ and $\alpha \approx 0.927295$ radians. Its sine form is $5\sin(\theta + \beta)$, where $\beta \approx 0.643501$ radians. Thus $2 + 3\cos\theta + 4\sin\theta$ ranges from -3 to 7 . For $3\cos\theta - 4\sin\theta$, use $5\cos(\theta + \alpha)$. For $-3\cos\theta + 4\sin\theta$, the cosine phase is in quadrant II, not the negative principal arctangent. To solve $3\cos\theta + 4\sin\theta = 2$ on $[0, 2\pi)$, put $\delta = \arccos(2/5) \approx 1.107148$. Then $\theta = \alpha \pm \delta + 2k\pi$; the two allowed values are about 2.086575 and 6.051201 radians. A right side of 6 would give no solution, since the amplitude is only 5 .



Test a tempting shortcut

- A plain $\arctan(b/a)$ can select the wrong quadrant and fails when $a=0$. Match both sine and cosine coefficients. The amplitude is $\sqrt{a^2 + b^2}$, not $|a| + |b|$. Do not treat the principal inverse cosine value as the only solution. The angle-mode unit must match the stated interval.

- When a shortcut fails, identify the assumption it breaks. Keep an exact value until the requested final rounding.

Watch out

The amplitude of $a \cos \theta + b \sin \theta$ is always $|a| + |b|$. This claim is false. Explain which definition or assumption it violates.

Interpret a new situation

- Expand your proposed $R \cos(\theta - \alpha)$ or $R \sin(\theta + \beta)$ to verify every coefficient and sign. For equations, compare the target with the range first, generate both phase branches and translate each by full turns into the original interval. At the upper or lower range endpoint the two branches coincide modulo 2π .
 - A complete solution gives the mathematical result and explains what it means. Check that it is possible in the stated context.
-

Use this in your course

- 7357 · A-level · E. Match the target tier and specification before assigning extensions.
- Give the method before the final answer, and use the paper's calculator and formula rules. Review a wrong answer by locating the first invalid step.

Key idea

The nonnegative size of an oscillation about its centre line. Choose the relationship, show the method, check its assumptions and interpret the result.

E · Quadratic trigonometric equations and complete root sets

An equation gives two possible sine values. How many angles do they create in one full turn?

- An equation gives two possible sine values. How many angles do they create in one full turn?
-

- This lesson studies **candidate** : A possible solution that still needs checking in the original equation and interval.

Choose the mathematical structure

- Replace one trig function by u , solve the algebraic equation, then solve each valid trig value on the stated interval. For sine/cosine, discard u outside $[-1,1]$. Tangent accepts every finite real u but excludes angles where cosine is zero. Factor before dividing by a trig function, so zero cases are retained.
- State the allowed inputs and units before calculating. An equation should express the relationship, not just record a calculator entry.

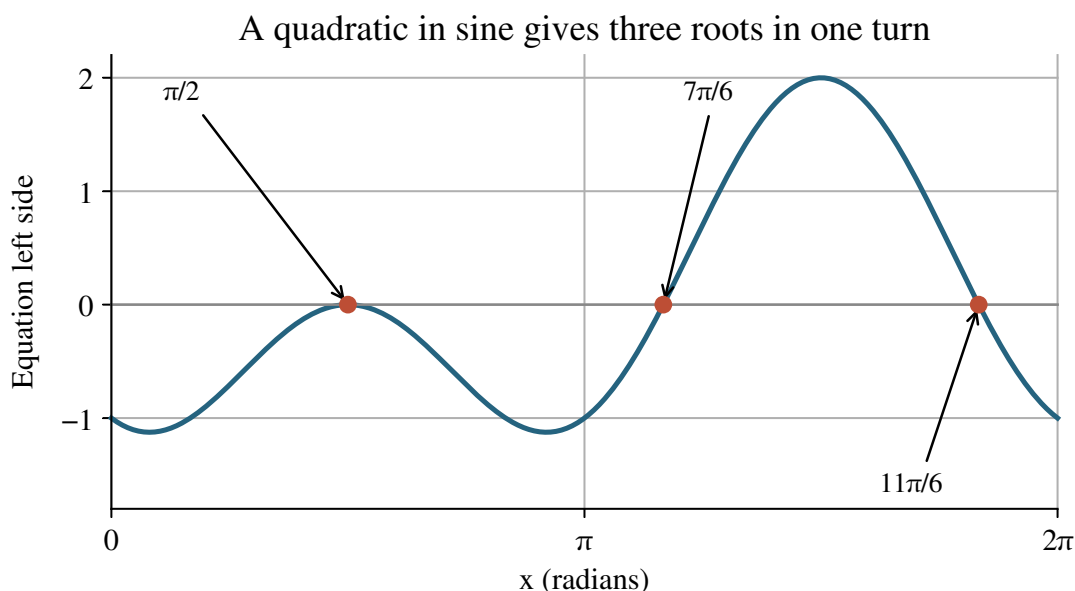
Work through a checked case

$$2 \sin^2 x - \sin x - 1 = (2 \sin x + 1)(\sin x - 1)$$

- Check the result against the starting quantities. Substitute into the original relation, or compare the graph and numerical answer where appropriate.

Worked example

On $0 \leq x < 2\pi$, solve $2\sin^2 x - \sin x - 1 = 0$. Put $u = \sin x$: $(2u+1)(u-1) = 0$, so $u = -1/2$ or 1 . The solutions are $7\pi/6$, $11\pi/6$ and $\pi/2$; sort them as $\pi/2$, $7\pi/6$, $11\pi/6$. For $2\cos^2 x - 5\cos x + 2 = 0$, the algebraic values are $1/2$ and 2 ; discard 2 , leaving $x = \pi/3, 5\pi/3$. For $\tan^2 x - 3\tan x + 2 = 0$, $\tan x = 1$ or 2 . Thus $x = \pi/4, 5\pi/4, \arctan 2, \pi + \arctan 2$, with all four checked in the original equation. For $\sin x \cos x = \sin x$, factor $\sin x(\cos x - 1) = 0$: $x = 0$ or π . Cancelling $\sin x$ would lose π .



Test a tempting shortcut

- Count angles rather than only algebraic roots: a sine value 1 has one angle per turn, while most interior values have two. Reject impossible sine/cosine outputs before taking inverse functions. Do not include 2π in a half-open interval, and do not list a repeated endpoint twice.
- When a shortcut fails, identify the assumption it breaks. Keep an exact value until the requested final rounding.

Watch out

Every real root of a quadratic in $\cos x$ is an allowed cosine value. This claim is false. Explain which definition or assumption it violates.

Interpret a new situation

- Use an identity first if both sine and cosine occur: $2\sin^2 x + \cos x - 2 = 0$ becomes $\cos x(1 - 2\cos x) = 0$, giving $x = \pi/2, 3\pi/2, \pi/3, 5\pi/3$ on $[0, 2\pi)$. A factorised product is zero when either factor is zero. Verify every candidate against the original domain and interval, especially after division or squaring.
- A complete solution gives the mathematical result and explains what it means. Check that it is possible in the stated context.

Use this in your course

- 7357 · A-level · E. Match the target tier and specification before assigning extensions.
- Give the method before the final answer, and use the paper's calculator and formula rules. Review a wrong answer by locating the first invalid step.

Key idea

A possible solution that still needs checking in the original equation and interval. Choose the relationship, show the method, check its assumptions and interpret the result.

E · Multiple-angle equations and transformed intervals

Doubling an angle doubles the number of cycles searched. How can we avoid losing half the solutions?

- Doubling an angle doubles the number of cycles searched. How can we avoid losing half the solutions?
- This lesson studies **transformed interval** : The interval for a substituted angle after applying the same input change to its endpoints.

Choose the mathematical structure

- For an argument $u=kx+c$, transform the whole interval for x into the corresponding interval for u . If k is negative, reverse endpoint order and preserve whether each endpoint is included. Solve in u using all repeated cycles, then recover $x=(u-c)/k$ and check the original interval. Sine/cosine repeat after 2π ; tangent repeats after π .
- State the allowed inputs and units before calculating. An equation should express the relationship, not just record a calculator entry.

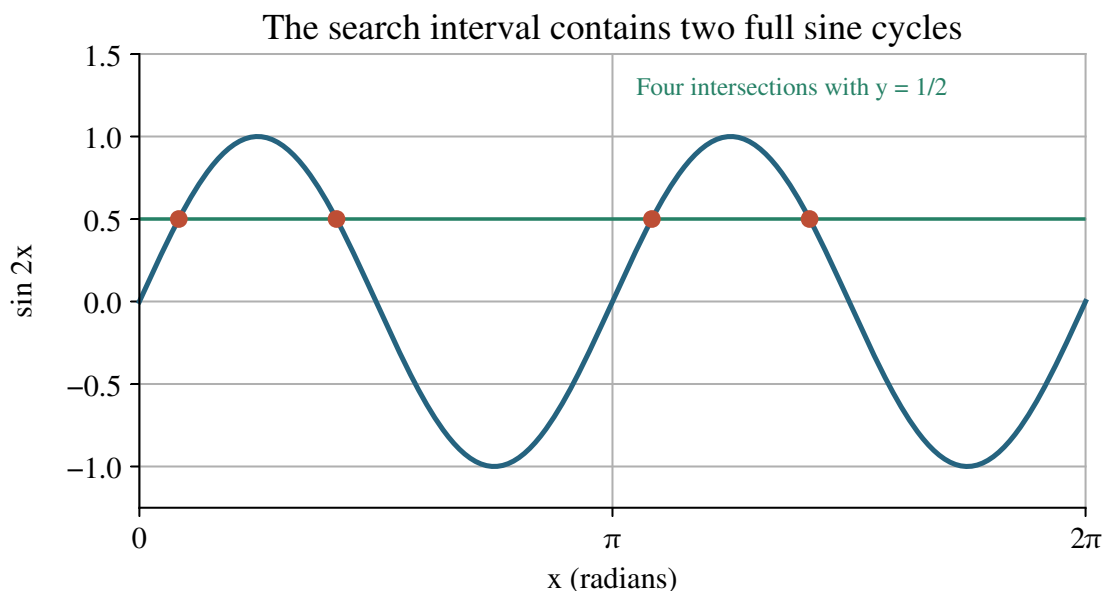
Work through a checked case

$$u = kx + c, \quad x = \frac{u - c}{k} \quad (k \neq 0)$$

- Check the result against the starting quantities. Substitute into the original relation, or compare the graph and numerical answer where appropriate.

Worked example

Solve $\sin 2x = 1/2$ on $0 \leq x < 2\pi$. Then $0 \leq u = 2x < 4\pi$, giving $u = \pi/6, 5\pi/6, 13\pi/6, 17\pi/6$. Divide by 2: $x = \pi/12, 5\pi/12, 13\pi/12, 17\pi/12$. For $\cos(2x - \pi/3) = 1/2$ on $[0, 2\pi)$, the new interval is $[-\pi/3, 11\pi/3)$. Its allowed u values are $-\pi/3, \pi/3, 5\pi/3, 7\pi/3$, hence $x = 0, \pi/3, \pi, 4\pi/3$. For $\tan 3x = 1$ on $[0, \pi)$, u ranges over $[0, 3\pi)$, so $x = \pi/12, 5\pi/12, 3\pi/4$. For $\sin(-2x) = 0$ on $[0, \pi)$, the transformed interval is $(-2\pi, 0]$; its roots $u = -\pi, 0$ give $x = \pi/2, 0$.



Test a tempting shortcut

- Solving only one cycle of u misses roots. An excluded upper endpoint for x need not be the upper endpoint for u when $k < 0$. Dividing by a trig factor can remove zero roots; squaring can introduce extra roots. Substitute into the original equation, whose reciprocal/tangent denominators may still exclude a candidate.
- When a shortcut fails, identify the assumption it breaks. Keep an exact value until the requested final rounding.

Watch out

Solving $\sin 2x$ on one full turn of $2x$ always finds every x solution on $[0, 2\pi)$. This claim is false. Explain which definition or assumption it violates.

Interpret a new situation

- Use general families such as $u = \alpha + 2j\pi$ or $\pi - \alpha + 2j\pi$ for sine, $u = \pm \arccos a + 2j\pi$ for cosine, and $u = \arctan a + j\pi$ for tangent, with integer j . Keep only values in the transformed interval. At sine/cosine extrema, deduplicate the families. Return the answer in the requested angle unit and original variable.
 - A complete solution gives the mathematical result and explains what it means. Check that it is possible in the stated context.
-

Use this in your course

- 7357 · A-level · E. Match the target tier and specification before assigning extensions.
- Give the method before the final answer, and use the paper's calculator and formula rules. Review a wrong answer by locating the first invalid step.

Key idea

The interval for a substituted angle after applying the same input change to its endpoints. Choose the relationship, show the method, check its assumptions and interpret the result.

E · Trigonometric components and motion models

A speed, force and height can each use sine or cosine. Which reference angle and axis decides the formula?

- A speed, force and height can each use sine or cosine. Which reference angle and axis decides the formula?
- This lesson studies **component** : The signed projection of a vector along a chosen coordinate direction.

Choose the mathematical structure

- A vector of magnitude R at angle θ anticlockwise from the positive horizontal axis has components $(R\cos\theta, R\sin\theta)$. A bearing β measured clockwise from north instead gives east $R\sin\beta$ and north $R\cos\beta$. Preserve units and signs. For a periodic height $c+a\sin(\omega t+\varphi)$, a full cycle has duration $2\pi/|\omega|$ when $\omega \neq 0$; interpret the time interval and physical assumptions.
 - State the allowed inputs and units before calculating. An equation should express the relationship, not just record a calculator entry.
-

Work through a checked case

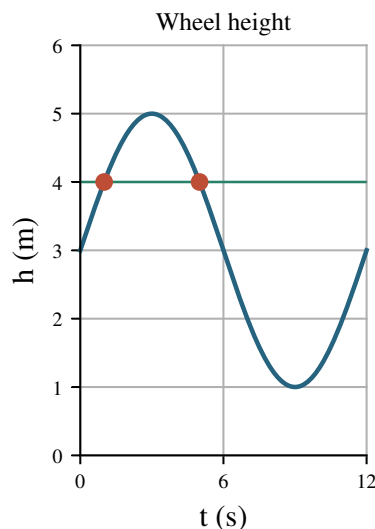
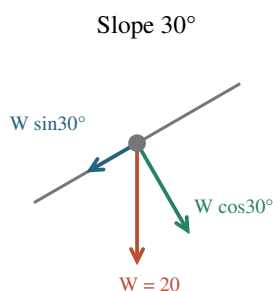
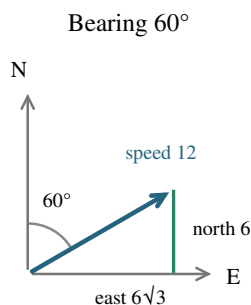
$$(v_x, v_y) = (R\cos\theta, R\sin\theta), \quad T = \frac{2\pi}{|\omega|}$$

- Check the result against the starting quantities. Substitute into the original relation, or compare the graph and numerical answer where appropriate.

Worked example

A velocity of 12 metres per second at bearing 60° has east component $6\sqrt{3}$ and north component 6 metres per second; its horizontal direction angle is 30° . For a 2 kg block on an ideal frictionless 30° slope, take $g=10$ metres per second squared: weight=20 N, downslope component= $20\sin 30^\circ=10$ N and normal component= $20\cos 30^\circ=10\sqrt{3}$ N. If there are no other normal forces, the reaction has that normal magnitude. For a uniformly rotating wheel, centre height 3 m and radius 2 m give $h(t)=3+2\sin(\pi t/6)$, with t in seconds and phase zero at $t=0$. Its period is 12 s and range 1–5 m. On $0 \leq t < 12$, height 4 m occurs at $t=1$ and 5 s; the first is 1 s. Differentiation gives vertical velocity $(\pi/3)\cos(\pi t/6)$, zero at the highest point $t=3$ s.

Define the reference angle, axes and physical interval



Test a tempting shortcut

- A bearing is not an angle from the horizontal. The downslope component of weight uses sine of the slope angle; the normal uses cosine. A reaction does not always equal the weight's normal component if other forces act normally. A position or height function is not itself a velocity. Negative components show direction, not negative magnitude.
- When a shortcut fails, identify the assumption it breaks. Keep an exact value until the requested final rounding.

Watch out

For every bearing, the north component is the magnitude multiplied by the sine of the bearing. This claim is false. Explain which definition or assumption it violates.

Interpret a new situation

- Draw and label axes, angle origin and the interval before projecting or solving. Recover magnitude with $\sqrt{(u^2 + v^2)}$ and choose the direction quadrant from both signed components. State the ideal assumptions: fixed rotation rate for the wheel; no friction or extra normal forces for the block. Restrict mathematical roots to physically allowed times and check dimensions.

- A complete solution gives the mathematical result and explains what it means. Check that it is possible in the stated context.
-

Use this in your course

- 7357 · A-level · E. Match the target tier and specification before assigning extensions.
- Give the method before the final answer, and use the paper's calculator and formula rules. Review a wrong answer by locating the first invalid step.

Key idea

The signed projection of a vector along a chosen coordinate direction. Choose the relationship, show the method, check its assumptions and interpret the result.

E · Sine rule, cosine rule and triangle area

A triangular plot has no right angle. Two known sides and their included angle can determine the remaining side without inventing a perpendicular side length.

- A triangular plot has no right angle. Two known sides and their included angle can determine the remaining side without inventing a perpendicular side length.
 - This lesson studies **included angle** : The angle between the two named sides.
-

Choose the mathematical structure

- Sine rule pairs opposite sides/angles: $a/\sin A = b/\sin B = c/\sin C$. Cosine rule $a^2 = b^2 + c^2 - 2bc \cos A$ uses the angle opposite a . Area is $ab \sin C / 2$ when C lies between a and b . Choose the rule from the known information. An inverse sine may give an acute angle and an obtuse supplement; check the angle sum and supplied sides before accepting either.
 - State the allowed inputs and units before calculating. An equation should express the relationship, not just record a calculator entry.
-

Work through a checked case

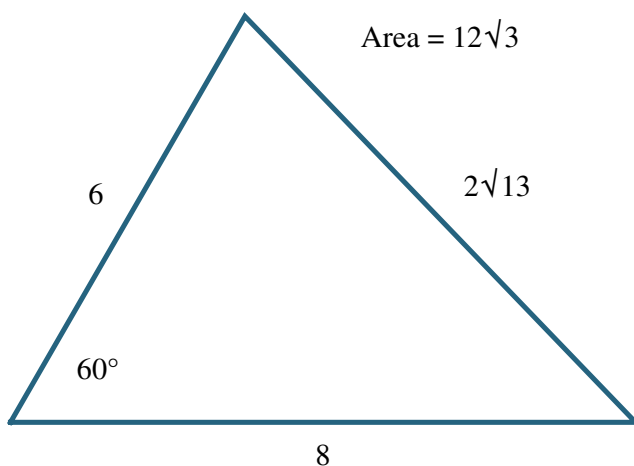
$$a^2 = b^2 + c^2 - 2bc \cos A, \quad A_{\triangle} = \frac{1}{2}ab \sin C$$

- Check the result against the starting quantities. Substitute into the original relation, or compare the graph and numerical answer where appropriate.

Worked example

With sides 6 and 8 enclosing 60° , $c^2 = 36 + 64 - 96 \times (1/2) = 52$, hence $c = 2\sqrt{13}$. Its area is $(1/2) \times 6 \times 8 \times \sin 60^\circ = 12\sqrt{3}$. For $a=4$ opposite $A=30^\circ$ and $B=45^\circ$, $b=4 \sin 45^\circ / \sin 30^\circ = 4\sqrt{2}$. If sides $a=7, b=5, c=6$, $\cos A = (25 + 36 - 49) / (2 \times 5 \times 6) = 1/5$, so $A \approx 78.5^\circ$. To find an angle from area 12 with enclosing sides 6 and 8, $\sin C = 24/48 = 1/2$; C could be 30° or 150° until the remaining data selects a shape. Label opposite pairs and check triangle inequalities to reject impossible side combinations.

Two sides and their included angle



Test a tempting shortcut

- Do not pair a side with its adjacent angle in the sine rule. The area angle must be included. A calculator's first inverse-sine answer need not be the only possible triangle.
- When a shortcut fails, identify the assumption it breaks. Keep an exact value until the requested final rounding.

Watch out

The sine rule pairs each side with any angle in the triangle. This claim is false. Explain which definition or assumption it violates.

Interpret a new situation

- AQA 7357 E1 includes sine/cosine rules and general-triangle area. Choose the rule from the known opposite pairs or included angle; preserve exact values and test all feasible triangle configurations. The dedicated ambiguous-case lesson develops the non-included-angle case.
 - A complete solution gives the mathematical result and explains what it means. Check that it is possible in the stated context.
-

Use this in your course

- 7357 · A-level · E. Match the target tier and specification before assigning extensions.
- Give the method before the final answer, and use the paper's calculator and formula rules. Review a wrong answer by locating the first invalid step.

Key idea

The angle between the two named sides. Choose the relationship, show the method, check its assumptions and interpret the result.

E · The ambiguous sine-rule case and valid triangle counts

Two sides and a non-included angle can describe two different plots of land. How do we check whether both triangles exist?

- Two sides and a non-included angle can describe two different plots of land. How do we check whether both triangles exist?
 - This lesson studies **ambiguous case** : Given triangle data that can produce two different valid triangles.
-

Choose the mathematical structure

- Let A be opposite a , with b another known side. The sine rule gives $\sin B = b \sin A / a$. If this ratio is above 1 there is no triangle. Otherwise test both $B = \arcsin(b \sin A / a)$ and its supplement $180^\circ - B$. Retain only positive $C = 180^\circ - A - B$; when $B = 90^\circ$ the two branches coincide. Every side must be positive.
 - State the allowed inputs and units before calculating. An equation should express the relationship, not just record a calculator entry.
-

Work through a checked case

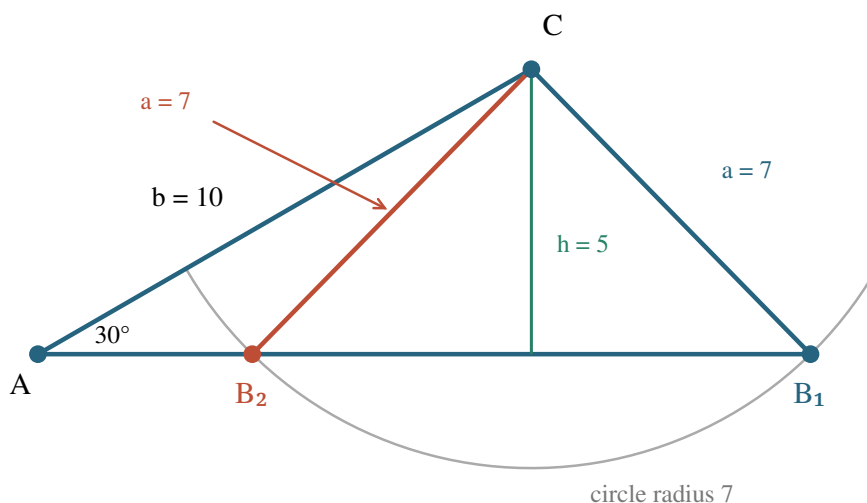
$$\sin B = \frac{b \sin A}{a}, \quad C = 180^\circ - A - B > 0$$

- Check the result against the starting quantities. Substitute into the original relation, or compare the graph and numerical answer where appropriate.

Worked example

Take $A = 30^\circ$, $a = 7$ and $b = 10$. Then $\sin B = 5/7$, so $B \approx 45.5847^\circ$ or 134.4153° . Both give positive C , about 104.4153° or 15.5847° . The corresponding third sides are $c = 5\sqrt{3} + 2\sqrt{6} \approx 13.5592$ and $c = 5\sqrt{3} - 2\sqrt{6} \approx 3.7613$. To derive them, place A at the origin and $C = (10\cos 30^\circ, 10\sin 30^\circ) = (5\sqrt{3}, 5)$; $B = (c, 0)$. The condition $BC = 7$ gives $(c - 5\sqrt{3})^2 + 25 = 49$. Both intersections lie on the positive baseline. Their areas are $bc \sin A / 2 = 25\sqrt{3} / 2 \pm 5\sqrt{6}$, so the same data does not determine one area.

The same a , b and A permit two positive triangles



Test a tempting shortcut

- A second inverse-sine value is only a candidate: it may make the remaining angle zero or negative. For acute A and $h = b \sin A$, the cases are $a < h$: none; $a = h$: one right triangle; $h < a < b$: two; $a \geq b$: one. At $a = b$ the extra baseline intersection is $c = 0$, a degenerate shape. For right or obtuse A , a must exceed b for one triangle; otherwise none.
- When a shortcut fails, identify the assumption it breaks. Keep an exact value until the requested final rounding.

Watch out

Two inverse-sine values always give two valid triangles. This claim is false. Explain which definition or assumption it violates.

Interpret a new situation

- In the example $h = 5$: changing a to 5 gives one right triangle, to 4 gives none, and to 10 gives one with $A = B = 30^\circ$. Check the angle sum, side order and cosine-rule reconstruction for every candidate. State both configurations when the given information does not select one, and keep exact lengths/areas where possible.

- A complete solution gives the mathematical result and explains what it means. Check that it is possible in the stated context.
-

Use this in your course

- 7357 · A-level · E. Match the target tier and specification before assigning extensions.
- Give the method before the final answer, and use the paper's calculator and formula rules. Review a wrong answer by locating the first invalid step.

Key idea

Given triangle data that can produce two different valid triangles. Choose the relationship, show the method, check its assumptions and interpret the result.

E · Key-point sketches and transformed tangent domains

A shifted and stretched trig graph can hide its key points. Which input angles locate its peaks, crossings and breaks?

- A shifted and stretched trig graph can hide its key points. Which input angles locate its peaks, crossings and breaks?
 - This lesson studies **asymptote** : A line approached by a graph while its values grow without bound or tend toward a limit.
-

Choose the mathematical structure

- For a sine/cosine graph a $f(kx+c)+d$, amplitude is $|a|$, centre line $y=d$ and period $2\pi/|k|$ for $k \neq 0$. Transform the five quarter-cycle base inputs to locate key points. A negative a reflects vertically. For tangent the period is $\pi/|k|$, there is no amplitude and the range is all real numbers; exclude every input giving a base-angle $\pi/2+j\pi$.
 - State the allowed inputs and units before calculating. An equation should express the relationship, not just record a calculator entry.
-

Work through a checked case

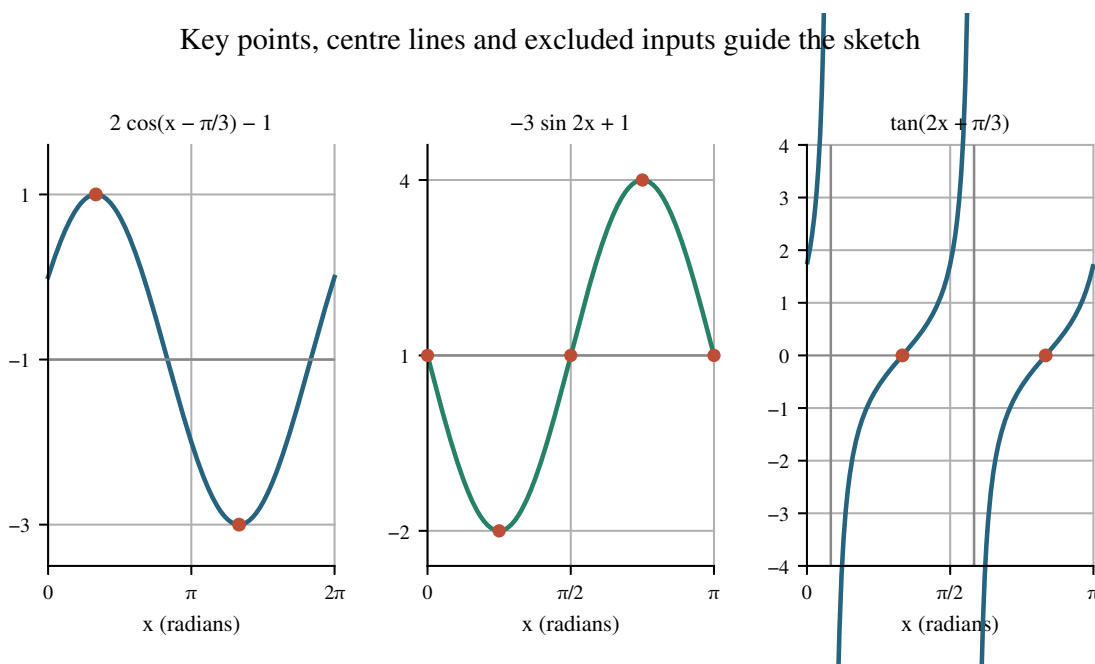
$$T_{\sin, \cos} = \frac{2\pi}{|k|}, \quad T_{\tan} = \frac{\pi}{|k|} \quad (k \neq 0)$$

- Check the result against the starting quantities. Substitute into the original relation, or compare the graph and numerical answer where appropriate.

Worked example

For $y = 2\cos(x - \pi/3) - 1$, one cycle has points $(\pi/3, 1), (5\pi/6, -1), (4\pi/3, -3), (11\pi/6, -1), (7\pi/3, 1)$. The range is $[-3, 1]$ and period 2π . For $y = -3\sin 2x + 1$, the points $x = 0, \pi/4, \pi/2, 3\pi/4, \pi$ give $y = 1, -2, 1, 4, 1$; the period is π and range $[-2, 4]$. For $y = \tan(2x + \pi/3)$, zeros satisfy $2x + \pi/3 = j\pi$, hence $x = j\pi/2 - \pi/6$. Asymptotes satisfy $2x + \pi/3 = \pi/2 + j\pi$, hence $x = \pi/12 + j\pi/2$. On $0 \leq x < \pi$, the zeros are $\pi/3$ and $5\pi/6$ and the asymptotes $\pi/12$ and $7\pi/12$.

Key points, centre lines and excluded inputs guide the sketch



Test a tempting shortcut

- The cosine maximum need not occur at $x=0$ after a horizontal shift. For the stated tangent function, draw separate increasing branches between asymptotes. A negative horizontal or vertical factor reverses the direction; never join branches across a break. Vertical translations change outputs but not the period or excluded x values. A displayed interval may contain only part of a complete shifted cycle.

- When a shortcut fails, identify the assumption it breaks. Keep an exact value until the requested final rounding.

Watch out

A transformed tangent graph has a finite amplitude and can be joined across its vertical asymptotes. This claim is false. Explain which definition or assumption it violates.

Interpret a new situation

- Find period and centre line first, then map exact base-angle key points. Add enough repeats to cover the requested interval and label scale, extremes and asymptotes. To find x-axis crossings, solve the output equation rather than assuming every centre-line crossing has $y=0$. Check a sample point in each tangent branch and preserve the original domain.
 - A complete solution gives the mathematical result and explains what it means. Check that it is possible in the stated context.
-

Use this in your course

- 7357 · A-level · E. Match the target tier and specification before assigning extensions.
- Give the method before the final answer, and use the paper's calculator and formula rules. Review a wrong answer by locating the first invalid step.

Key idea

A line approached by a graph while its values grow without bound or tend toward a limit. Choose the relationship, show the method, check its assumptions and interpret the result.

F · Exponentials, logarithms and modelling

Why does a decay model stay positive?

- A medicine concentration falls by the same percentage each hour. A constant subtraction would eventually predict a negative amount.
- This lesson studies **half-life** : The time for a decaying quantity to fall to half its initial value.

Choose the mathematical structure

- For $y=Ae^{kt}$, k is a proportional rate. Taking logs gives $\ln y = \ln A + kt$. Logarithms require positive arguments, and $\log(x+y)$ is not $\log x + \log y$.
 - State the allowed inputs and units before calculating. An equation should express the relationship, not just record a calculator entry.
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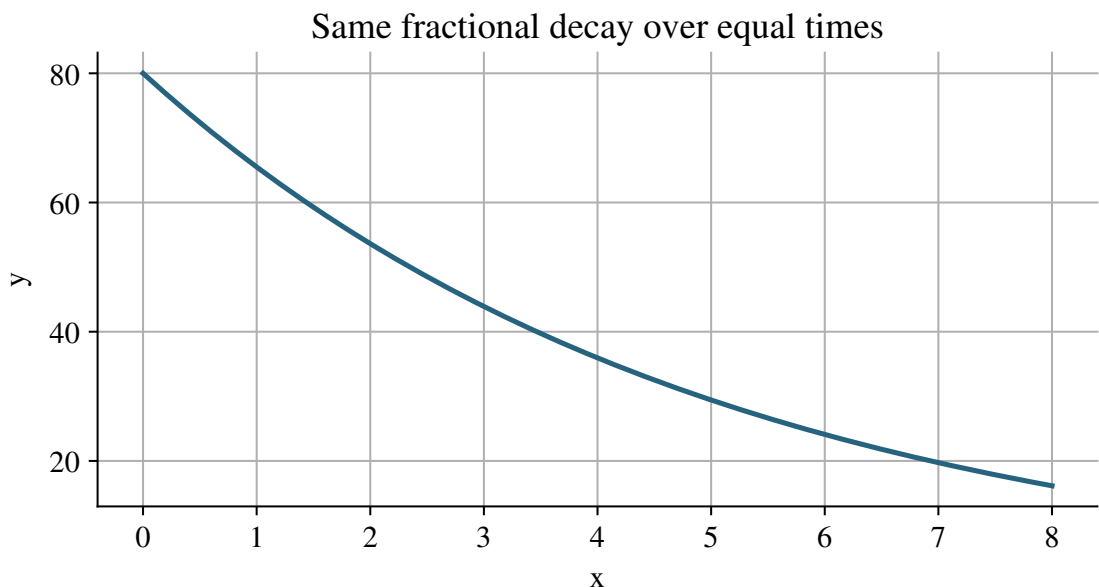
Work through a checked case

$$y = Ae^{kt}, \quad \ln y = \ln A + kt$$

- Check the result against the starting quantities. Substitute into the original relation, or compare the graph and numerical answer where appropriate.

Worked example

If $y=80e^{(-0.2t)}$, $y=40$ gives $e^{(-0.2t)}=1/2$. Hence $t=\ln 2/0.2 \approx 3.466$. For $3^x=20$, $x=\ln 20/\ln 3$. Plotting $\ln y$ against t linearises this exponential model.



Test a tempting shortcut

- A fitted exponential is a model, not a guarantee. Specify the time units and range of use. A negative k describes decay; a negative starting amount usually contradicts the context.
- When a shortcut fails, identify the assumption it breaks. Keep an exact value until the requested final rounding.

Watch out

For positive x and y , $\ln(x+y) = \ln x + \ln y$. This claim is false. Explain which definition or assumption it violates.

Interpret a new situation

- Compare actual observations with the model. Systematic departures may indicate changing conditions. In a report, explain what the rate and initial value mean, rather than giving a bare equation.
 - A complete solution gives the mathematical result and explains what it means. Check that it is possible in the stated context.
-

Use this in your course

- 7357 · A-level · F. Match the target tier and specification before assigning extensions.
- Give the method before the final answer, and use the paper's calculator and formula rules. Review a wrong answer by locating the first invalid step.

Key idea

The time for a decaying quantity to fall to half its initial value. Choose the relationship, show the method, check its assumptions and interpret the result.

F · Exponential graphs and logarithmic inverses

An exponential maps an exponent to a positive amount. Which graph reverses that input-output pair?

- An exponential maps an exponent to a positive amount. Which graph reverses that input-output pair?
 - This lesson studies **logarithm** : The exponent to which a stated valid base must be raised to produce a positive input.
-

Choose the mathematical structure

- For $a > 0$, $y = a^x$ has every real input and positive outputs. It passes through $(0, 1)$. For $a > 1$ it increases; for $0 < a < 1$ it decreases. For $a = 1$ it is the constant 1, with no inverse function. For $a > 0$ and $a \neq 1$, $y = \log_a x$ is the inverse of a^x : domain $x > 0$, range all real numbers. $\ln x$ is $\log_e x$ and reverses e^x .
 - State the allowed inputs and units before calculating. An equation should express the relationship, not just record a calculator entry.
-

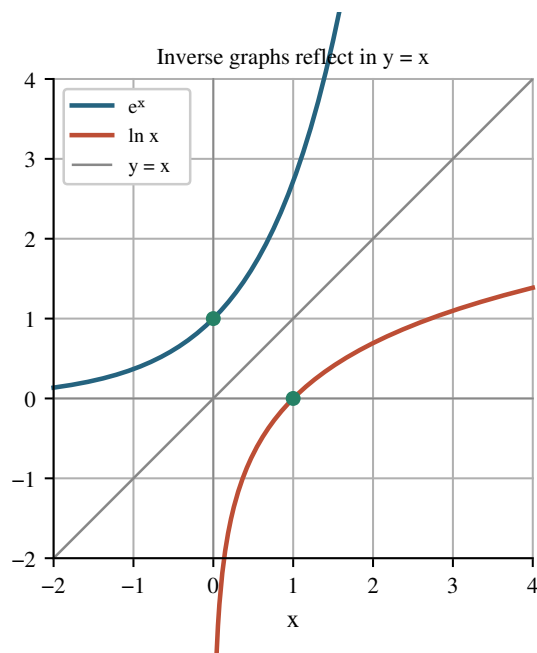
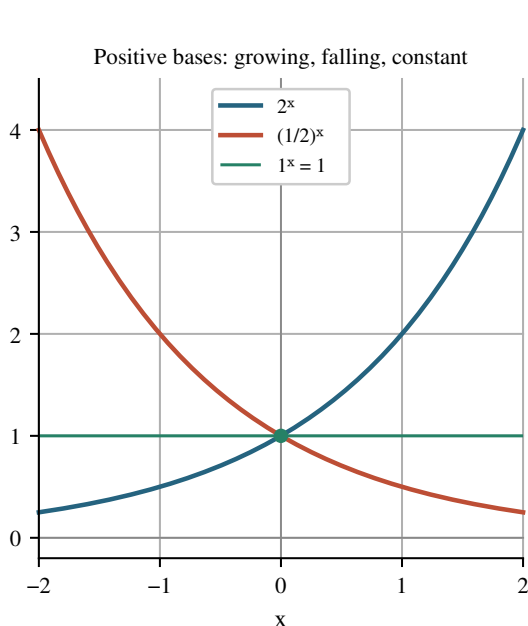
Work through a checked case

$$y = a^x \iff x = \log_a y \quad (a > 0, a \neq 1, y > 0)$$

- Check the result against the starting quantities. Substitute into the original relation, or compare the graph and numerical answer where appropriate.

Worked example

The graph 2^x contains $(-1, 1/2), (0, 1), (1, 2), (2, 4)$. Its inverse $\log_2 x$ contains $(1/2, -1), (1, 0), (2, 1), (4, 2)$, reflected in $y = x$. The graph $(1/2)^x$ gives $2, 1, 1/2$ at $x = -1, 0, 1$, while its inverse $\log_{1/2} x$ is decreasing. For e^x and $\ln x$, the points $(0, 1)$ and $(1, 0)$ are reflected partners; $\ln e = 1$ and $\ln(e^3) = 3$. For any real u , $\ln(e^u) = u$; for $x > 0$, $e^{(\ln x)} = x$. Logarithms approach but do not reach their vertical boundary $x = 0$; e^x approaches $y = 0$ as x tends to negative infinity.



Test a tempting shortcut

- Neither $\ln 0$ nor the logarithm of a negative real number exists in the real-number course. A negative logarithm output is allowed: $\ln(1/e) = -1$. Base 1 is permitted for an exponential constant but not for a logarithm. For a base below 1 the exponential approaches zero as x tends to positive infinity, and the inverse has the opposite increasing/decreasing direction from the $a > 1$ case.
- When a shortcut fails, identify the assumption it breaks. Keep an exact value until the requested final rounding.

Watch out

The constant exponential 1^x has a logarithmic inverse for all positive inputs. This claim is false. Explain which definition or assumption it violates.

Interpret a new situation

- State the base, domain and range before taking an inverse. Swap the coordinates of exact exponential points to construct the log graph. Preserve open boundaries and asymptotes. The source F3 prints $x \geq 0$; the valid logarithm input condition used here is $x > 0$, since a^u is positive and never zero.

- A complete solution gives the mathematical result and explains what it means. Check that it is possible in the stated context.
-

Use this in your course

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- Give the method before the final answer, and use the paper's calculator and formula rules. Review a wrong answer by locating the first invalid step.

Key idea

The exponent to which a stated valid base must be raised to produce a positive input. Choose the relationship, show the method, check its assumptions and interpret the result.

F · Logarithm laws, negative powers and equation domains

Combining two logs can produce a polynomial with an extra root. Which original input condition rejects it?

- Combining two logs can produce a polynomial with an extra root. Which original input condition rejects it?
 - This lesson studies **argument** : The input inside a logarithm, which must be positive for a real logarithm.
-

Choose the mathematical structure

- For $a > 0$, $a \neq 1$ and positive u, v : $\log_a(uv) = \log_a u + \log_a v$, $\log_a(u/v) = \log_a u - \log_a v$, and $\log_a(u^k) = k \log_a u$ for real k . These follow from exponent multiplication/division/powers. In particular $k = -1$ gives a reciprocal and $k = -1/2$ gives an inverse square root. Solve $a^x = b$, with $b > 0$, by $x = \ln b / \ln a$.
 - State the allowed inputs and units before calculating. An equation should express the relationship, not just record a calculator entry.
-

Work through a checked case

$$\log_a(uv) = \log_a u + \log_a v, \quad x = \frac{\ln b}{\ln a} \quad (a^x = b)$$

- Check the result against the starting quantities. Substitute into the original relation, or compare the graph and numerical answer where appropriate.

Worked example

For $x > 0$, $\ln(x^{-1}) = -\ln x$ and $\ln(x^{-1/2}) = -(1/2)\ln x$. Solve $\ln(x-1) + \ln(x+1) = \ln 8$: the original domain is $x > 1$. Combining gives $\ln(x^2-1) = \ln 8$, hence $x^2 = 9$ and candidates ± 3 ; only $x = 3$ satisfies the original arguments. For $3^x = 20$, $x = \ln 20 / \ln 3$. If $2^{2x} - 2^x - 2 = 0$, let $u = 2^x > 0$: $(u-2)(u+1) = 0$, so $u = 2$ gives $x = 1$, while $u = -1$ is impossible. Also $\ln(2+3) = \ln 5$, whereas $\ln 2 + \ln 3 = \ln 6$; a sum inside a log cannot be split this way.

A log-law step preserves the original positive arguments

$$\ln(x-1) + \ln(x+1) = \ln 8$$

$x > 1$ before combining

$$x^2 - 1 = 8 \Rightarrow x = \pm 3$$

Accept $x = 3$; reject $x = -3$

Test a tempting shortcut

- Combining or expanding logs does not erase their original domains. $\ln(u^2) = 2\ln|u|$ for $u \neq 0$; writing $2\ln u$ additionally needs $u > 0$. Do not take a real logarithm of a

zero or negative right-hand side in $a^x=b$. For base 1, $1^x=b$ has all real x when $b=1$ and no solutions otherwise; the log quotient is not available.

- When a shortcut fails, identify the assumption it breaks. Keep an exact value until the requested final rounding.

Watch out

Every polynomial root obtained after combining logarithms satisfies the original log equation. This claim is false. Explain which definition or assumption it violates.

Interpret a new situation

- List positivity restrictions first, use one common base, then apply a valid law. After algebra, substitute every candidate into the original logarithmic/exponential equation. Keep log quotients exact unless a decimal is requested. Changing the log base does not change an exponential equation's solution when both quotient logs use the same valid base.
 - A complete solution gives the mathematical result and explains what it means. Check that it is possible in the stated context.
-

Use this in your course

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- Give the method before the final answer, and use the paper's calculator and formula rules. Review a wrong answer by locating the first invalid step.

Key idea

The input inside a logarithm, which must be positive for a real logarithm. Choose the relationship, show the method, check its assumptions and interpret the result.

F · Logarithmic graphs for power and exponential fits

A curve can become a straight line after taking logs. Which horizontal axis distinguishes a power model from an exponential one?

- A curve can become a straight line after taking logs. Which horizontal axis distinguishes a power model from an exponential one?
 - This lesson studies **intercept** : The transformed vertical value when the chosen horizontal variable is zero.
-

Choose the mathematical structure

- For positive observations and parameters, $y = ax^p$ with $x > 0$ gives $\ln y = \ln a + p \ln x$. Plot $\ln y$ against $\ln x$: gradient p , intercept $\ln a$. For $y = kb^x$ with $k, b > 0$, $\ln y = \ln k + x \ln b$. Plot $\ln y$ against x : gradient $\ln b$, intercept $\ln k$. Use the same log base for both logarithmic axes and recover parameters with that base's inverse.
 - State the allowed inputs and units before calculating. An equation should express the relationship, not just record a calculator entry.
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Work through a checked case

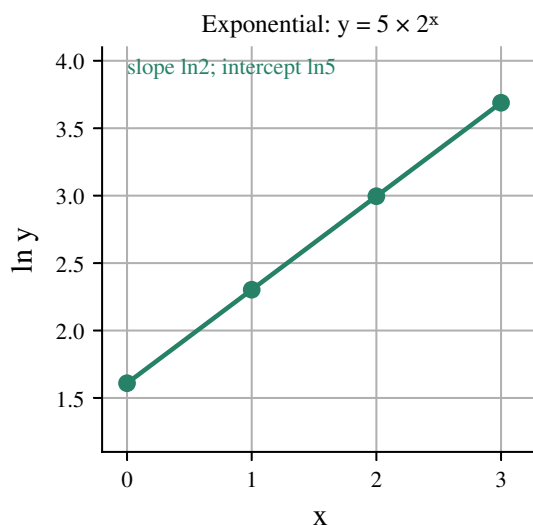
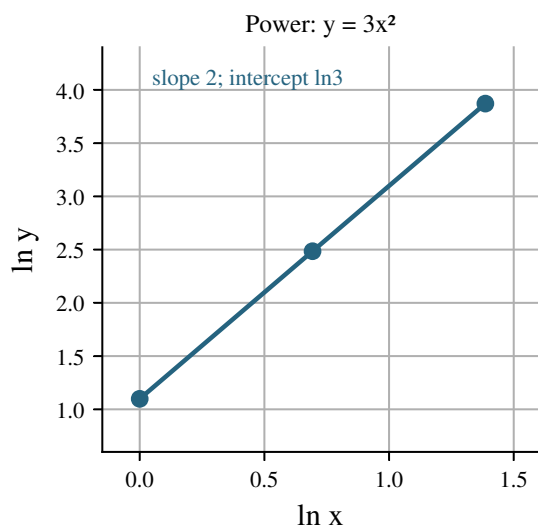
$$\ln y = \ln a + p \ln x, \quad \ln y = \ln k + x \ln b$$

- Check the result against the starting quantities. Substitute into the original relation, or compare the graph and numerical answer where appropriate.

Worked example

Power data $(x, y) = (1, 3), (2, 12), (4, 48)$ give gradient $[\ln 48 - \ln 3] / [\ln 4 - \ln 1] = 2$, intercept $\ln 3$, hence $y = 3x^2$. Exponential data $(0, 5), (1, 10), (2, 20), (3, 40)$ give a semilog gradient $\ln 2$ and intercept $\ln 5$, hence $y = 5 \times 2^x$. With base-10 logs the same exponential gradient is $\log_{10} 2 \approx 0.30103$, so $b = 10^{0.30103} \approx 2$ rather than $e^{0.30103}$. A new power-data observation $(3, 30)$ exceeds the fitted prediction 27 by residual 3; a perfectly straight calibration set does not prove the model beyond its observed range.

Different horizontal axes recover different parameters



Test a tempting shortcut

- For the power fit the gradient is the exponent, not its logarithm. For the exponential fit the gradient is the logarithm of the base, not the base itself. The power graph's intercept corresponds to $x=1$ because $\ln x=0$ there; the exponential intercept corresponds to $x=0$. Nonpositive observations cannot be logged directly. An exponential log-y plot may include $x=0$ or negative x ; the log-log power plot cannot.
- When a shortcut fails, identify the assumption it breaks. Keep an exact value until the requested final rounding.

Watch out

Plotting $\ln y$ against x linearises every power model $y=ax^p$. This claim is false. Explain which definition or assumption it violates.

Interpret a new situation

- Label transformed axes and estimate gradient from well-separated points on the fitted line, not from a single data point. Recover each intercept parameter before returning to original units. Substitute predictions into the original model and examine residuals there. State the fitted range, measurement uncertainty and

whether extrapolation is justified; systematic departures can suggest a different model.

- A complete solution gives the mathematical result and explains what it means. Check that it is possible in the stated context.
-

Use this in your course

- 7357 · A-level · F. Match the target tier and specification before assigning extensions.
- Give the method before the final answer, and use the paper's calculator and formula rules. Review a wrong answer by locating the first invalid step.

Key idea

The transformed vertical value when the chosen horizontal variable is zero. Choose the relationship, show the method, check its assumptions and interpret the result.

F · Proportional rates, half-life and model limits

A decaying amount loses more per hour when more remains. Why does one fixed half-life not mean a fixed subtraction?

- A decaying amount loses more per hour when more remains. Why does one fixed half-life not mean a fixed subtraction?
 - This lesson studies **proportional rate** : A rate of change equal to a constant multiplied by the current amount.
-

Choose the mathematical structure

- For $y = Ae^{kt}$, $dy/dt = ke^{kt}A = ky$. $A > 0$ is the initial amount; k is a constant proportional rate in inverse time units. Positive k gives growth, negative k decay, zero k a constant. Over elapsed time Δt the multiplier is $e^{k\Delta t}$. Doubling time is $\ln 2/k$ for $k > 0$; half-life is $\ln 2/|k|$ for $k < 0$.
 - State the allowed inputs and units before calculating. An equation should express the relationship, not just record a calculator entry.
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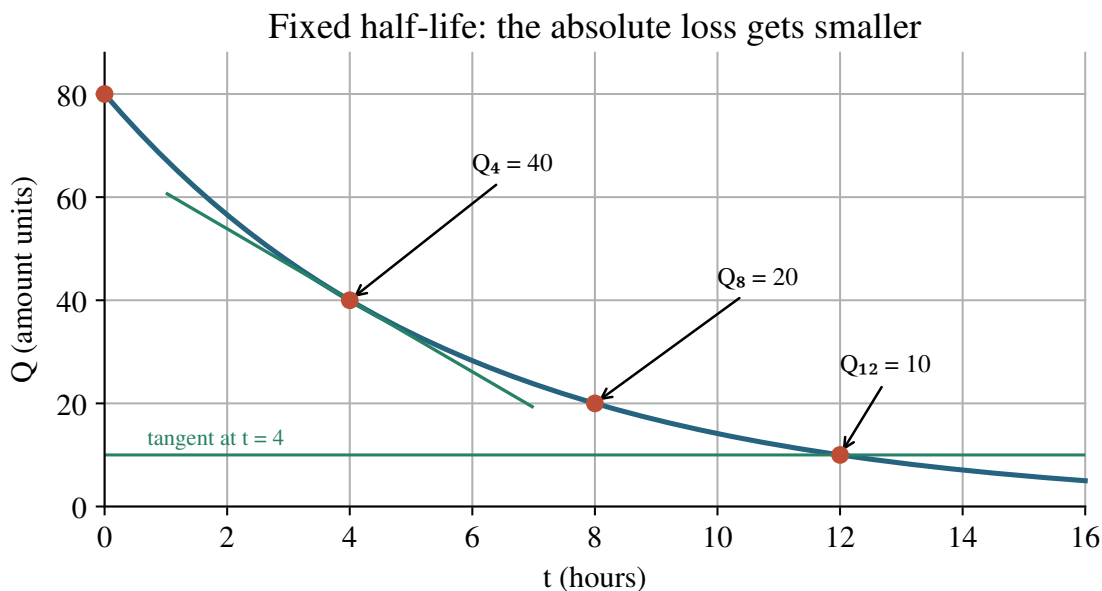
Work through a checked case

$$y = Ae^{kt}, \quad \frac{dy}{dt} = ky, \quad T_{1/2} = \frac{\ln 2}{|k|} \quad (k < 0)$$

- Check the result against the starting quantities. Substitute into the original relation, or compare the graph and numerical answer where appropriate.

Worked example

Use an illustrative decay $Q = 80e^{(-\lambda t)}$, t in hours and $\lambda = \ln 2/4$. Then $Q_4 = 40$, $Q_8 = 20$ and $Q_{12} = 10$: each four-hour step halves the amount. At $t = 4$ the rate is $Q' = -40\lambda = -10\ln 2$ amount units per hour. Solving $Q \leq 10$ gives $t \geq 12$; the strict condition $Q < 10$ instead needs $t > 12$. For an illustrative mathematical account $P = 100e^{(0.1t)}$, t in years, $P_1 \approx 110.517$ and the one-year effective growth is about 10.517%, not 10%. If the given effective yearly growth is 10%, use $k = \ln 1.1$ instead. Continuous compounding and a stated annual multiplier must not be confused.



Test a tempting shortcut

- The exponent must be dimensionless: changing hours to minutes changes k by a factor $1/60$. A half-life is not the time to reach zero; positive A times an exponential stays positive for finite time. Absolute losses vary with the amount although the proportional rate is fixed. A first whole-period threshold needs adjacent integer checks rather than only a continuous crossing time.
- When a shortcut fails, identify the assumption it breaks. Keep an exact value until the requested final rounding.

Watch out

A constant exponential proportional rate means the same absolute amount is lost in each equal time interval. This claim is false. Explain which definition or assumption it violates.

Interpret a new situation

- State time/amount units, initial conditions and the constant-rate assumption. Compare observations with predictions before extrapolating. Limited resources, changing environmental conditions, fees or withdrawals can invalidate a simple growth model; changing decay conditions can require a varying-rate or multi-stage model. These are illustrative mathematical models, not product terms or treatment instructions.
 - A complete solution gives the mathematical result and explains what it means. Check that it is possible in the stated context.
-

Use this in your course

- 7357 · A-level · F. Match the target tier and specification before assigning extensions.
- Give the method before the final answer, and use the paper's calculator and formula rules. Review a wrong answer by locating the first invalid step.

Key idea

A rate of change equal to a constant multiplied by the current amount. Choose the relationship, show the method, check its assumptions and interpret the result.

G · Derivatives and stationary points

What is the slope at one point?

- A curved road has different slopes at different positions. An average gradient cannot describe every point.
 - This lesson studies **derivative** : The instantaneous rate of change, also the gradient of a tangent.
-

Choose the mathematical structure

- For $y=ax^n$, $dy/dx=nax^{n-1}$. A stationary point satisfies $dy/dx=0$. Check the sign change of the derivative, or the second derivative when it is nonzero, to classify it.
 - State the allowed inputs and units before calculating. An equation should express the relationship, not just record a calculator entry.
-

Work through a checked case

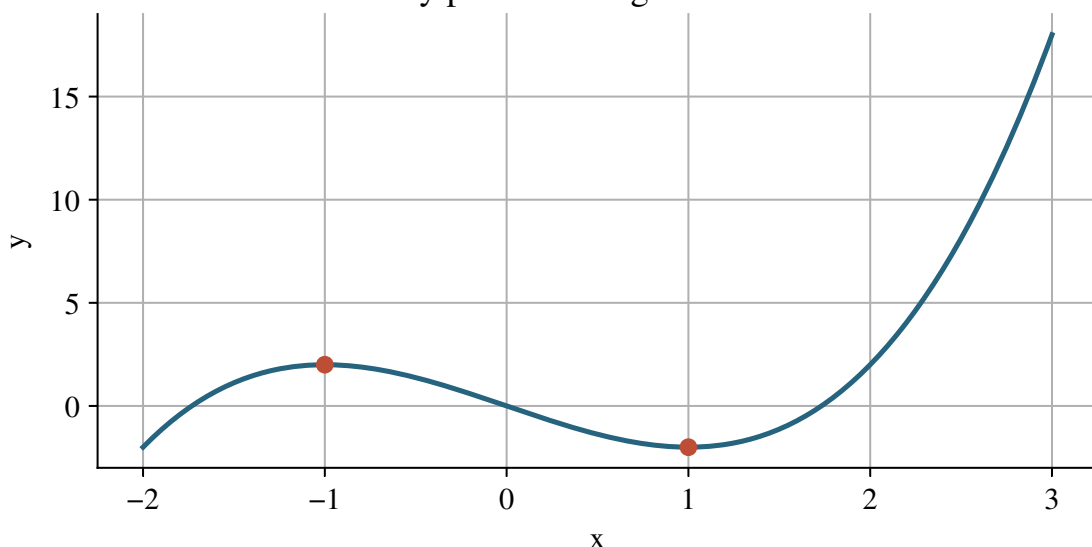
$$\frac{d}{dx}(ax^n) = anx^{n-1}, \quad f'(x) = 0$$

- Check the result against the starting quantities. Substitute into the original relation, or compare the graph and numerical answer where appropriate.

Worked example

For $y=x^3-3x$, $dy/dx=3x^2-3$. At $x=1$, the gradient is 0 and $y=-2$. The second derivative is $6x$, positive at $x=1$, so this is a local minimum. At $x=-1$, $y=2$ and the second derivative is negative, giving a local maximum.

Stationary points and signed accumulation



Test a tempting shortcut

- A zero derivative does not always mean a maximum or minimum: $y=x^3$ is stationary at 0 but continues increasing. An endpoint can also produce an extreme value on a restricted domain.
- When a shortcut fails, identify the assumption it breaks. Keep an exact value until the requested final rounding.

Watch out

Every point with zero derivative is a local maximum or minimum. This claim is false. Explain which definition or assumption it violates.

Interpret a new situation

- At GCSE/IGCSE use only the polynomial scope allowed by the tier; do not add chain, product or quotient rules there. At advanced level, connect the derivative to rates and optimization with a valid domain.
 - A complete solution gives the mathematical result and explains what it means. Check that it is possible in the stated context.
-

Use this in your course

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- Give the method before the final answer, and use the paper's calculator and formula rules. Review a wrong answer by locating the first invalid step.

Key idea

The instantaneous rate of change, also the gradient of a tangent. Choose the relationship, show the method, check its assumptions and interpret the result.

G · Chain, product, quotient and implicit differentiation

Does the inside expression change too?

- A cost curve is a power of a changing expression. Differentiating the outer power alone misses the rate of its input.
- This lesson studies **chain rule** : The rule that multiplies the outer derivative by the inner derivative for a composite function.

Choose the mathematical structure

- For $y=f(g(x))$, $y' = f'(g(x))g'(x)$. For uv , differentiate to $u'v + uv'$. For u/v , use $(u'v - uv')/v^2$, where $v \neq 0$.
- State the allowed inputs and units before calculating. An equation should express the relationship, not just record a calculator entry.

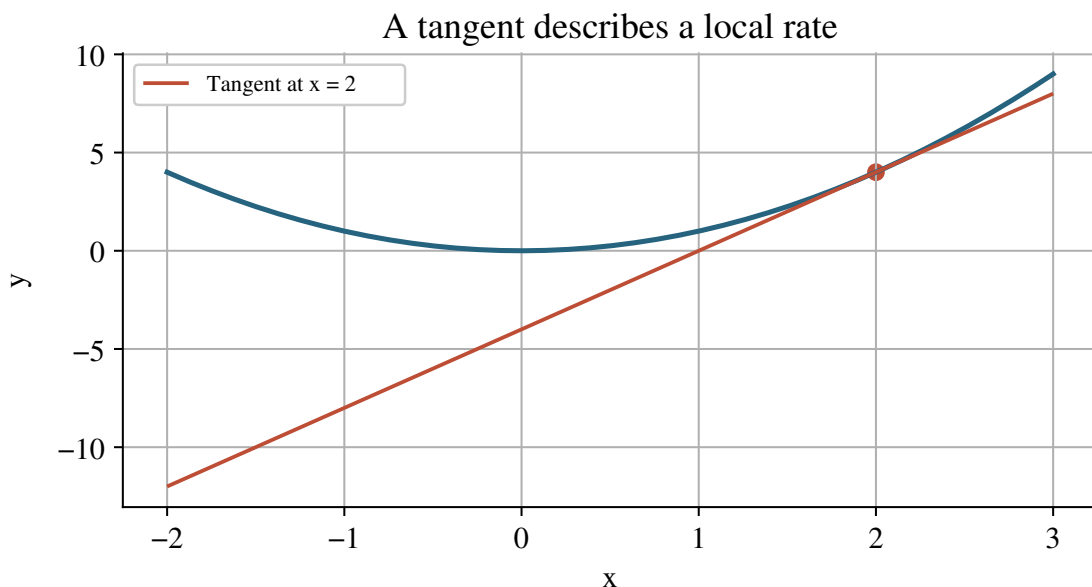
Work through a checked case

$$\frac{d}{dx}f(g(x)) = f'(g(x))g'(x)$$

- Check the result against the starting quantities. Substitute into the original relation, or compare the graph and numerical answer where appropriate.

Worked example

For $y=(3x+1)^4$, $y' = 4(3x+1)^3 \times 3 = 12(3x+1)^3$. At $x=0$, the gradient is 12.
 For $x^2+y^2=25$, differentiate implicitly: $2x+2y y' = 0$, so $y' = -x/y$ when $y \neq 0$.



Test a tempting shortcut

- A derivative of a product is not the product of the derivatives. In implicit differentiation, every differentiated function of y brings a dy/dx factor. A quotient's denominator is squared.
- When a shortcut fails, identify the assumption it breaks. Keep an exact value until the requested final rounding.

Watch out

The derivative of $u(x)v(x)$ is always u' times v' . This claim is false. Explain which definition or assumption it violates.

Interpret a new situation

- Choose a useful form before differentiating: expanding a short polynomial may be simpler. For related rates, write the relation in symbols, differentiate with respect to time, then substitute measured values.

- A complete solution gives the mathematical result and explains what it means. Check that it is possible in the stated context.
-

Use this in your course

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- Give the method before the final answer, and use the paper's calculator and formula rules. Review a wrong answer by locating the first invalid step.

Key idea

The rule that multiplies the outer derivative by the inner derivative for a composite function. Choose the relationship, show the method, check its assumptions and interpret the result.

G · Secants, limits and first-principles power derivatives

A distance sensor gives two positions a short time apart. What happens to their average rate as the interval shrinks?

- A distance sensor gives two positions a short time apart. What happens to their average rate as the interval shrinks?
 - This lesson studies **difference quotient** : The change in function output divided by a nonzero change in input.
-

Choose the mathematical structure

- For $h \neq 0$, the secant gradient is $[f(x+h)-f(x)]/h$. The derivative $f'(x)$ is its limit as h tends to zero, if the two-sided finite limit exists. It gives the tangent gradient and instantaneous rate. The variable h may approach zero from either sign; simplify before taking the limit.
 - State the allowed inputs and units before calculating. An equation should express the relationship, not just record a calculator entry.
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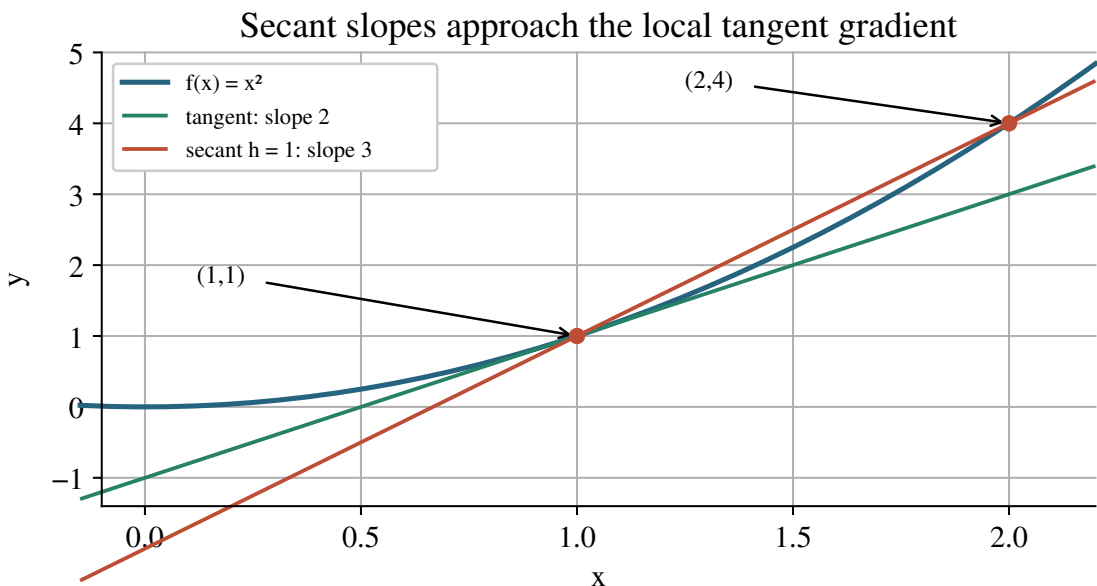
Work through a checked case

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

- Check the result against the starting quantities. Substitute into the original relation, or compare the graph and numerical answer where appropriate.

Worked example

For $f(x)=x^2$, $[(x+h)^2-x^2]/h=2x+h$, so $f'(x)=2x$. At $x=1$, $h=1, 0.5, -0.5$ give secant gradients 3, 2.5, 1.5, tending to 2 from both sides. The tangent through $(1,1)$ is $y-1=2(x-1)$. For $f(x)=x^3$, expanding gives $3x^2+3xh+h^2$, hence $f'(x)=3x^2$. A constant gives zero and $f(x)=x$ gives one. For any positive integer n , the binomial expansion leaves nx^{n-1} plus terms containing h , so the limit is nx^{n-1} . This argument alone does not prove the rule for rational powers.



Test a tempting shortcut

- Substituting $h=0$ into the unsimplified quotient gives $0/0$, not a gradient. Cancelling h is valid for $h \neq 0$ before the limit. Continuity alone is insufficient: at $x=0$,

$|x|$ has right quotient 1 and left quotient -1 , so no derivative there. The tangent is a local line, not an exact replacement for the entire curve.

- When a shortcut fails, identify the assumption it breaks. Keep an exact value until the requested final rounding.

Watch out

A difference quotient can be evaluated by setting $h=0$ before simplification. This claim is false. Explain which definition or assumption it violates.

Interpret a new situation

- Write the difference quotient, expand, cancel only with $h \neq 0$ stated, and take the limit. Interpret derivative units as output units divided by input units. When a question says first principles, merely quoting $\widehat{nx^{(n-1)}}$ does not establish the limit. At a corner check both approach directions.
 - A complete solution gives the mathematical result and explains what it means. Check that it is possible in the stated context.
-

Use this in your course

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- Give the method before the final answer, and use the paper's calculator and formula rules. Review a wrong answer by locating the first invalid step.

Key idea

The change in function output divided by a nonzero change in input. Choose the relationship, show the method, check its assumptions and interpret the result.

G · First-principles sine and cosine derivatives

A rotating point has sine and cosine coordinates. How do its tiny coordinate changes lead to exact derivative formulas?

- A rotating point has sine and cosine coordinates. How do its tiny coordinate changes lead to exact derivative formulas?
 - This lesson studies **radian limit** : A small-angle limit used with angles measured in radians.
-

Choose the mathematical structure

- In radians, $\sin h/h$ tends to 1 and $(\cos h - 1)/h$ tends to 0 as h tends to zero. Using angle-addition formulas in the difference quotient gives $d(\sin x)/dx = \cos x$ and $d(\cos x)/dx = -\sin x$. These limits determine the tangent gradients; they require radian input.
 - State the allowed inputs and units before calculating. An equation should express the relationship, not just record a calculator entry.
-

Work through a checked case

$$\lim_{h \rightarrow 0} \frac{\sin h}{h} = 1, \quad (\sin x)' = \cos x, \quad (\cos x)' = -\sin x$$

- Check the result against the starting quantities. Substitute into the original relation, or compare the graph and numerical answer where appropriate.

Worked example

For $\sin x$, $[\sin(x+h) - \sin x]/h = \sin x[(\cos h - 1)/h] + \cos x[\sin h/h]$, tending to $\cos x$. For $\cos x$ the quotient is $\cos x[(\cos h - 1)/h] - \sin x[\sin h/h]$, tending to $-\sin x$. Near zero, the unit-circle inequality $\sin h < h < \tan h$ for $0 < h < \pi/2$ bounds $\sin h/h$ between $\cos h$ and 1; negative h has the same ratio. Also $(\cos h - 1)/h = -2\sin^2(h/2)/h$ tends to zero, using $\sin(h/2)/(h/2) \rightarrow 1$. At $x = \pi/6$, the sine gradient is $\sqrt{3}/2$ and the cosine gradient is $-1/2$. For sine at $x=0$, $h=0.1$ gives $\sin(0.1)/0.1 \approx 0.998334$; at $h=-0.1$ the ratio is the same.

Separate the fixed x factors from the shrinking h factors

$$\sin(x+h) - \sin x = \sin x(\cos h - 1) + \cos x \sin h$$

$$\cos(x+h) - \cos x = \cos x(\cos h - 1) - \sin x \sin h$$

Divide by $h \neq 0$, then take the radian limits:

$$\sin h / h \rightarrow 1; (\cos h - 1) / h \rightarrow 0$$

$$(\sin x)' = \cos x; (\cos x)' = -\sin x$$

Test a tempting shortcut

- A small-angle approximation is not an exact finite-h equality. The cosine derivative has a minus sign. If a variable θ is measured in degrees, $d \sin(\pi\theta/180)/d\theta = (\pi/180)\cos(\pi\theta/180)$; the radian formula cannot be copied without that factor. Do not use the derivative formula itself to prove the starting limit.
- When a shortcut fails, identify the assumption it breaks. Keep an exact value until the requested final rounding.

Watch out

The derivative of $\cos x$ is positive $\sin x$ for radian x . This claim is false. Explain which definition or assumption it violates.

Interpret a new situation

- Start with the original difference quotient and use angle addition before applying the two radian limits. Retain the fixed $\sin x/\cos x$ factors and their signs. Check exact-point gradients against whether the graph is rising or falling; the proof applies at any real radian x , not only near $x=0$.

- A complete solution gives the mathematical result and explains what it means. Check that it is possible in the stated context.
-

Use this in your course

- 7357 · A-level · G. Match the target tier and specification before assigning extensions.
- Give the method before the final answer, and use the paper's calculator and formula rules. Review a wrong answer by locating the first invalid step.

Key idea

A small-angle limit used with angles measured in radians. Choose the relationship, show the method, check its assumptions and interpret the result.

G · Gradient graphs, concavity and inflection tests

Two curves can both have zero slope, yet one turns and the other keeps rising. Which gradient graph tells the difference?

- Two curves can both have zero slope, yet one turns and the other keeps rising. Which gradient graph tells the difference?
 - This lesson studies **point of inflection**: A point where a continuous curve changes its direction of concavity.
-

Choose the mathematical structure

- The sign of f' identifies increasing or decreasing intervals. Zeros of f' are stationary candidates. A change $+$ to $-$ gives a local maximum; $-$ to $+$ gives a local minimum. The second derivative f'' measures how the gradient changes: positive means an increasing gradient and a convex (concave-up) curve; negative means a decreasing gradient and a concave (concave-down) curve. An inflection needs a change of concavity, not only $f''=0$.
 - State the allowed inputs and units before calculating. An equation should express the relationship, not just record a calculator entry.
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Work through a checked case

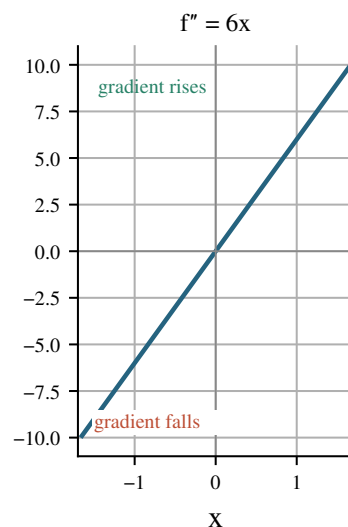
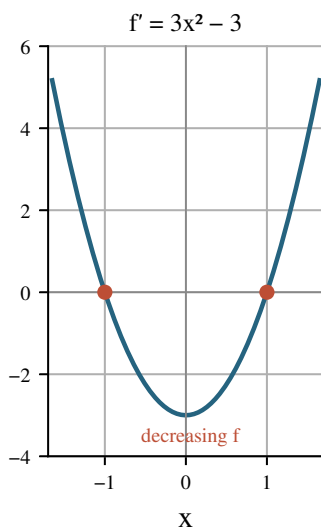
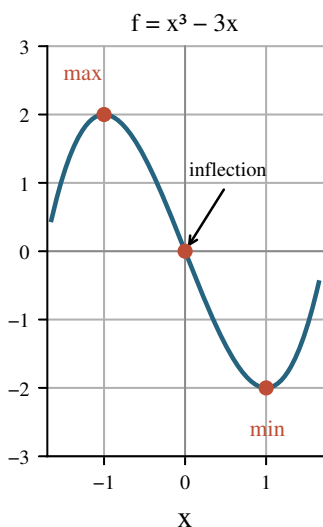
$$f = x^3 - 3x, \quad f' = 3x^2 - 3, \quad f'' = 6x$$

- Check the result against the starting quantities. Substitute into the original relation, or compare the graph and numerical answer where appropriate.

Worked example

For $f=x^3-3x$, $f'=3x^2-3$ and $f''=6x$. The gradient graph is above zero for $x<-1$ or $x>1$ and below for $-1<x<1$. Thus $(-1,2)$ is a local maximum and $(1,-2)$ a local minimum. The second derivative changes from negative to positive at $x=0$, so $(0,0)$ is an inflection with nonzero gradient -3 . For $g=x^3$, $g'=3x^2$ stays positive on both sides of zero: $(0,0)$ is a stationary inflection, not a turning point. For $q=x^4$, $q'=4x^3$ changes $-$ to $+$ and $q''=12x^2$ is nonnegative: $(0,0)$ is a minimum, but not an inflection even though $q''(0)=0$.

Sign changes distinguish extrema from inflections



Test a tempting shortcut

- Do not confuse the height of f with its gradient f' . A decreasing curve can have an increasing gradient while staying below zero in the gradient graph. The test $f''(a)=0$ is inconclusive; inspect signs on each side. A point of inflection need not

be stationary, and a stationary point need not be an extremum. Endpoint extrema on a restricted interval also require checking the endpoints.

- When a shortcut fails, identify the assumption it breaks. Keep an exact value until the requested final rounding.

Watch out

Every point at which the second derivative is zero is a point of inflection. This claim is false. Explain which definition or assumption it violates.

Interpret a new situation

- Align the x-axes of the original curve and gradient graph. Mark zeros of f' and build sign intervals before naming extrema. Read concavity from whether the gradient graph rises or falls. For a twice-differentiable polynomial, use an actual sign change of f'' to establish an inflection and evaluate the original function for its coordinates.
 - A complete solution gives the mathematical result and explains what it means. Check that it is possible in the stated context.
-

Use this in your course

- 7357 · A-level · G. Match the target tier and specification before assigning extensions.
- Give the method before the final answer, and use the paper's calculator and formula rules. Review a wrong answer by locating the first invalid step.

Key idea

A point where a continuous curve changes its direction of concavity. Choose the relationship, show the method, check its assumptions and interpret the result.

G · Rational-power derivatives and valid domains

A square-root curve has a finite height at zero but an unbounded slope nearby. Why must a derivative have its own domain?

- A square-root curve has a finite height at zero but an unbounded slope nearby. Why must a derivative have its own domain?
 - This lesson studies **rational power** : An exponent that can be written as a fraction of integers, interpreted with a valid real root.
-

Choose the mathematical structure

- For a rational exponent p , $d(x^p)/dx = p x^{p-1}$ wherever the real function is defined and differentiable. Multiply by constant coefficients; differentiate sums and differences term by term. A constant has derivative zero. Rewrite radicals and reciprocals as powers before subtracting one from the exponent. Positive x is valid for all rational powers; negative and zero inputs need separate checks.
 - State the allowed inputs and units before calculating. An equation should express the relationship, not just record a calculator entry.
-

Work through a checked case

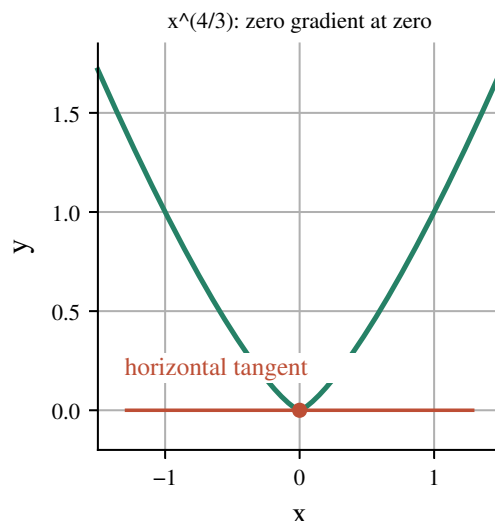
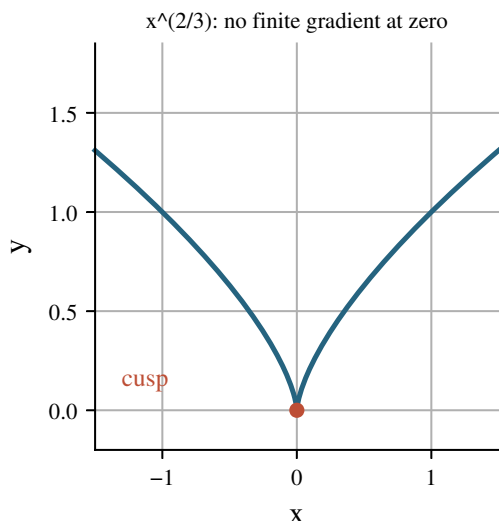
$$\frac{d}{dx} x^p = p x^{p-1} \quad \text{on its differentiable real domain}$$

- Check the result against the starting quantities. Substitute into the original relation, or compare the graph and numerical answer where appropriate.

Worked example

For $x > 0$, $f = 3\sqrt{x} - 4/x + 1/\sqrt{x}$ gives $f' = (3/2)x^{-1/2} + 4x^{-2} - (1/2)x^{-3/2}$. At $x=4$ this is $3/4 + 1/4 - 1/16 = 15/16$. For $q = x^{2/3} = (\text{cube root } x)^2$, negative x is allowed; $q' = (2/3)x^{-1/3}$ for $x \neq 0$, so $q'(-8) = -1/3$. At zero q has a cusp: the difference quotient $|h|^{2/3}/h$ diverges with opposite signs. For $r = x^{4/3}$, $[r(h) - r(0)]/h = \text{cube root } h$ tends to zero, so $r'(0) = 0$. Thus a fractional power does not automatically exclude zero from the derivative.

A finite function value does not decide differentiability



Test a tempting shortcut

- The original root domain and derivative domain can differ. $\sqrt{x}$ is defined at $x=0$, but $1/(2\sqrt{x})$ has no finite value there; $1/\sqrt{x}$ is not defined there at all. A real negative base needs a reduced rational exponent with an odd denominator. For $x<0$, $x^{2/3}$ means the square of its real cube root, not an invalid real logarithm calculation. Do not differentiate a reciprocal by keeping the same exponent or dropping the minus sign.
- When a shortcut fails, identify the assumption it breaks. Keep an exact value until the requested final rounding.

Watch out

Every rational-power function defined at zero has a finite derivative there. This claim is false. Explain which definition or assumption it violates.

Interpret a new situation

- State the original domain first, rewrite each term, multiply its coefficient by the exponent and subtract one from that exponent. Return to radicals if requested. Check boundary inputs with the original difference quotient when the formal deriva-

tive is undefined or inconclusive. Keep exact fractional coefficients until the final evaluation.

- A complete solution gives the mathematical result and explains what it means. Check that it is possible in the stated context.
-

Use this in your course

- 7357 · A-level · G. Match the target tier and specification before assigning extensions.
- Give the method before the final answer, and use the paper's calculator and formula rules. Review a wrong answer by locating the first invalid step.

Key idea

An exponent that can be written as a fraction of integers, interpreted with a valid real root. Choose the relationship, show the method, check its assumptions and interpret the result.

G · Exponential, logarithmic and trigonometric derivatives

A wave with twice the input frequency has a steeper initial slope. Which factor must appear in its derivative?

- A wave with twice the input frequency has a steeper initial slope. Which factor must appear in its derivative?
 - This lesson studies **inner rate** : The derivative of the changing input inside a composite function.
-

Choose the mathematical structure

- For constant k and $a > 0$: $(e^{kx})' = k e^{kx}$, $(a^{kx})' = k \ln(a) a^{kx}$, $(\sin kx)' = k \cos kx$, $(\cos kx)' = -k \sin kx$ and $(\tan kx)' = k \sec^2(kx)$, where $\sec u = 1/\cos u$. Trigonometric inputs are radians. $(\ln x)' = 1/x$ for $x > 0$. Add differentiated terms and retain their coefficients.

- State the allowed inputs and units before calculating. An equation should express the relationship, not just record a calculator entry.

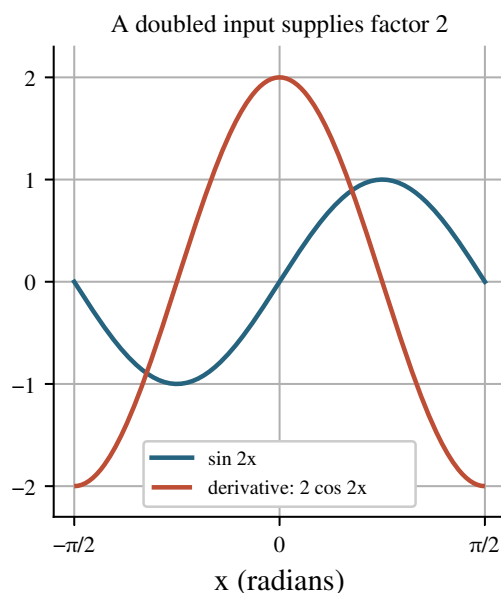
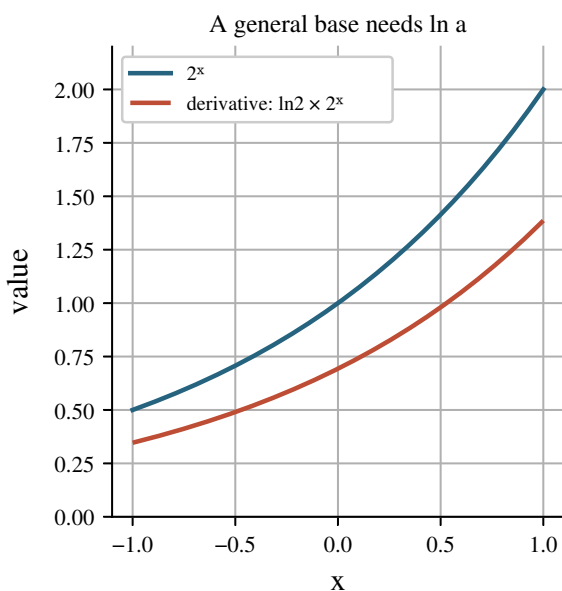
Work through a checked case

$$(a^{kx})' = k \ln(a) a^{kx}, \quad (\tan kx)' = k \sec^2(kx), \quad (\ln x)' = \frac{1}{x}$$

- Check the result against the starting quantities. Substitute into the original relation, or compare the graph and numerical answer where appropriate.

Worked example

For $x > 0$ and $\cos(x/2) \neq 0$, let $f = 2e^{(3x)} + 3^{(2x)} - 4\ln x + \sin 2x - 2\cos 3x + \tan(x/2)$. Then $f' = 6e^{(3x)} + 2\ln 3 \cdot 3^{(2x)} - 4/x + 2\cos 2x + 6\sin 3x + (1/2)\sec^2(x/2)$. The term $-2\cos 3x$ gives a positive $6\sin 3x$ because its coefficient and derivative sign are both negative. For $g = 5^{(-2x)}$, $g' = -2\ln 5 \cdot 5^{(-2x)}$, so $g'(0) = -2\ln 5$. For $h = \sin 2x - 2\cos 3x + \tan(x/2)$, $h'(0) = 2 + 0 + 1/2 = 5/2$. For $p = 2e^{(3x)}$, $p'(0) = 6$. Base $a = 1$ gives the constant 1 and derivative zero; $k = 0$ also produces constant functions.



Test a tempting shortcut

- The exponential base is constant but its exponent changes: $a^{(kx)}$ needs $\ln a$ as well as k . The derivative of $\ln x$ is $1/x$, not $\ln x/x$. Tangent is undefined where $\cos(kx)=0$ and its derivative retains those exclusions. Derivative formulas in radians need $\pi/180$ factors for degree variables. A negative k changes the inner-rate sign; it must not disappear.
- When a shortcut fails, identify the assumption it breaks. Keep an exact value until the requested final rounding.

Watch out

The derivative of $a^{(kx)}$ is $k a^{(kx)}$ for every positive base a . This claim is false. Explain which definition or assumption it violates.

Interpret a new situation

- Identify the function family, preserve its coefficient, differentiate the outer function and multiply by k . List log and tangent domain exclusions before evaluating at a point. Check the sign against an increasing/declining exponential or the local wave direction. Derivatives of linear combinations are obtained by addition, not by multiplying terms.
 - A complete solution gives the mathematical result and explains what it means. Check that it is possible in the stated context.
-

Use this in your course

- 7357 · A-level · G. Match the target tier and specification before assigning extensions.
- Give the method before the final answer, and use the paper's calculator and formula rules. Review a wrong answer by locating the first invalid step.

Key idea

The derivative of the changing input inside a composite function. Choose the relationship, show the method, check its assumptions and interpret the result.

G · Product and quotient rules with retained exclusions

Both factors in a changing area can vary. Why does multiplying their separate derivatives miss most of the rate?

- Both factors in a changing area can vary. Why does multiplying their separate derivatives miss most of the rate?
 - This lesson studies **quotient rule** : The rule for differentiating a ratio of two changing functions when the denominator is nonzero.
-

Choose the mathematical structure

- For differentiable u, v , $(uv)' = u'v + uv'$. For $v \neq 0$, $(u/v)' = (u'v - uv')/v^2$. The product rule has two contributions: change in either factor while the other provides its current value. The quotient rule uses the derivative of the numerator times the denominator, minus the numerator times the derivative of the denominator, all over v^2 .
 - State the allowed inputs and units before calculating. An equation should express the relationship, not just record a calculator entry.
-

Work through a checked case

$$(uv)' = u'v + uv', \quad \left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$$

- Check the result against the starting quantities. Substitute into the original relation, or compare the graph and numerical answer where appropriate.

Worked example

For $y = x^2 e^x$, $y' = 2xe^x + x^2 e^x = e^x(x^2 + 2x)$, so $y'(1) = 3e$. For $z = (x^2 + 1)/(x - 1)$, $x \neq 1$, $z' = [2x(x - 1) - (x^2 + 1)]/(x - 1)^2 = (x^2 - 2x - 1)/(x - 1)^2$; at $x = 2$ it is -1 . Polynomial division gives $z = x + 1 + 2/(x - 1)$, hence $z' = 1 - 2/(x - 1)^2$, an independent check. For $w = (x^2 - 1)/(x - 1)$, $x \neq 1$, cancellation gives $w = x + 1$ and $w' = 1$ only on that original domain. The missing point at $x = 1$ is not restored; its continuous extension would be a different function.

Two product contributions; quotient order and domain matter

Product: $(x^2 e^x)' = 2x e^x + x^2 e^x$

Quotient: $z = (x^2 + 1)/(x - 1), x \neq 1$

$$z' = [2x(x - 1) - (x^2 + 1)]/(x - 1)^2$$

Check: $z = x + 1 + 2/(x - 1)$

Hence $z' = 1 - 2/(x - 1)^2$, still $x \neq 1$

Test a tempting shortcut

- Multiplying $u'v'$ is not the derivative of uv . Reversing the quotient numerator flips its sign. The squared denominator does not eliminate the original pole or hole. Simplifying before differentiating can help, but it must preserve exclusions. A stationary quotient point must also lie in the original domain.
- When a shortcut fails, identify the assumption it breaks. Keep an exact value until the requested final rounding.

Watch out

Cancelling a factor from a quotient automatically restores every excluded input. This claim is false. Explain which definition or assumption it violates.

Interpret a new situation

- Name u, v and write u', v' before substitution. Keep brackets around both quotient numerator terms until expansion is complete. Factor a product derivative only after collecting both contributions. Compare an expanded or divided form when available and substitute into the original domain before evaluating or solving $y' = 0$.
- A complete solution gives the mathematical result and explains what it means. Check that it is possible in the stated context.

Use this in your course

- 7357 · A-level · G. Match the target tier and specification before assigning extensions.
- Give the method before the final answer, and use the paper's calculator and formula rules. Review a wrong answer by locating the first invalid step.

Key idea

The rule for differentiating a ratio of two changing functions when the denominator is nonzero. Choose the relationship, show the method, check its assumptions and interpret the result.

G · Chain rules for nested functions and restricted inputs

A sensor converts a changing input through several formulas. How do we keep every link in its rate of change?

- A sensor converts a changing input through several formulas. How do we keep every link in its rate of change?
- This lesson studies **composite derivative** : The outer derivative evaluated at the inner output, multiplied by the derivative of that inner input.

Choose the mathematical structure

- For $y=f(u)$ and $u=g(x)$, $dy/dx=f'(g(x))g'(x)$. Evaluate the outer derivative at the actual inner expression. For several nested functions, multiply every inner-rate factor. Domain restrictions come from each stage; the differentiated expression cannot restore a forbidden original input.
- State the allowed inputs and units before calculating. An equation should express the relationship, not just record a calculator entry.

Work through a checked case

$$\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx}, \quad (e^{x^2})' = 2xe^{x^2}$$

- Check the result against the starting quantities. Substitute into the original relation, or compare the graph and numerical answer where appropriate.

Worked example

For $y=(3x-1)^4$, set $u=3x-1$: $y'=4u^3 \times 3=12(3x-1)^3$, giving $y'(0)=-12$. For $z=e^{(x^2)}$, $z'=e^{(x^2)} \times 2x$; $z'(0)=0$ although the outer exponential derivative is never zero. For $w=\ln(2x+1)$, $x>-1/2$, $w'=2/(2x+1)$; $w'(1)=2/3$. For $r=\sin((2x+1)^2)$, first set $u=(2x+1)^2$: $r'=\cos((2x+1)^2) \times 2(2x+1) \times 2=4(2x+1)\cos((2x+1)^2)$. For $s=\sqrt{5-x^2}$, the original domain is $-\sqrt{5} \leq x \leq \sqrt{5}$, but $s'=-x/\sqrt{5-x^2}$ is finite only in the open interval.

Every nested input contributes its changing rate

$$y = \sin((2x + 1)^2)$$

$$\text{Outer derivative: } \cos((2x + 1)^2)$$

$$\text{Middle rate: } 2(2x + 1)$$

$$\text{Inner rate: } 2$$

$$\text{Multiply: } y' = 4(2x + 1) \cos((2x + 1)^2)$$

Test a tempting shortcut

- Do not stop after differentiating the outer power or exponential. For $\sin(u^2)$, the cosine input stays u^2 ; it is not replaced by $2u$. Missing one link in a nested derivative loses a factor. A zero inner derivative can give zero total derivative, so dividing by it without checking can lose valid cases. A square root can exist at an endpoint without a finite derivative there.

- When a shortcut fails, identify the assumption it breaks. Keep an exact value until the requested final rounding.

Watch out

The derivative of $e^{\wedge}(x^2)$ is $e^{\wedge}(x^2)$ without an inner-rate factor. This claim is false. Explain which definition or assumption it violates.

Interpret a new situation

- Work from the outside inward and record each input change. Multiply the factors, then simplify. For a short polynomial, expansion gives an independent alternative check. Retain every log/root restriction and interpret whether a zero rate comes from the outer function, the inner input or both.
 - A complete solution gives the mathematical result and explains what it means. Check that it is possible in the stated context.
-

Use this in your course

- 7357 · A-level · G. Match the target tier and specification before assigning extensions.
- Give the method before the final answer, and use the paper's calculator and formula rules. Review a wrong answer by locating the first invalid step.

Key idea

The outer derivative evaluated at the inner output, multiplied by the derivative of that inner input. Choose the relationship, show the method, check its assumptions and interpret the result.

G · Inverse derivatives at corresponding points

A calibration converts length to a reading. How do we reverse its local rate without taking the reciprocal of its output?

- A calibration converts length to a reading. How do we reverse its local rate without taking the reciprocal of its output?
-

- This lesson studies **inverse derivative** : The rate of the inverse output with respect to its input, evaluated at the corresponding original point.

Choose the mathematical structure

- Let f be one-to-one on a stated interval and $g=f^{-1}$. The identity $f(g(v))=v$ gives $f'(g(v))g'(v)=1$. When the derivatives exist and $f'(g(v))\neq 0$, $g'(v)=1/f'(g(v))$. The inverse is evaluated at the original output v , not automatically at the same original input x . The reciprocal $1/f(x)$ is a different function.
- State the allowed inputs and units before calculating. An equation should express the relationship, not just record a calculator entry.

Work through a checked case

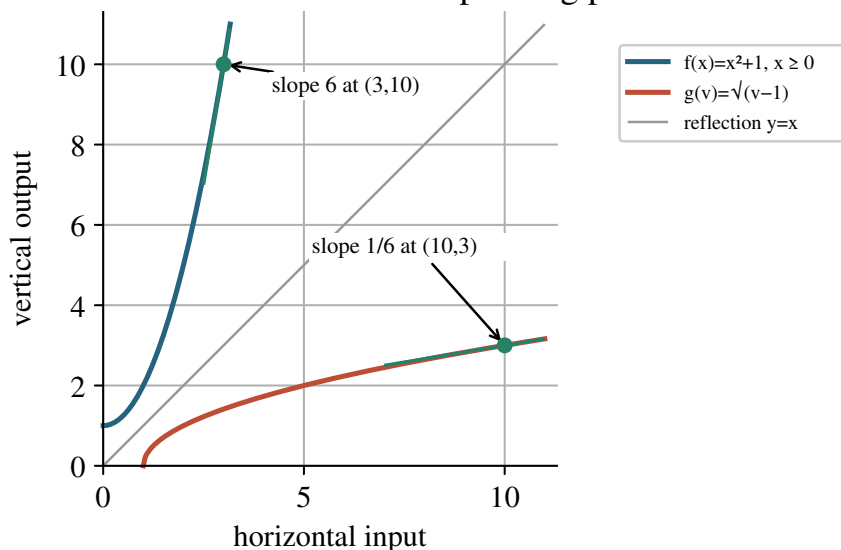
$$(f^{-1})'(v) = \frac{1}{f'(f^{-1}(v))}$$

- Check the result against the starting quantities. Substitute into the original relation, or compare the graph and numerical answer where appropriate.

Worked example

For $f(x)=x^2+1$ on $x\geq 0$, $g(v)=\sqrt{v-1}$, $v\geq 1$. At $v=10$, $g(10)=3$ and $f'(3)=6$, so $g'(10)=1/6$. On $x\leq 0$ the inverse branch is $-\sqrt{v-1}$, with $g(10)=-3$ and $g'(10)=-1/6$. For $f=e^x$, $g=\ln v$: $f'(g(v))=e^{\ln v}=v$, hence $g'=1/v$ for $v>0$. For $f=\sin x$ on $-\pi/2\leq x\leq \pi/2$, $g=\arcsin v$. Within $-1<v<1$, $\cos(g(v))=\sqrt{1-v^2}>0$, so $g'=1/\sqrt{1-v^2}$. The nonnegative cosine follows from the restricted sine branch; it is not an arbitrary sign choice.

Inverse rates use reflected corresponding points



Test a tempting shortcut

- Do not replace the corresponding-point denominator $f'(g(v))$ by $f'(v)$. One-to-one behaviour alone does not ensure a finite inverse gradient: x^3 is invertible, but its derivative is zero at zero and cube root has no finite derivative there. Root/arcsine inverse endpoints can also lack finite gradients. Different inverse branches can give different signs.
- When a shortcut fails, identify the assumption it breaks. Keep an exact value until the requested final rounding.

Watch out

The derivative of an inverse is always $1/f'(v)$, evaluated at the inverse input v . This claim is false. Explain which definition or assumption it violates.

Interpret a new situation

- Restrict the original domain, find the inverse input/output pair, differentiate f and evaluate its derivative at the recovered original input. Then take the reciprocal only if that finite gradient is nonzero. Swap horizontal and vertical units for the inverse rate. Check the result by differentiating an explicit inverse when one is available.

- A complete solution gives the mathematical result and explains what it means. Check that it is possible in the stated context.
-

Use this in your course

- 7357 · A-level · G. Match the target tier and specification before assigning extensions.
- Give the method before the final answer, and use the paper's calculator and formula rules. Review a wrong answer by locating the first invalid step.

Key idea

The rate of the inverse output with respect to its input, evaluated at the corresponding original point. Choose the relationship, show the method, check its assumptions and interpret the result.

G · Connected rates, geometry and time units

A spherical object gains volume at a steady rate. Does its radius increase at a steady rate too?

- A spherical object gains volume at a steady rate. Does its radius increase at a steady rate too?
 - This lesson studies **connected rates** : Rates linked by a relationship between quantities changing with the same time variable.
-

Choose the mathematical structure

- If $y=f(x)$ and x varies with time t , $dy/dt=(dy/dx)(dx/dt)$. Derive a relation between changing quantities, differentiate with respect to time, then substitute the measured values at that instant. Keep the time variable and each rate's sign and units. For a sphere $V=(4/3)\pi r^3$ and $A=4\pi r^2$, so $dV/dt=4\pi r^2 dr/dt$ and $dA/dt=8\pi r dr/dt$.
 - State the allowed inputs and units before calculating. An equation should express the relationship, not just record a calculator entry.
-

Work through a checked case

$$\frac{dy}{dt} = \frac{dy}{dx} \frac{dx}{dt}, \quad \frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$$

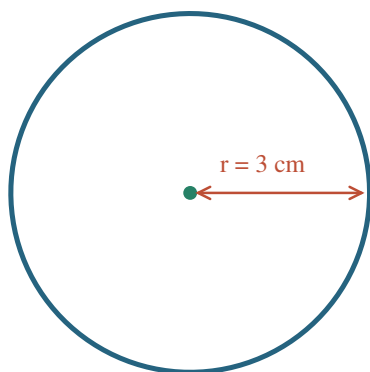
- Check the result against the starting quantities. Substitute into the original relation, or compare the graph and numerical answer where appropriate.

Worked example

A sphere has radius 3 cm and gains volume at $18\pi \text{ cm}^3/\text{s}$. Its instantaneous radius rate is $18\pi/(4\pi \times 9) = 1/2 \text{ cm/s}$, and surface-area rate is $8\pi \times 3 \times (1/2) = 12\pi \text{ cm}^2/\text{s}$. This is a local rate; as r grows, the same volume rate gives a smaller radius rate. In a conical vessel whose full radius is half its full height, similarity gives $r = h/2$ at every water level. Thus $V = \pi h^3/12$ and $dV/dt = (\pi h^2/4) dh/dt$. At $h = 4 \text{ cm}$ and inflow $8\pi \text{ cm}^3/\text{s}$, $dh/dt = 2 \text{ cm/s}$. For a fixed 5 m ladder, $x^2 + y^2 = 25$; at $(x, y) = (3, 4) \text{ m}$ with $dx/dt = 0.2 \text{ m/s}$, $dy/dt = -(3/4) \times 0.2 = -0.15 \text{ m/s}$: the top descends.

Sphere: current radius sets the rate

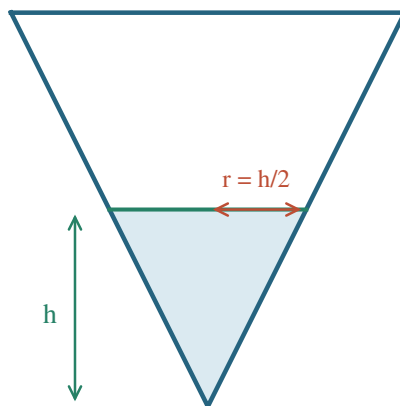
$$dV/dt = 18\pi \text{ cm}^3/\text{s}$$



$$dr/dt = 1/2 \text{ cm/s}$$

Cone: r changes together with h

fixed vessel ratio: $R/H = 1/2$



$$V = \pi h^3/12$$

Test a tempting shortcut

- Do not substitute the current radius or height before differentiating: that would falsely turn the relation into a constant. dV/dt is not dV/dr . A changing cone radius must follow the similarity relation, not stay fixed at its current value. Convert a per-minute inflow to a per-second rate before combining it with second-based data. Negative rate means decrease, not a negative length.

- When a shortcut fails, identify the assumption it breaks. Keep an exact value until the requested final rounding.

Watch out

Substitute a measured radius into the volume formula before differentiating it with respect to time. This claim is false. Explain which definition or assumption it violates.

Interpret a new situation

- Draw the geometry, identify constants and state the valid time/measurement units. Eliminate an extra changing variable using similarity or a constraint, then differentiate. Use the values for the same instant and describe the result's direction. Physical assumptions, such as a fixed ladder length or a true sphere, limit the model; rates need not remain constant after that instant.
 - A complete solution gives the mathematical result and explains what it means. Check that it is possible in the stated context.
-

Use this in your course

- 7357 · A-level · G. Match the target tier and specification before assigning extensions.
- Give the method before the final answer, and use the paper's calculator and formula rules. Review a wrong answer by locating the first invalid step.

Key idea

Rates linked by a relationship between quantities changing with the same time variable. Choose the relationship, show the method, check its assumptions and interpret the result.

G · Implicit first derivatives and tangent domains

A circle has an upper and lower branch. How can we find its tangent gradient without solving for a square root first?

- A circle has an upper and lower branch. How can we find its tangent gradient without solving for a square root first?
 - This lesson studies **implicit relation** : An equation relating x and y without isolating one as an explicit function of the other.
-

Choose the mathematical structure

- Treat y as a differentiable local function of x . Differentiate the whole relation with respect to x : a function of y brings a dy/dx factor by the chain rule. Collect terms containing dy/dx and solve when its coefficient is nonzero. If the differentiated relation becomes $A(x,y)+B(x,y)y'=0$, then $y'=-A/B$ where $B \neq 0$; the gradient belongs to a chosen local branch.
 - State the allowed inputs and units before calculating. An equation should express the relationship, not just record a calculator entry.
-

Work through a checked case

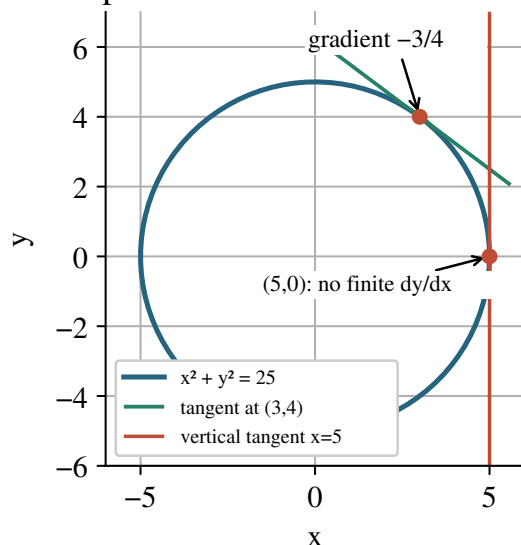
$$2x + 2y \frac{dy}{dx} = 0, \quad \frac{dy}{dx} = -\frac{x}{y} \quad (y \neq 0)$$

- Check the result against the starting quantities. Substitute into the original relation, or compare the graph and numerical answer where appropriate.

Worked example

For $x^2+y^2=25$, differentiate to $2x+2y y'=0$, hence $y'=-x/y$ for $y \neq 0$. At $(3,4)$, the tangent gradient is $-3/4$ and $y-4=-(3/4)(x-3)$; the normal gradient is $4/3$. At $(5,0)$ the tangent is vertical, $x=5$, and no finite dy/dx exists. For $x^2+xy+y^2=7$, differentiate the product xy to $y+x y'$. Then $2x+y+(x+2y)y'=0$, so $y'=- (2x+y)/(x+2y)$. At $(1,2)$, which satisfies the relation, the gradient is $-4/5$. For $e^y+x=3$, $e^y y'+1=0$, giving $y'=-e^{-y}$; at $(2,0)$ the gradient is -1 .

Implicit slopes describe a chosen local curve branch



Test a tempting shortcut

- Differentiating y^2 gives $2y y'$, not $2y$ or $2(y')^2$. The product xy needs both y and $x y'$. Verify a supplied point lies on the original relation before assigning its tangent. A zero denominator may indicate a vertical tangent or a singular point; the fraction alone cannot classify every case. For example $x^2 + y^2 = 0$ has only one isolated real point, not a smooth circle tangent.
- When a shortcut fails, identify the assumption it breaks. Keep an exact value until the requested final rounding.

Watch out

Differentiating y^2 with respect to x gives $2y$ without a dy/dx factor. This claim is false. Explain which definition or assumption it violates.

Interpret a new situation

- Differentiate every term, retain all chain/product factors, collect the y' terms and state the resulting denominator condition. Evaluate at the original point and use point-slope form for the tangent. A horizontal tangent has zero finite gradient and a vertical normal; a vertical tangent needs its own line equation. This lesson uses first derivatives only, matching G5.

- A complete solution gives the mathematical result and explains what it means. Check that it is possible in the stated context.
-

Use this in your course

- 7357 · A-level · G. Match the target tier and specification before assigning extensions.
- Give the method before the final answer, and use the paper's calculator and formula rules. Review a wrong answer by locating the first invalid step.

Key idea

An equation relating x and y without isolating one as an explicit function of the other. Choose the relationship, show the method, check its assumptions and interpret the result.

G · Parametric first derivatives and vertical tangents

Two coordinates depend on the same slider. How do their separate rates combine into the curve's slope?

- Two coordinates depend on the same slider. How do their separate rates combine into the curve's slope?
 - This lesson studies **parametric gradient** : The ratio of the y -rate to the x -rate along a parametrised curve, when the x -rate is nonzero.
-

Choose the mathematical structure

- For $x=x(t)$, $y=y(t)$, the chain rule gives $dy/dt=(dy/dx)(dx/dt)$, so $dy/dx=(dy/dt)/(dx/dt)$ when $dx/dt \neq 0$. Differentiate the two coordinates separately, then divide in y -over- x order. The point is obtained from the same parameter value. A parameter need not represent physical time.
 - State the allowed inputs and units before calculating. An equation should express the relationship, not just record a calculator entry.
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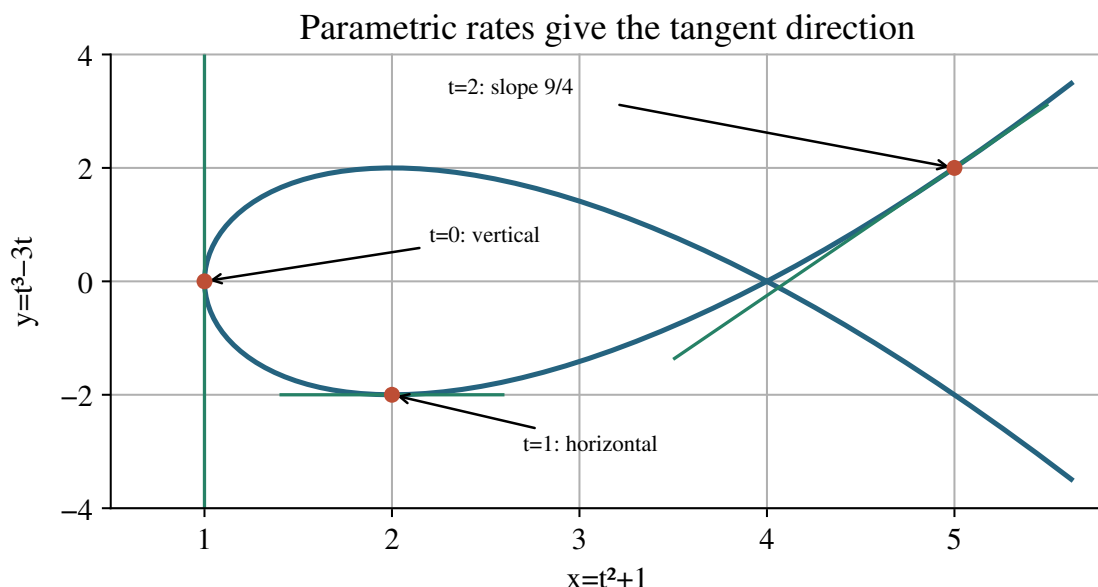
Work through a checked case

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} \quad (dx/dt \neq 0)$$

- Check the result against the starting quantities. Substitute into the original relation, or compare the graph and numerical answer where appropriate.

Worked example

For $x=t^2+1$, $y=t^3-3t$, $dx/dt=2t$ and $dy/dt=3t^2-3$. At $t=2$ the point is $(5,2)$ and gradient is $9/4$, so the tangent is $y-2=(9/4)(x-5)$. At $t=1$, the point $(2,-2)$ has horizontal tangent $y=-2$ because $dy/dt=0$ while $dx/dt=2$. At $t=0$, $(1,0)$ has vertical tangent $x=1$: $dx/dt=0$ but $dy/dt=-3 \neq 0$. A regular local parametrisation then changes y while x has zero instantaneous change. For $x=2\cos t, y=3\sin t$, at $t=\pi/4$ the point is $(\sqrt{2}, 3\sqrt{2}/2)$ and slope is $-3/2$. At $t=0$ the ellipse tangent is vertical, $x=2$.



Test a tempting shortcut

- The ratio is dy/dt divided by dx/dt , not its reciprocal. If both rates vanish, $0/0$ is inconclusive: $x=t^3, y=t^3$ still traces $y=x$ with slope 1 through zero. If $dx/dt=0$

alone, inspect the local curve rather than report a finite numerical gradient. Repeated x values can correspond to different y values or branches; keep the stated parameter value.

- When a shortcut fails, identify the assumption it breaks. Keep an exact value until the requested final rounding.

Watch out

When both parametric coordinate rates are zero, the curve must have no tangent. This claim is false. Explain which definition or assumption it violates.

Interpret a new situation

- Find the coordinate pair, calculate both first rates, then test whether the ratio is valid. State horizontal/vertical tangent lines separately when appropriate and preserve the parameter range. Check by eliminating the parameter when that gives a simple local relation. G5 requires first derivatives only; a second parametric derivative is outside this lesson's stated scope.
 - A complete solution gives the mathematical result and explains what it means. Check that it is possible in the stated context.
-

Use this in your course

- 7357 · A-level · G. Match the target tier and specification before assigning extensions.
- Give the method before the final answer, and use the paper's calculator and formula rules. Review a wrong answer by locating the first invalid step.

Key idea

The ratio of the y -rate to the x -rate along a parametrised curve, when the x -rate is nonzero. Choose the relationship, show the method, check its assumptions and interpret the result.

G · Tangent and normal equations and gradient conditions

A road's tangent gives its local direction; a cross-road follows the normal. Why must both lines use the same curve point?

- A road's tangent gives its local direction; a cross-road follows the normal. Why must both lines use the same curve point?
 - This lesson studies **normal line** : A line through the curve point perpendicular to its local tangent.
-

Choose the mathematical structure

- At a differentiable point $(a, f(a))$, the tangent gradient is $m=f'(a)$ and its equation is $y-f(a)=m(x-a)$. For a finite nonzero m , the normal gradient is $-1/m$. Find points satisfying a given gradient condition by solving for $f'(x)$, then evaluate the original f . Parallel lines have equal gradients; finite nonzero perpendicular gradients multiply to -1 .
 - State the allowed inputs and units before calculating. An equation should express the relationship, not just record a calculator entry.
-

Work through a checked case

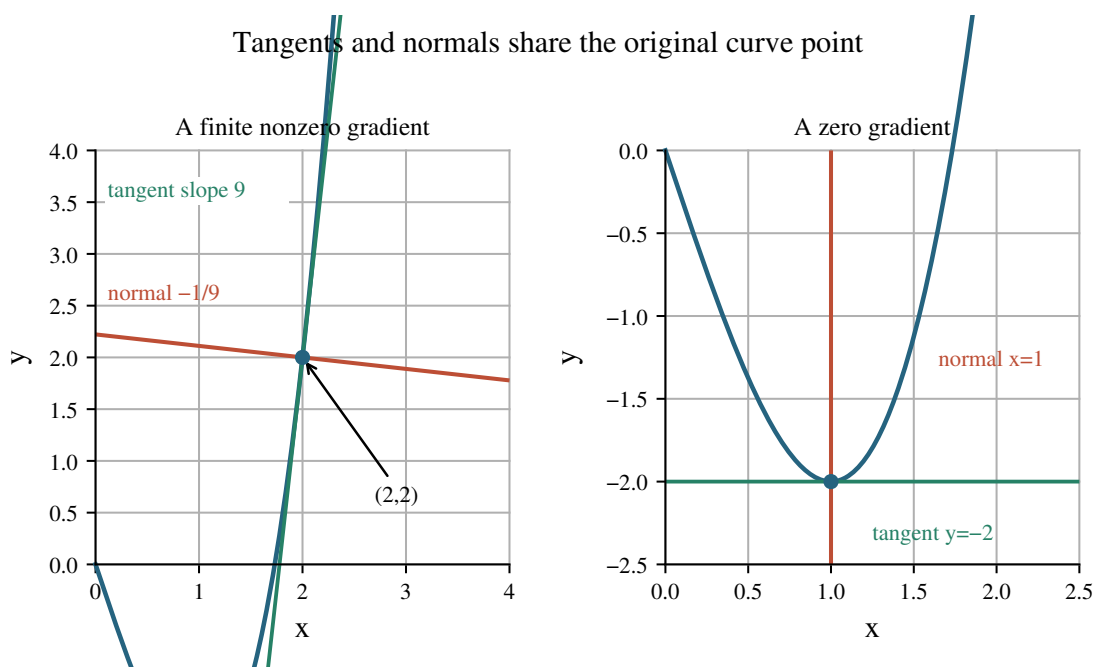
$$y - f(a) = f'(a)(x - a), \quad m_{\text{normal}} = -\frac{1}{f'(a)} \quad (f'(a) \neq 0)$$

- Check the result against the starting quantities. Substitute into the original relation, or compare the graph and numerical answer where appropriate.

Worked example

For $f=x^3-3x$ at $x=2$, $f(2)=2$ and $f'(2)=9$. The tangent is $y-2=9(x-2)$, or $y=9x-16$. The normal is $y-2=-(1/9)(x-2)$. Tangents parallel to $y=9x+4$ satisfy $3x^2-3=9$, giving $x=\pm 2$: points $(2,2)$ and $(-2,-2)$. The second tangent is $y+2=9(x+2)$. At $x=1$ the tangent is horizontal $y=-2$ and the normal is vertical $x=1$; no finite negative reciprocal of zero exists. For the circle at $(5,0)$, the tangent is vertical $x=5$ and the

normal horizontal $y=0$. The secant through $(-2,-2)$ and $(2,2)$ has gradient 1, unlike either local tangent gradient 9.



Test a tempting shortcut

- The derivative supplies a gradient, not the y-coordinate. Use the original curve for the point. The normal's reciprocal includes a minus sign and is unavailable for a horizontal tangent; give a vertical line instead. A tangent can meet the curve again elsewhere. A given line's intercept does not affect a parallel-gradient condition, but both tangent and normal must pass through the stated curve point.
- When a shortcut fails, identify the assumption it breaks. Keep an exact value until the requested final rounding.

Watch out

A normal to a horizontal tangent has finite gradient zero. This claim is false. Explain which definition or assumption it violates.

Interpret a new situation

- Find the point and gradient separately, then use point-slope form. If a gradient condition gives several inputs, retain every allowed point and line. State horizontal/vertical cases without inventing an infinite numerical gradient. Verify the line passes through the point and that a finite nonzero tangent/normal gradient pair multiplies to -1 .
 - A complete solution gives the mathematical result and explains what it means. Check that it is possible in the stated context.
-

Use this in your course

- 7357 · A-level · G. Match the target tier and specification before assigning extensions.
- Give the method before the final answer, and use the paper's calculator and formula rules. Review a wrong answer by locating the first invalid step.

Key idea

A line through the curve point perpendicular to its local tangent. Choose the relationship, show the method, check its assumptions and interpret the result.

G · Optimisation with constraints, boundaries and rejected roots

Cutting larger corner squares makes a box taller but its base smaller. Which cut actually gives the largest valid volume?

- Cutting larger corner squares makes a box taller but its base smaller. Which cut actually gives the largest valid volume?
 - This lesson studies **feasible domain** : The allowed input values after the original physical or mathematical constraints are applied.
-

Choose the mathematical structure

- Express the quantity to optimise in one variable, with its feasible domain. Solve a derivative-zero condition for interior candidates, reject invalid roots, and classify using derivative signs or a nonzero second derivative. A global maximum/minimum also needs boundary values or limiting behaviour; local classification alone does not settle a restricted-domain question.
 - State the allowed inputs and units before calculating. An equation should express the relationship, not just record a calculator entry.
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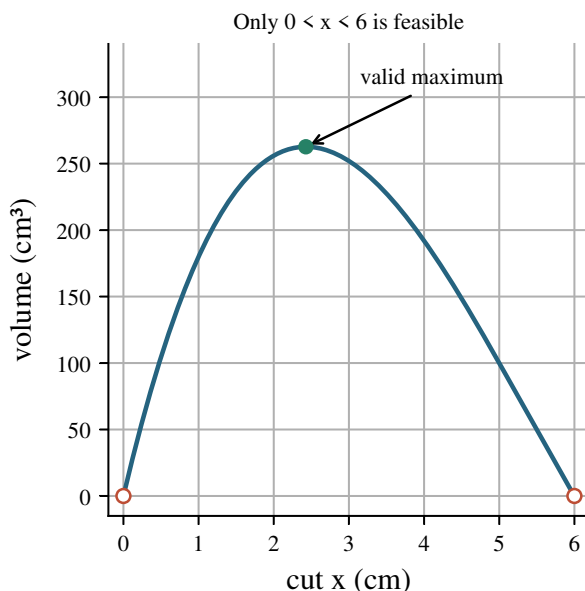
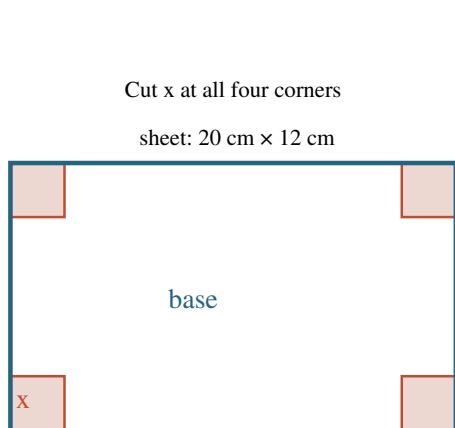
Work through a checked case

$$V = x(20 - 2x)(12 - 2x), \quad 0 < x < 6, \quad V' = 240 - 128x + 12x^2$$

- Check the result against the starting quantities. Substitute into the original relation, or compare the graph and numerical answer where appropriate.

Worked example

Cut squares of side x cm from a 20 cm by 12 cm sheet and fold an open box. Its dimensions are $x, 20 - 2x, 12 - 2x$, so $V = 240x - 64x^2 + 4x^3$ and $0 < x < 6$. $V' = 240 - 128x + 12x^2$ gives roots $x = (16 \pm 2\sqrt{19})/3$. The smaller root $\alpha \approx 2.4274$ lies in the domain; the larger ≈ 8.2393 makes the short base side negative and is rejected. V' is positive on $(0, \alpha)$ and negative on $(\alpha, 6)$, while V tends to zero at both ends. Thus α gives the global maximum. A separate rectangle with perimeter 24 cm has sides $x, 12 - x$, area $A = 12x - x^2$ and $0 < x < 12$; its maximum is 36 cm^2 at $x = 6$. For $f = (x - 1)^2$ on $[0, 3]$, the interior stationary value is the minimum 0, but the global maximum is the endpoint value $f(3) = 4$.



Test a tempting shortcut

- A stationary root outside the feasible domain is not an alternative design. A second-derivative test gives a local classification, not every global comparison. Open endpoints are limiting values, not permitted designs. For a closed interval check actual endpoint values. If dimensions must use a specified discrete step, compare the neighbouring allowed choices rather than reporting an unattainable continuous optimum.
- When a shortcut fails, identify the assumption it breaks. Keep an exact value until the requested final rounding.

Watch out

Every solution of $V'=0$ is a feasible global maximum of the box volume. This claim is false. Explain which definition or assumption it violates.

Interpret a new situation

- Write units and constraints before calculus, keep exact roots through classification, and give the requested final dimensions or quantity rather than only the input. State idealisations such as negligible material thickness and exact folds. If the

model has discrete cuts or measurement tolerance, explain how the final feasible choice and its comparison would change.

- A complete solution gives the mathematical result and explains what it means. Check that it is possible in the stated context.
-

Use this in your course

- 7357 · A-level · G. Match the target tier and specification before assigning extensions.
- Give the method before the final answer, and use the paper's calculator and formula rules. Review a wrong answer by locating the first invalid step.

Key idea

The allowed input values after the original physical or mathematical constraints are applied. Choose the relationship, show the method, check its assumptions and interpret the result.

G · Constructing differential equations from rate laws

A population grows in proportion to its current size while a fixed number leaves each day. Which parts belong in the rate equation?

- A population grows in proportion to its current size while a fixed number leaves each day. Which parts belong in the rate equation?
 - This lesson studies **differential equation** : A relationship involving an unknown function and one or more of its derivatives.
-

Choose the mathematical structure

- Choose independent and dependent variables and convert the stated rate law into a derivative. Proportional to y means ky ; proportional to y^2 means ky^2 . A fixed inflow minus proportional loss gives $dy/dt = a - ky$; a rate proportional to the gap from a target A gives $dy/dt = k(A - y)$. An initial condition fixes a function value,

not a derivative formula. Constants must have units making both sides the same kind of rate.

- State the allowed inputs and units before calculating. An equation should express the relationship, not just record a calculator entry.

Work through a checked case

$$\frac{dP}{dt} = 0.04P - 10, \quad P(0) = 500$$

- Check the result against the starting quantities. Substitute into the original relation, or compare the graph and numerical answer where appropriate.

Worked example

For an illustrative population P , t in days, births occur at $0.04P$ per day and 10 individuals leave per day. The model is $dP/dt = 0.04P - 10$, $P(0) = 500$. Initially the rate is 10 individuals/day; $P = 250$ is an equilibrium, and below it the same equation predicts decline. The model must stop or be refined if it predicts a negative population. For a velocity v in m/s with t in seconds, acceleration $6 - 0.2v$ gives $dv/dt = 6 - 0.2v$, $v(0) = 0$: initial acceleration is 6 m/s^2 and equilibrium speed 30 m/s . In a hypothetical demand model, quantity Q falls at a price-rate proportional to Q . At price $p = 12$, $Q = 200$ and $dQ/dp = -10$, so $dQ/dp = -0.05Q$ with $Q(12) = 200$; the independent variable is price, not time. In pure maths, slope $2xy$ with $y(1) = 3$ gives $dy/dx = 2xy$ and initial slope 6.

The rate equation and initial condition do different jobs

Changing population P ; independent variable t (days)

Birth rate $0.04P$ – fixed departures 10

$$dP/dt = 0.04P - 10; P(0) = 500$$

$$\text{Initial rate: } 0.04 \times 500 - 10 = 10 \text{ per day}$$

$$\text{Equilibrium: } 0.04P - 10 = 0 \rightarrow P = 250$$

Test a tempting shortcut

- Proportional rate is not constant absolute change. Fixed departures need a subtraction term, not a second multiplier of P . Do not replace a changing P with its initial value throughout the equation. An equilibrium solves the rate-equals-zero condition; it need not equal the initial value or be approached from every start. A price derivative and a time derivative are different quantities with different units.
- When a shortcut fails, identify the assumption it breaks. Keep an exact value until the requested final rounding.

Watch out

A statement that the rate is proportional to the current amount means the same absolute change at all times. This claim is false. Explain which definition or assumption it violates.

Interpret a new situation

- Define each variable, translate every inflow/outflow or proportional term with its sign, state constant assumptions and attach the initial condition separately. Evaluate the initial rate as a check and find any equilibrium before interpreting the model. These are illustrative constructions, not compulsory context lists: G6 examples are optional. Solving the resulting equation is a separate integration task.
 - A complete solution gives the mathematical result and explains what it means. Check that it is possible in the stated context.
-

Use this in your course

- 7357 · A-level · G. Match the target tier and specification before assigning extensions.
- Give the method before the final answer, and use the paper's calculator and formula rules. Review a wrong answer by locating the first invalid step.

Key idea

A relationship involving an unknown function and one or more of its derivatives. Choose the relationship, show the method, check its assumptions and interpret the

result.

H · Integrals, area and accumulation

How much change has accumulated?

- A velocity graph tells us motion at an instant. How can we recover the displacement accumulated over time?
 - This lesson studies **antiderivative** : A function whose derivative equals the given integrand.
-

Choose the mathematical structure

- For $n \neq -1$, the integral of ax^n is $ax^{(n+1)}/(n+1)+C$. A definite integral is signed accumulation. Split at sign changes when total area or distance is required.
 - State the allowed inputs and units before calculating. An equation should express the relationship, not just record a calculator entry.
-

Work through a checked case

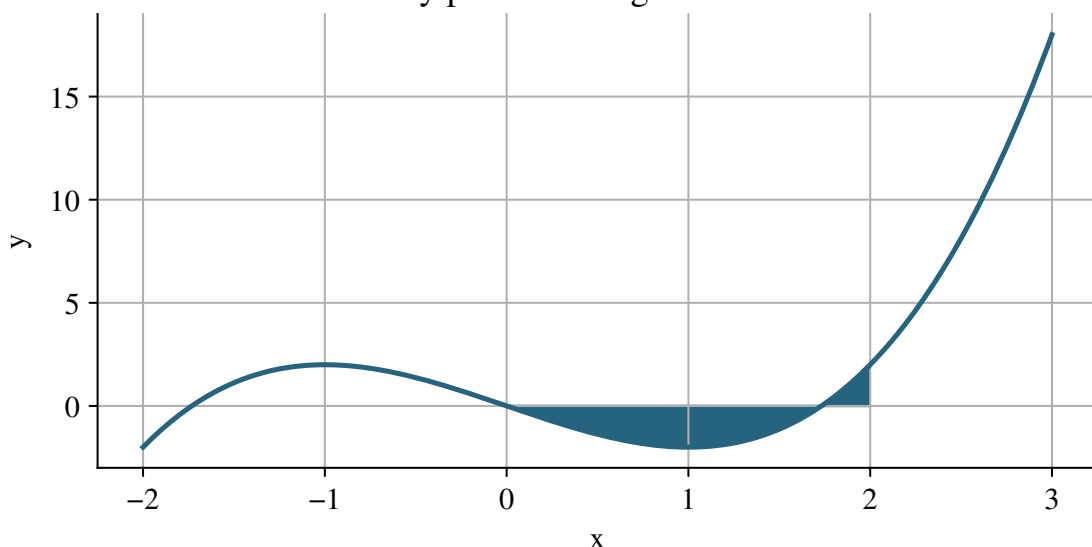
$$\int ax^n dx = \frac{ax^{n+1}}{n+1} + C \quad (n \neq -1)$$

- Check the result against the starting quantities. Substitute into the original relation, or compare the graph and numerical answer where appropriate.

Worked example

For $v(t)=3t^2-3$ over $0 \leq t \leq 2$, displacement= $[t^3-3t]_0^2=2$. Since v changes sign at $t=1$, distance= $-[t^3-3t]_0^1+[t^3-3t]_1^2=2+4=6$. The constants cancel only for a definite integral.

Stationary points and signed accumulation



Test a tempting shortcut

- The integral of $1/x$ is $\ln|x|+C$, not the power rule with $n=-1$. Signed area can be zero even when the total enclosed area is positive.
- When a shortcut fails, identify the assumption it breaks. Keep an exact value until the requested final rounding.

Watch out

A definite integral always equals the total positive area, even when the graph crosses the axis. This claim is false. Explain which definition or assumption it violates.

Interpret a new situation

- Identify what the integral means and include the correct units. A rate measured per second integrates to the underlying quantity, not to another rate. Differentiate an antiderivative to check it.
- A complete solution gives the mathematical result and explains what it means. Check that it is possible in the stated context.

Use this in your course

- 7357 · A-level · H. Match the target tier and specification before assigning extensions.
- Give the method before the final answer, and use the paper's calculator and formula rules. Review a wrong answer by locating the first invalid step.

Key idea

A function whose derivative equals the given integrand. Choose the relationship, show the method, check its assumptions and interpret the result.

H · Substitution, parts and partial fractions

Can we integrate the two factors separately?

- A simple-looking product such as $x e^x$ cannot be integrated by separately integrating each factor.
 - This lesson studies **integration by parts** : An integration method based on the derivative of a product.
-

Choose the mathematical structure

- Use substitution when an inner derivative appears as a factor. By parts, $\int u v' = uv - \int u' v$. For a rational function, divide first if needed, then decompose into partial fractions.
 - State the allowed inputs and units before calculating. An equation should express the relationship, not just record a calculator entry.
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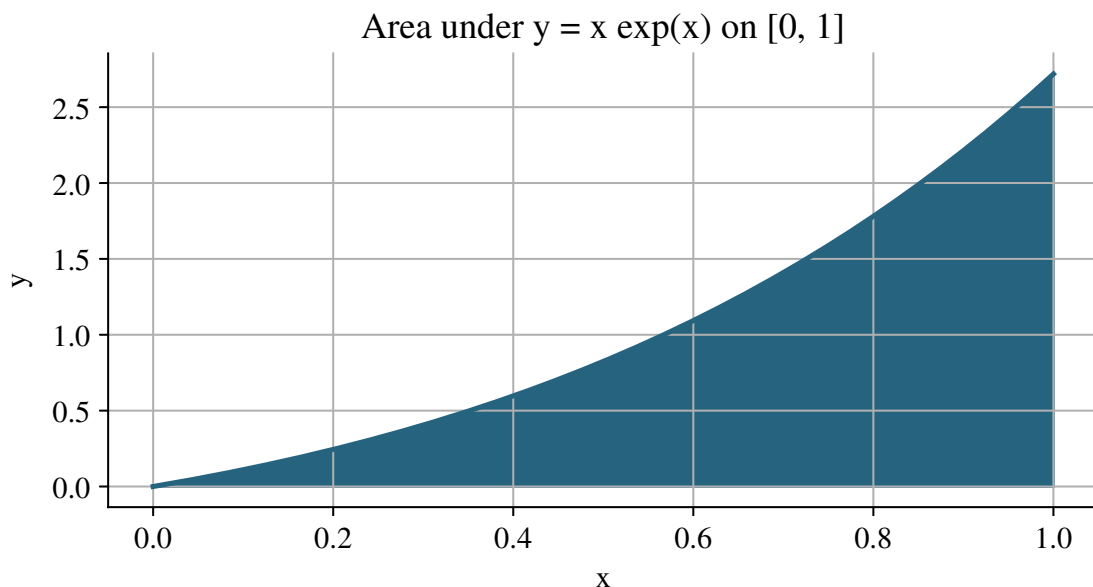
Work through a checked case

$$\int uv' dx = uv - \int u'v dx$$

- Check the result against the starting quantities. Substitute into the original relation, or compare the graph and numerical answer where appropriate.

Worked example

For integral $x e^x dx$, take $u=x$ and $v' = e^x$. Then the integral is $x e^x - e^x + C$.
 For integral $2x/(x^2+1) dx$, substitute $w=x^2+1$, $dw=2x dx$; the answer is $\ln(x^2+1)+C$.



Test a tempting shortcut

- Choosing u and v' well matters: the remaining integral should become simpler. In a definite substitution, either change the limits or return to x before applying the original limits.
- When a shortcut fails, identify the assumption it breaks. Keep an exact value until the requested final rounding.

Watch out

The integral of a product equals the product of its separate integrals. This claim is false. Explain which definition or assumption it violates.

Interpret a new situation

- AQA H5 introduces substitution and parts; H6 integrates partial fractions with linear denominators. The single worked parts/substitution examples are introductory support, not complete repeated-parts or partial-fraction coverage.

- A complete solution gives the mathematical result and explains what it means. Check that it is possible in the stated context.
-

Use this in your course

- 7357 · A-level · H. Match the target tier and specification before assigning extensions.
- Give the method before the final answer, and use the paper's calculator and formula rules. Review a wrong answer by locating the first invalid step.

Key idea

An integration method based on the derivative of a product. Choose the relationship, show the method, check its assumptions and interpret the result.

H · Differential equations and numerical solutions

What does the starting value decide?

- A growing population's rate is proportional to its current size. The rate equation describes a whole family until an initial population is supplied.
 - This lesson studies **initial condition** : A specified value that selects a particular solution from a family.
-

Choose the mathematical structure

- Separate the variables in $dy/dx=ky$, integrate both sides and apply an initial condition. Include any equilibrium solution excluded when dividing by y .
 - State the allowed inputs and units before calculating. An equation should express the relationship, not just record a calculator entry.
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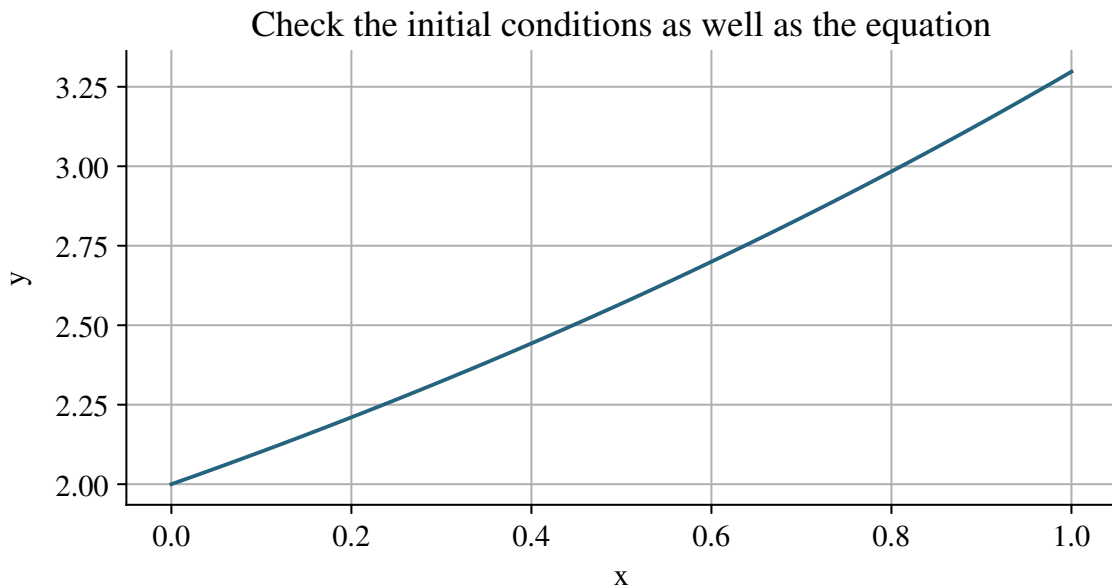
Work through a checked case

$$\frac{dy}{dx} = ky, \quad y = Ae^{kx}$$

- Check the result against the starting quantities. Substitute into the original relation, or compare the graph and numerical answer where appropriate.

Worked example

For $\frac{dy}{dx}=0.5y$ and $y(0)=2$, separation gives $\ln|y|=0.5x+C$, hence $y=Ae^{(0.5x)}$. The initial value gives $A=2$. Differentiate the result to check the equation and substitute $x=0$ to check the initial condition.



Test a tempting shortcut

- An initial condition determines the integration constant; it is not a replacement for integrating. Dividing by y can omit an equilibrium solution $y=0$. Euler accuracy depends on step size and the equation.
- When a shortcut fails, identify the assumption it breaks. Keep an exact value until the requested final rounding.

Watch out

An initial condition never changes the constant in a differential-equation solution. This claim is false. Explain which definition or assumption it violates.

Interpret a new situation

- This unit uses first-order separation and initial conditions. Euler numerical integration and second-order complementary functions are excluded from this lesson.
 - A complete solution gives the mathematical result and explains what it means. Check that it is possible in the stated context.
-

Use this in your course

- 7357 · A-level · H. Match the target tier and specification before assigning extensions.
- Give the method before the final answer, and use the paper's calculator and formula rules. Review a wrong answer by locating the first invalid step.

Key idea

A specified value that selects a particular solution from a family. Choose the relationship, show the method, check its assumptions and interpret the result.

H · The Fundamental Theorem and accumulated change

A flow meter reports a changing rate. How does the accumulated volume recover that same rate when differentiated?

- A flow meter reports a changing rate. How does the accumulated volume recover that same rate when differentiated?
 - This lesson studies **accumulation function** : A function whose value is the signed integral from a fixed starting point to a variable endpoint.
-

Choose the mathematical structure

- For continuous f on an interval, $A(x)=\text{integral from } a \text{ to } x \text{ of } f(t) \, dt$ has $A'(x)=f(x)$. If $F'=f$, the definite integral from a to b is $F(b)-F(a)$. These two directions connect

accumulation and differentiation. Dummy variable t is integrated; x is the moving endpoint. For $A(g(x))$, apply the chain rule: its derivative is $f(g(x))g'(x)$.

- State the allowed inputs and units before calculating. An equation should express the relationship, not just record a calculator entry.

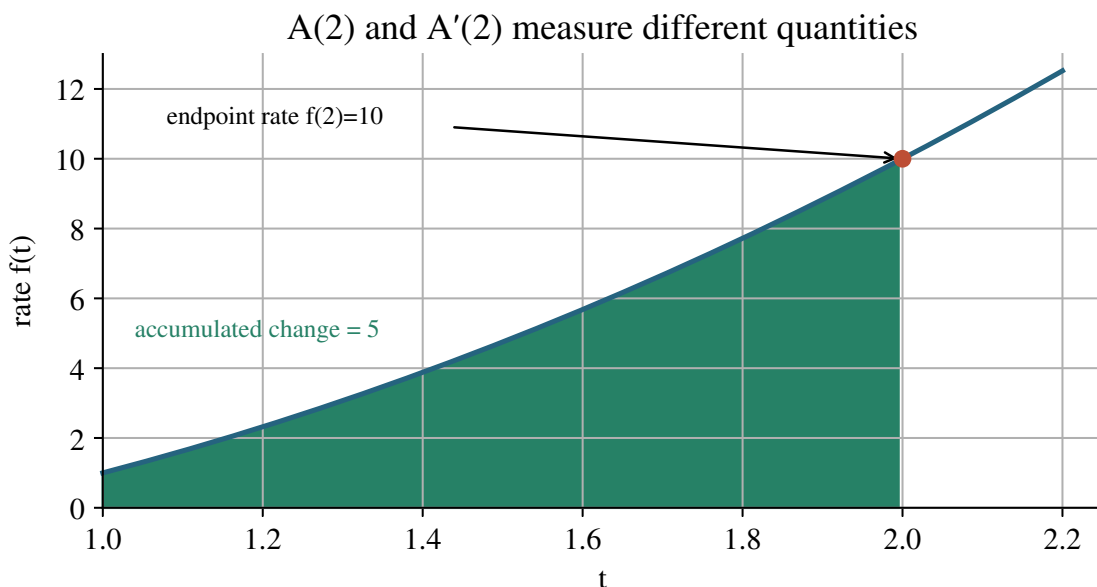
Work through a checked case

$$\frac{d}{dx} \int_a^x f(t) dt = f(x), \quad \int_a^b f(x) dx = F(b) - F(a)$$

- Check the result against the starting quantities. Substitute into the original relation, or compare the graph and numerical answer where appropriate.

Worked example

Let $A(x) = \text{integral from 1 to } x \text{ of } (3t^2 - 2) dt$. An antiderivative is $t^3 - 2t$, so $A(x) = x^3 - 2x + 1$, $A(1) = 0$, $A(2) = 5$ and $A'(2) = 10$. Thus accumulated change 5 differs from endpoint rate 10. For $B(x) = \text{integral from 1 to } x^2 \text{ of } (3t^2 - 2) dt$, $B'(x) = (3x^4 - 2)2x$; at $x = 1$ this is 2. Independently, if $F' = 3x^2 - 2$ and $F(1) = 4$, then $F = x^3 - 2x + 5$. Its change $F(2) - F(1) = 5$ agrees with $A(2)$, although its initial value is not zero.



Test a tempting shortcut

- Subtract the lower antiderivative value even when the lower limit is not zero. A definite integral is signed change, not automatically total positive area. $A'(x)$ is the integrand at the endpoint, not its antiderivative. Continuity on the relevant interval is a sufficient hypothesis; do not integrate across a pole such as $1/t$ at zero. A moving endpoint x^2 also contributes its chain-rule factor.
- When a shortcut fails, identify the assumption it breaks. Keep an exact value until the requested final rounding.

Watch out

The derivative of an accumulation function is always its accumulated total. This claim is false. Explain which definition or assumption it violates.

Interpret a new situation

- Use rate units multiplied by input units for accumulated change. Check $A(a)=0$ and differentiate the final expression to recover f . Initial values determine constants in a state function; definite changes do not depend on an arbitrary antiderivative constant. This lesson gives the theorem and elementary variable-bound examples; improper integrals are outside this treatment.
 - A complete solution gives the mathematical result and explains what it means. Check that it is possible in the stated context.
-

Use this in your course

- 7357 · A-level · H. Match the target tier and specification before assigning extensions.
- Give the method before the final answer, and use the paper's calculator and formula rules. Review a wrong answer by locating the first invalid step.

Key idea

A function whose value is the signed integral from a fixed starting point to a variable endpoint. Choose the relationship, show the method, check its assumptions and interpret the result.

H · Integration as the limit of rectangle sums

A finite collection of rectangles misses part of a curved region. What changes as the strip width tends to zero?

- A finite collection of rectangles misses part of a curved region. What changes as the strip width tends to zero?
 - This lesson studies **Riemann sum** : A sum of signed rectangle contributions that approaches an integral as the partition becomes finer.
-

Choose the mathematical structure

- For n equal strips on $[a, b]$, width $h = (b - a)/n$. A right-end sum is h times the sum of $f(a + kh)$ for $k = 1, \dots, n$; a left-end sum uses $k = 0, \dots, n - 1$. For continuous f , both limits as n tends to infinity equal the integral. A finite sum is an approximation; the limit is the exact signed accumulation.
 - State the allowed inputs and units before calculating. An equation should express the relationship, not just record a calculator entry.
-

Work through a checked case

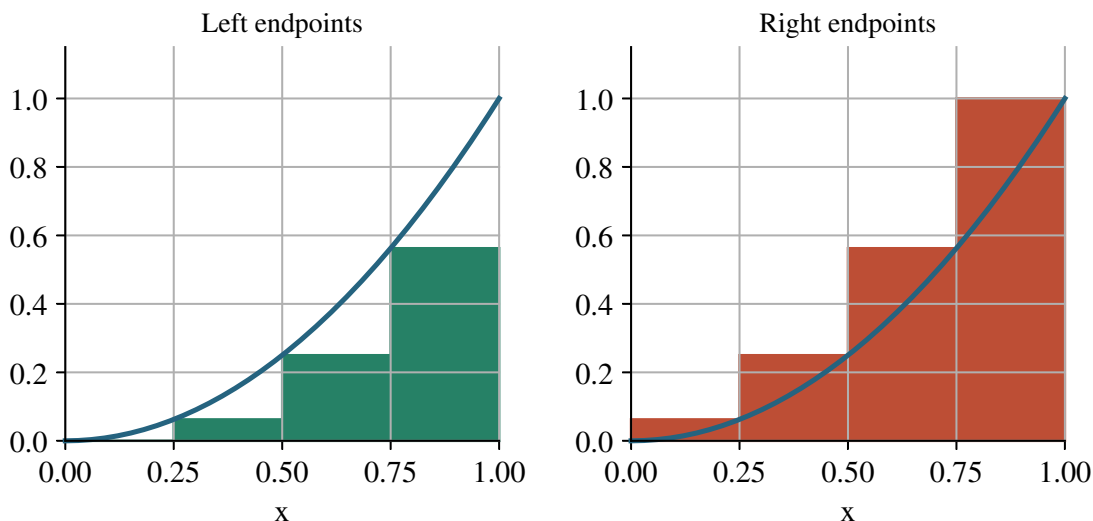
$$\lim_{n \rightarrow \infty} \frac{1}{n^3} \sum_{k=1}^n k^2 = \frac{1}{3} = \int_0^1 x^2 dx$$

- Check the result against the starting quantities. Substitute into the original relation, or compare the graph and numerical answer where appropriate.

Worked example

For $f(x) = x^2$ on $[0, 1]$, $h = 1/n$. The right sum $R_n = (1/n^3) \sum_{k=1}^n k^2$ from $k=1$ to n of $k^2 = (n+1)(2n+1)/(6n^2) = 1/3 + 1/(2n) + 1/(6n^2)$. The left sum $L_n = (n-1)n(2n-1)/(6n^3) = 1/3 - 1/(2n) + 1/(6n^2)$. With $n=2$, $L_2 = 1/8$ and $R_2 = 5/8$; both bound the exact integral $1/3$ because x^2 increases on this interval. Their difference is $1/n$, tending to zero. This same definition gives signed contributions: for $f = -2$ on $[0, 1]$, every such sum is -2 , while positive geometric area is 2 .

Finite rectangles approach the same signed integral



Test a tempting shortcut

- Multiply by strip width: adding heights alone gives the wrong units and limit. The left sum includes $k=0$ and excludes $k=n$; the right sum does the reverse. Increasing functions give left/right bounds, but the rule must be checked on the actual interval. Rectangle sums and the trapezium rule are different finite approximations. Negative heights contribute negative signed area.
- When a shortcut fails, identify the assumption it breaks. Keep an exact value until the requested final rounding.

Watch out

A finite right-end rectangle sum is already the exact integral of every continuous curve. This claim is false. Explain which definition or assumption it violates.

Interpret a new situation

- Write the partition, evaluation points and sigma limits before simplifying. Use a finite-sum identity, then evaluate the limit. Compare finite bounds with the antiderivative result as an independent check. This is H4 exact limit reasoning; numerical trapezia belong to I3 and are not a substitute for the limiting argument.

- A complete solution gives the mathematical result and explains what it means. Check that it is possible in the stated context.
-

Use this in your course

- 7357 · A-level · H. Match the target tier and specification before assigning extensions.
- Give the method before the final answer, and use the paper's calculator and formula rules. Review a wrong answer by locating the first invalid step.

Key idea

A sum of signed rectangle contributions that approaches an integral as the partition becomes finer. Choose the relationship, show the method, check its assumptions and interpret the result.

H · Elementary antiderivatives, coefficients and domains

Differentiation multiplies an exponential or trigonometric argument by its coefficient. How must integration undo that factor?

- Differentiation multiplies an exponential or trigonometric argument by its coefficient. How must integration undo that factor?
 - This lesson studies **antiderivative** : A function whose derivative is the specified integrand on the stated interval.
-

Choose the mathematical structure

- For real powers on an allowed differentiable domain, integrate x^n as $x^{n+1}/(n+1)+C$ when $n \neq -1$. Integrate $1/x$ as $\ln|x|+C$ on an interval excluding zero. For $k \neq 0$, integrals of e^{kx} , $\sin(kx)$, $\cos(kx)$ are e^{kx}/k , $-\cos(kx)/k$, $\sin(kx)/k$ respectively, plus constants. Apply linearity to sums and constant multiples; verify by differentiation.
-

- State the allowed inputs and units before calculating. An equation should express the relationship, not just record a calculator entry.

Work through a checked case

$$\int \cos(kx) dx = \frac{\sin(kx)}{k} + C \quad (k \neq 0), \quad \int \frac{1}{x} dx = \ln|x| + C$$

- Check the result against the starting quantities. Substitute into the original relation, or compare the graph and numerical answer where appropriate.

Worked example

For $f(x) = 6x^2 - 4e^{2x} + 3\cos(3x) - 2\sin(2x) + 5/x$ with $x > 0$,
 $F = 2x^3 - 2e^{2x} + \sin(3x) + \cos(2x) + 5\ln x + C$. Each derivative recovers its matching term. On $x < 0$, integrate $5/x$ as $5\ln(-x)$, or $5\ln|x|$; constants can differ on disconnected intervals. The integral of $3\cos(3x)$ from 0 to $\pi/6$ is $\sin(\pi/2) - \sin 0 = 1$. For $F' = 6x^2$ and $F(1) = 7$, $F = 2x^3 + 5$. If $k = 0$, $e^{kx} = 1$ and $\cos(kx) = 1$ integrate to $x + C$, while $\sin(kx) = 0$ integrates to C ; never divide by $k = 0$.

Differentiate each result to recover its integrand

Integrand Antiderivative (plus C)

$$6x^2 \quad 2x^3$$

$$-4 \exp(2x) \quad -2 \exp(2x)$$

$$3 \cos(3x) \quad \sin(3x)$$

$$-2 \sin(2x) \quad \cos(2x)$$

$$5/x \quad 5 \ln|x| \quad (x \neq 0)$$

Test a tempting shortcut

- The $n=-1$ power is a separate logarithmic case, not division by zero in the power formula. Keep the sign in the sine antiderivative and divide by the argument coefficient. $\ln x$ alone does not cover negative x . An indefinite integral needs a constant; a definite integral has a numerical value when valid limits are supplied. A real fractional power may restrict the domain.
- When a shortcut fails, identify the assumption it breaks. Keep an exact value until the requested final rounding.

Watch out

Integrating $\cos(kx)$ always gives $k \sin(kx)$, including $k=0$. This claim is false. Explain which definition or assumption it violates.

Interpret a new situation

- Differentiate term by term to check coefficients and signs, and state the interval before using logarithms or fractional powers. The H2 named elementary forms are powers, e^{kx} , $1/x$, $\sin(kx)$ and $\cos(kx)$. Other forms require separate justification and are not asserted as additional compulsory H2 forms.
 - A complete solution gives the mathematical result and explains what it means. Check that it is possible in the stated context.
-

Use this in your course

- 7357 · A-level · H. Match the target tier and specification before assigning extensions.
- Give the method before the final answer, and use the paper's calculator and formula rules. Review a wrong answer by locating the first invalid step.

Key idea

A function whose derivative is the specified integrand on the stated interval. Choose the relationship, show the method, check its assumptions and interpret the result.

I · Root finding and numerical integration

How reliable is an approximation?

- A nonlinear equation has no convenient exact solution. An approximation still needs evidence that it is reliable.
 - This lesson studies **iteration** : A repeated update in which each new approximation is calculated from the previous one.
-

Choose the mathematical structure

- A continuous function with opposite signs at two endpoints has a root between them. Newton's method uses $x_{\text{next}} = x - f(x)/f'(x)$, with a nonzero derivative. The trapezium rule approximates a definite integral using endpoint heights.
 - State the allowed inputs and units before calculating. An equation should express the relationship, not just record a calculator entry.
-

Work through a checked case

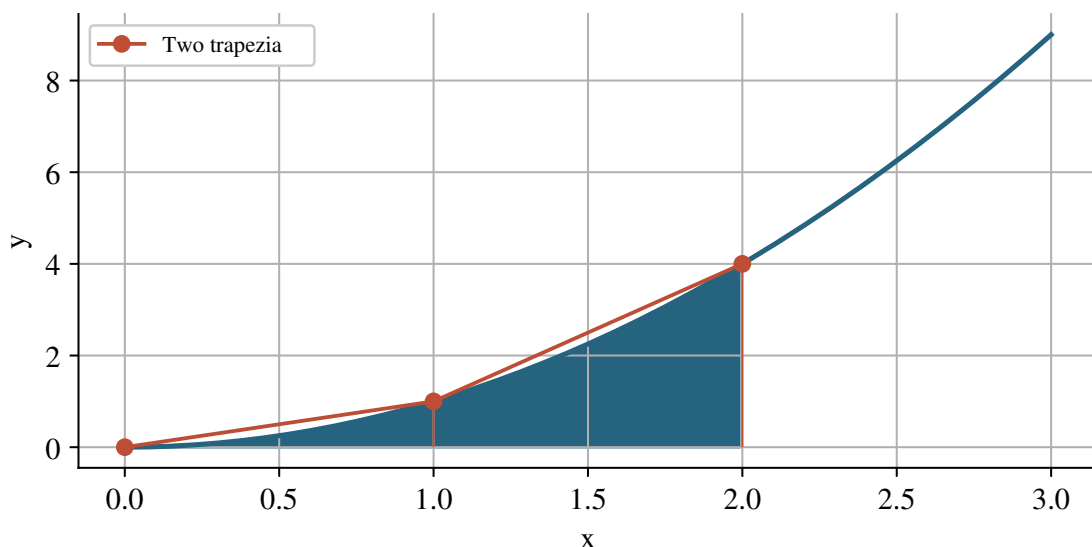
$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

- Check the result against the starting quantities. Substitute into the original relation, or compare the graph and numerical answer where appropriate.

Worked example

For $f(x) = x^2 - 2$ and $x_0 = 1.5$, Newton gives $x_1 = 1.5 - (2.25 - 2)/3 = 1.4166667$. With $y = x^2$ on $[0, 2]$ and two equal strips, $h = 1$ and trapezium area $= (1/2)(0 + 2 \times 1 + 4) = 3$; the exact area is $8/3$.

Accumulation under a curve



Test a tempting shortcut

- A sign change across a discontinuity does not prove a root. Iteration can diverge or cycle. The trapezium rule's overestimate or underestimate depends on curvature, not just whether the function increases.
- When a shortcut fails, identify the assumption it breaks. Keep an exact value until the requested final rounding.

Watch out

Every sign change proves a root, including one across a discontinuity. This claim is false. Explain which definition or assumption it violates.

Interpret a new situation

- Give a stopping criterion and report sensible accuracy. Confirm the approximated root by a sign bracket around the stated rounded answer; explain any failure of the chosen iteration.
- A complete solution gives the mathematical result and explains what it means. Check that it is possible in the stated context.

Use this in your course

- 7357 · A-level · I. Match the target tier and specification before assigning extensions.
- Give the method before the final answer, and use the paper's calculator and formula rules. Review a wrong answer by locating the first invalid step.

Key idea

A repeated update in which each new approximation is calculated from the previous one. Choose the relationship, show the method, check its assumptions and interpret the result.

J · Vectors and transformation geometry

Why is displacement shorter than the walk?

- Walking 4 m east and 3 m north gives a displacement of 5 m, even though the travelled distance is 7 m.
- This lesson studies **resultant** : The vector sum representing the combined displacement or force.

Choose the mathematical structure

- Add corresponding vector components and subtract position vectors to find a displacement. A translation moves every point by the same vector; a scalar multiple changes length and possibly direction.
- State the allowed inputs and units before calculating. An equation should express the relationship, not just record a calculator entry.

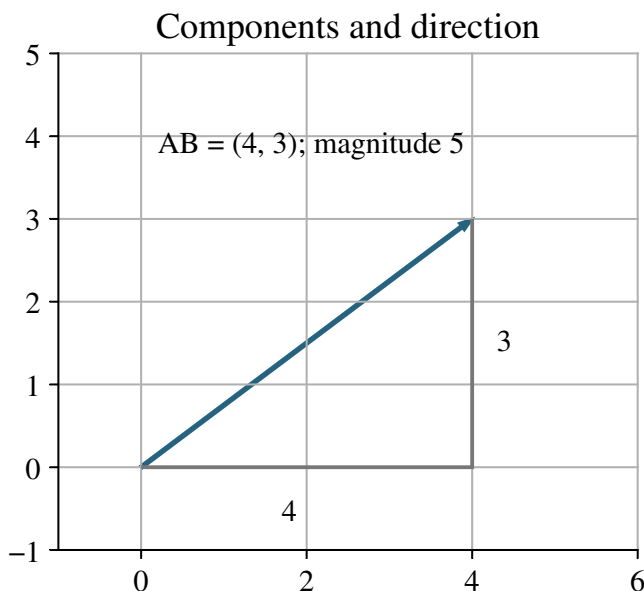
Work through a checked case

$$\overrightarrow{AB} = \mathbf{b} - \mathbf{a}, \quad |\mathbf{v}| = \sqrt{v_x^2 + v_y^2}$$

- Check the result against the starting quantities. Substitute into the original relation, or compare the graph and numerical answer where appropriate.

Worked example

With $a=(4,1)$ and $b=(1,3)$, $a+b=(5,4)$. From $A=(1,2)$ to $B=(5,5)$, displacement $AB=(4,3)$. Its magnitude is $\sqrt{4^2+3^2}=5$.



Test a tempting shortcut

- The order of subtraction matters: $BA=-AB$. Proving parallelism needs a scalar-multiple relation; a sketch alone is insufficient. Negative enlargement reverses position about its centre.
- When a shortcut fails, identify the assumption it breaks. Keep an exact value until the requested final rounding.

Watch out

AB and BA always have the same components. This claim is false. Explain which definition or assumption it violates.

Interpret a new situation

- Use vector components, scalar multiples and magnitudes for displacements and forces. This unit does not require the scalar product.
- A complete solution gives the mathematical result and explains what it means. Check that it is possible in the stated context.

Use this in your course

- 7357 · A-level · J. Match the target tier and specification before assigning extensions.
- Give the method before the final answer, and use the paper's calculator and formula rules. Review a wrong answer by locating the first invalid step.

Key idea

The vector sum representing the combined displacement or force. Choose the relationship, show the method, check its assumptions and interpret the result.

J · Reflections, rotations and enlargements

Where is the enlargement centre?

- A logo is enlarged around a point away from the origin. Multiplying its coordinates alone puts it in the wrong place.
- This lesson studies **centre of enlargement** : The point from which each point's displacement is multiplied by a scale factor.

Choose the mathematical structure

- A translation adds a vector. A reflection reverses signed perpendicular distance from a mirror line. A rotation needs a centre, angle and direction. For enlargement from C, use new $P = C + k(P - C)$.
- State the allowed inputs and units before calculating. An equation should express the relationship, not just record a calculator entry.

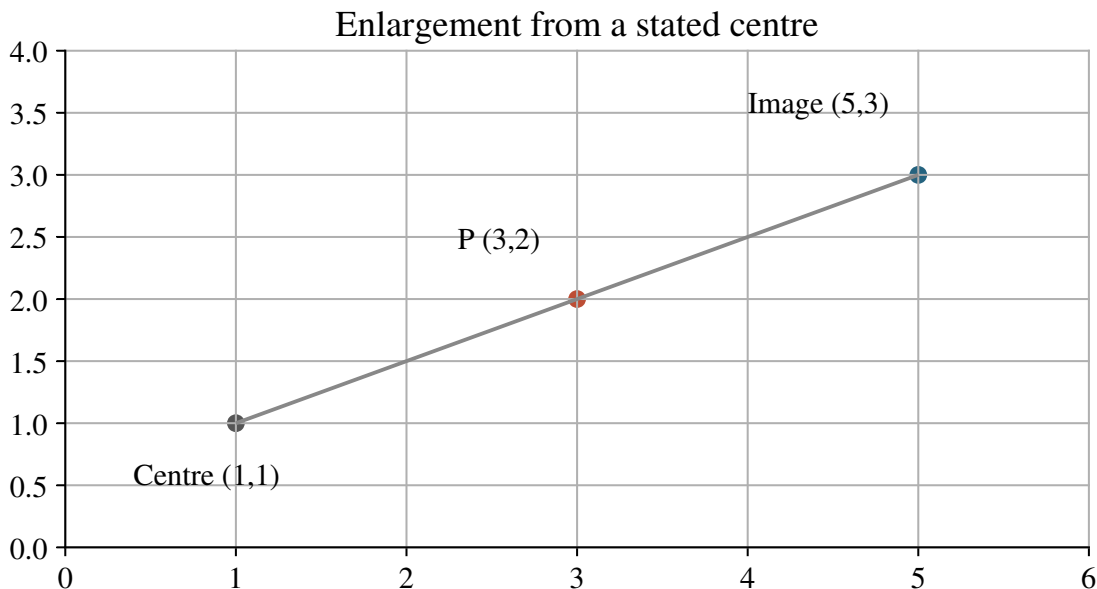
Work through a checked case

$$\mathbf{p}_{\text{image}} = \mathbf{c} + k(\mathbf{p} - \mathbf{c})$$

- Check the result against the starting quantities. Substitute into the original relation, or compare the graph and numerical answer where appropriate.

Worked example

For $C=(1,1)$, $P=(3,2)$, $k=2$, $P-C=(2,1)$, so new $P=(1,1)+2(2,1)=(5,3)$. Reflection of $(3,2)$ in the y -axis gives $(-3,2)$. A 90° anticlockwise rotation about the origin gives $(-2,3)$.



Test a tempting shortcut

- A rotation needs its centre and direction, not just an angle. A negative enlargement factor places the image on the opposite side of the centre. A translation does not change orientation or size.
- When a shortcut fails, identify the assumption it breaks. Keep an exact value until the requested final rounding.

Watch out

Every enlargement is centred at the origin unless its scale factor is negative. This claim is false. Explain which definition or assumption it violates.

Interpret a new situation

- Describe a transformation completely before constructing the image. Check corresponding distances and angles. For combined transformations, apply them in the stated order; they usually do not commute.
 - A complete solution gives the mathematical result and explains what it means. Check that it is possible in the stated context.
-

Use this in your course

- 7357 · A-level · J. Match the target tier and specification before assigning extensions.
- Give the method before the final answer, and use the paper's calculator and formula rules. Review a wrong answer by locating the first invalid step.

Key idea

The point from which each point's displacement is multiplied by a scale factor. Choose the relationship, show the method, check its assumptions and interpret the result.

K · Sampling and a large data set

Who is missing from the data?

- A weather database has a missing entry and several stations. Treating each row as identical can distort a comparison.
 - This lesson studies **sampling frame** : The list or population definition from which a sample is selected.
-

Choose the mathematical structure

- Identify the population, sampling unit and frame. Distinguish random, systematic, stratified, quota and opportunity sampling. Missing data is not zero; verify units, dates and variable definitions before comparing samples.
 - State the allowed inputs and units before calculating. An equation should express the relationship, not just record a calculator entry.
-

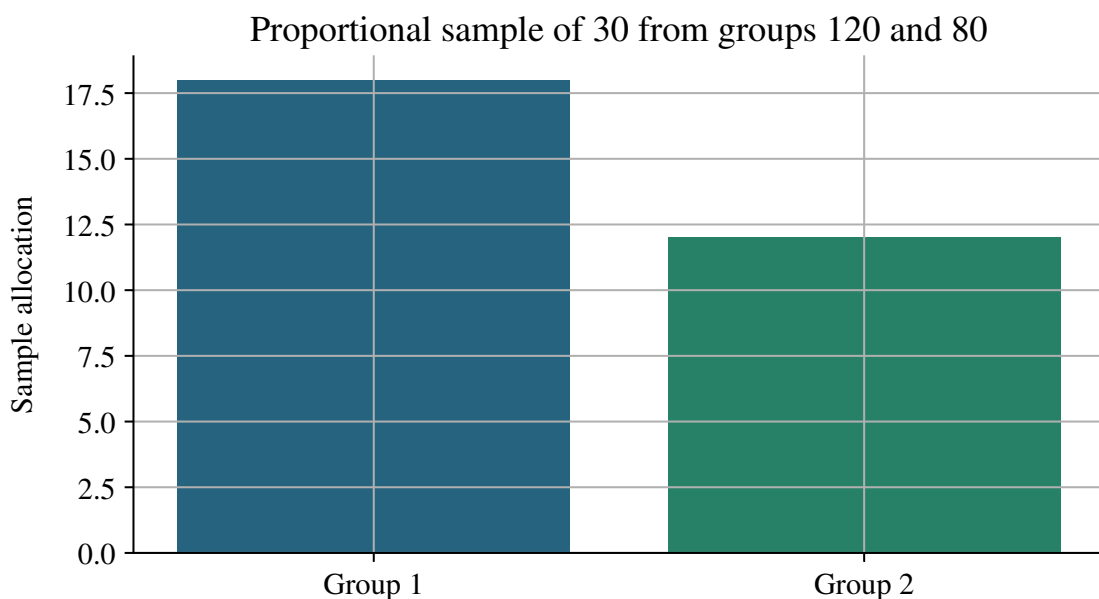
Work through a checked case

$$n_i = n \frac{N_i}{N}$$

- Check the result against the starting quantities. Substitute into the original relation, or compare the graph and numerical answer where appropriate.

Worked example

A population has 120 students in one group and 80 in another. A proportional stratified sample of 30 needs $30 \times 120 / 200 = 18$ from the first and 12 from the second. Random selection is then needed within each group.



Test a tempting shortcut

- A large biased sample remains biased. Stratification is not the same as selecting whoever is available from each group. AQA large-data-set familiarity requires the actual supplied data and metadata, not invented weather values.
- When a shortcut fails, identify the assumption it breaks. Keep an exact value until the requested final rounding.

Watch out

Doubling a biased sample automatically removes its selection bias. This claim is false. Explain which definition or assumption it violates.

Interpret a new situation

- For AQA, use the official large data set in a supervised spreadsheet task: identify a variable, justify a comparison, inspect missing values, create a display and explain a limitation. Save the decisions with the analysis.
 - A complete solution gives the mathematical result and explains what it means. Check that it is possible in the stated context.
-

Use this in your course

- 7357 · A-level · K. Match the target tier and specification before assigning extensions.
- Give the method before the final answer, and use the paper's calculator and formula rules. Review a wrong answer by locating the first invalid step.

Key idea

The list or population definition from which a sample is selected. Choose the relationship, show the method, check its assumptions and interpret the result.

L · Data summaries, histograms and interpretation

Can one average tell the whole story?

- Two groups have the same median but different spread. One summary cannot describe both location and consistency.
 - This lesson studies **frequency density** : Frequency divided by class width, used as histogram height.
-

Choose the mathematical structure

- Compare an appropriate average and spread in context. A histogram uses area for frequency, so $\text{height} = \text{frequency} / \text{class width}$. Grouped estimates assume representative values within intervals.
-

- State the allowed inputs and units before calculating. An equation should express the relationship, not just record a calculator entry.

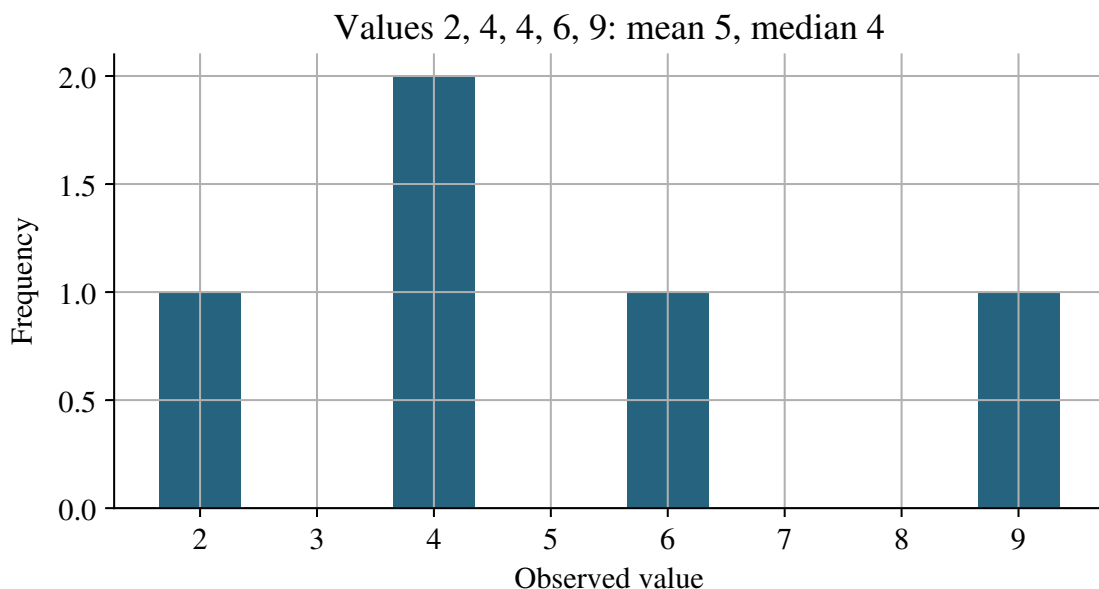
Work through a checked case

$$\bar{x} = \frac{\sum x_i}{n}, \quad \text{density} = \frac{\text{frequency}}{\text{class width}}$$

- Check the result against the starting quantities. Substitute into the original relation, or compare the graph and numerical answer where appropriate.

Worked example

A class from 10 to 20 with frequency 30 has density $30/10=3$. A class from 20 to 40 with frequency 20 has density $20/20=1$. Its wider bar must not be mistaken for a larger density. For values 2,4,4,6,9, the median is 4 and mean is 5.



Test a tempting shortcut

- The tallest histogram bar need not contain the most observations. A grouped mean is an estimate. Correlation does not prove causation, and extrapolation extends beyond the observed range.

- When a shortcut fails, identify the assumption it breaks. Keep an exact value until the requested final rounding.

Watch out

A histogram bar's height always equals its frequency, even with unequal class widths. This claim is false. Explain which definition or assumption it violates.

Interpret a new situation

- Choose a display that fits the data type. Give both a numerical comparison and what it means for the population; do not infer more precision than the sample supports.
 - A complete solution gives the mathematical result and explains what it means. Check that it is possible in the stated context.
-

Use this in your course

- 7357 · A-level · L. Match the target tier and specification before assigning extensions.
- Give the method before the final answer, and use the paper's calculator and formula rules. Review a wrong answer by locating the first invalid step.

Key idea

Frequency divided by class width, used as histogram height. Choose the relationship, show the method, check its assumptions and interpret the result.

L · Regression, financial models and residuals

Is the prediction supported by the data?

- A line predicts bus travel time from distance. A small calculation error matters less than using a model outside its evidence.
 - This lesson studies **residual** : The observed value minus the value predicted by a fitted model.
-

Choose the mathematical structure

- A linear model $y=a+bx$ has intercept a and slope b with contextual units. Inspect residuals and the data range. For repeated percentage change use a geometric model; for a loan distinguish principal, rate, repayment and period.
 - State the allowed inputs and units before calculating. An equation should express the relationship, not just record a calculator entry.
-

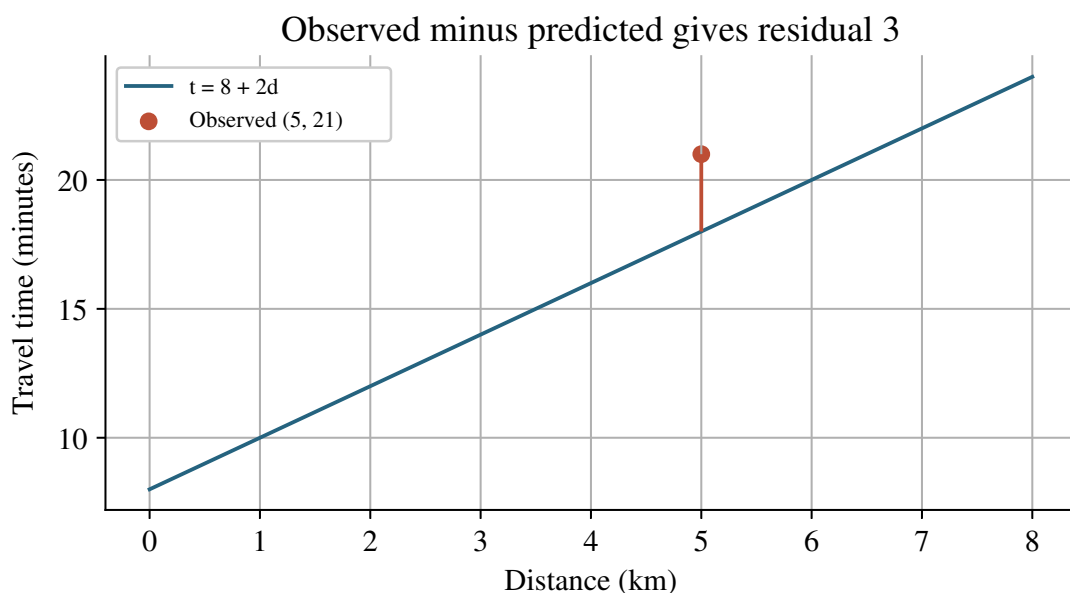
Work through a checked case

$$\hat{t} = 8 + 2d, \quad e = t - \hat{t}$$

- Check the result against the starting quantities. Substitute into the original relation, or compare the graph and numerical answer where appropriate.

Worked example

The fitted model $t=8+2d$ predicts 18 minutes at $d=5$ km. If the observed time is 21, the residual is 3 minutes.



Test a tempting shortcut

- A good fit does not prove a causal mechanism. A correlation coefficient measures linear association, not the gradient. Calculator output must be translated into a model, checked and interpreted.
- When a shortcut fails, identify the assumption it breaks. Keep an exact value until the requested final rounding.

Watch out

A high correlation permits reliable extrapolation to any distance. This claim is false. Explain which definition or assumption it violates.

Interpret a new situation

- Use the AQA large data set and suitable statistical displays. Interpret residuals, correlation and model limitations; association alone does not establish causation.
 - A complete solution gives the mathematical result and explains what it means. Check that it is possible in the stated context.
-

Use this in your course

- 7357 · A-level · L. Match the target tier and specification before assigning extensions.
- Give the method before the final answer, and use the paper's calculator and formula rules. Review a wrong answer by locating the first invalid step.

Key idea

The observed value minus the value predicted by a fitted model. Choose the relationship, show the method, check its assumptions and interpret the result.

M · Probability, trees and conditional reasoning

What changes after the first draw?

- A bag has 3 red and 2 blue counters. Taking two without replacement changes the chance of the second colour.

- This lesson studies **conditional probability** : The probability of an event after restricting the sample space to a stated condition.

Choose the mathematical structure

- Multiply along a tree branch and add disjoint branches. With replacement, the composition stays fixed. Conditional probability is $P(A \text{ given } B) = P(A \cap B) / P(B)$, for $P(B) > 0$.
- State the allowed inputs and units before calculating. An equation should express the relationship, not just record a calculator entry.

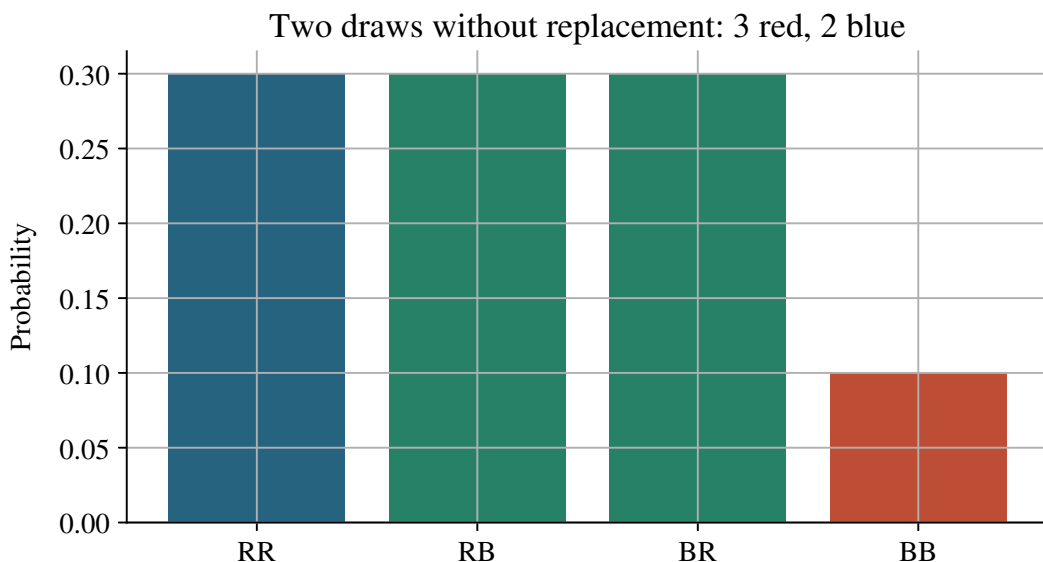
Work through a checked case

$$P(A | B) = \frac{P(A \cap B)}{P(B)}, \quad P(B) > 0$$

- Check the result against the starting quantities. Substitute into the original relation, or compare the graph and numerical answer where appropriate.

Worked example

Without replacement, $P(\text{two red}) = 3/5 \times 2/4 = 3/10$. $P(\text{one of each}) = 3/5 \times 2/4 + 2/5 \times 3/4 = 3/5$. If $P(A \cap B) = 0.12$ and $P(B) = 0.3$, $P(A \text{ given } B) = 0.4$.



Test a tempting shortcut

- Mutually exclusive means no overlap; independent means that knowing one event does not change the other's probability. Two disjoint events with positive probability are not independent.
- When a shortcut fails, identify the assumption it breaks. Keep an exact value until the requested final rounding.

Watch out

Mutually exclusive events with positive probabilities must be independent. This claim is false. Explain which definition or assumption it violates.

Interpret a new situation

- A two-way table makes the restricted denominator visible. Before using a product $P(A)P(B)$, justify independence from the context or the supplied information.
 - A complete solution gives the mathematical result and explains what it means. Check that it is possible in the stated context.
-

Use this in your course

- 7357 · A-level · M. Match the target tier and specification before assigning extensions.
- Give the method before the final answer, and use the paper's calculator and formula rules. Review a wrong answer by locating the first invalid step.

Key idea

The probability of an event after restricting the sample space to a stated condition. Choose the relationship, show the method, check its assumptions and interpret the result.

N · Binomial, normal and Poisson models

What makes a count predictable?

- A quality inspector counts defective items. The number is random, but a model can describe its likely range.
 - This lesson studies **expected value** : The probability-weighted mean of a random variable.
-

Choose the mathematical structure

- A binomial model needs fixed n , independent trials, two outcomes and constant p . $E(X)=np$ and $\text{Var}(X)=np(1-p)$. For a normal model use $z=(x-\mu)/\sigma$. State the event before calculating a tail probability.
 - State the allowed inputs and units before calculating. An equation should express the relationship, not just record a calculator entry.
-

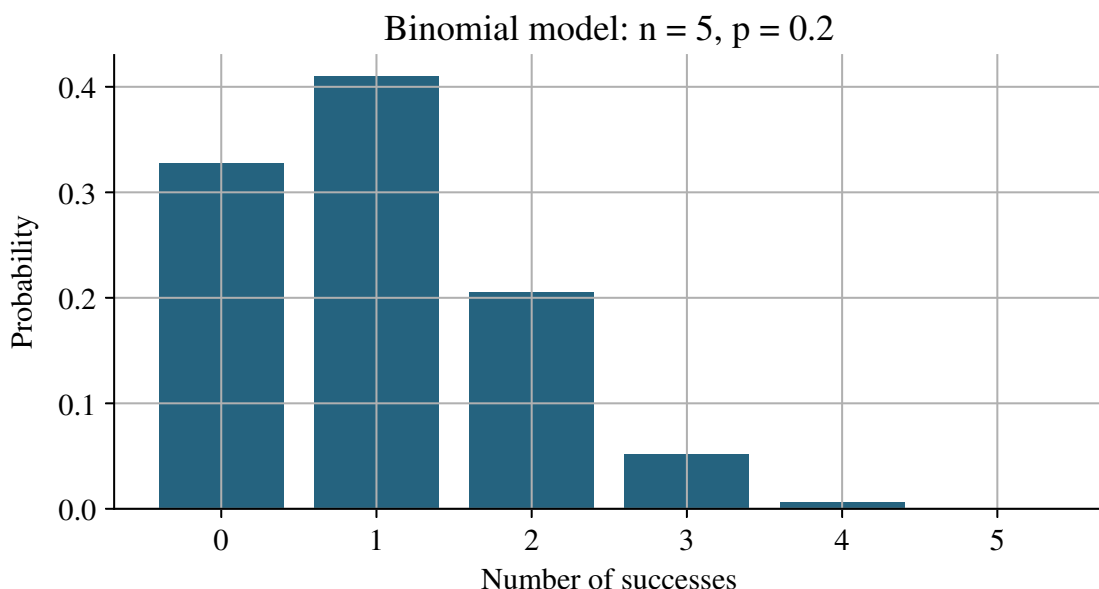
Work through a checked case

$$P(X = k) = \binom{n}{k} p^k (1 - p)^{n-k}, \quad X \sim B(n, p)$$

- Check the result against the starting quantities. Substitute into the original relation, or compare the graph and numerical answer where appropriate.

Worked example

For X binomial(5,0.2), $P(X=0)=0.8^5=0.32768$, $E(X)=1$ and $\text{Var}(X)=0.8$. For a normal quantity with $\mu=100, \sigma=15$, the value 130 has $z=2$. Poisson is not part of this AQA 7357 lesson.



Test a tempting shortcut

- Not every count is binomial: changing p or dependence can invalidate it. For a continuous variable, the probability of one exact value is zero. Continuity correction matters when approximating a discrete distribution by a normal one.
- When a shortcut fails, identify the assumption it breaks. Keep an exact value until the requested final rounding.

Watch out

Every count of successes has a binomial distribution regardless of dependence. This claim is false. Explain which definition or assumption it violates.

Interpret a new situation

- Write the event as an inequality before using calculator distribution functions. Distinguish $P(X < k)$, $P(X \leq k)$ and a tail complement. State assumptions in context.
 - A complete solution gives the mathematical result and explains what it means. Check that it is possible in the stated context.
-

Use this in your course

- 7357 · A-level · N. Match the target tier and specification before assigning extensions.
- Give the method before the final answer, and use the paper's calculator and formula rules. Review a wrong answer by locating the first invalid step.

Key idea

The probability-weighted mean of a random variable. Choose the relationship, show the method, check its assumptions and interpret the result.

O · Hypothesis testing and contextual conclusions

Could chance explain the result?

- A factory claims that only 10% of items are defective. A sample contains more defects, but chance alone may explain some difference.
 - This lesson studies **significance level** : The chosen probability threshold for rejecting a null hypothesis.
-

Choose the mathematical structure

- State H_0 and H_1 in population parameters before inspecting the outcome. Calculate the appropriate tail probability under H_0 . Reject H_0 when the evidence meets the specified significance rule; otherwise say there is insufficient evidence.
 - State the allowed inputs and units before calculating. An equation should express the relationship, not just record a calculator entry.
-

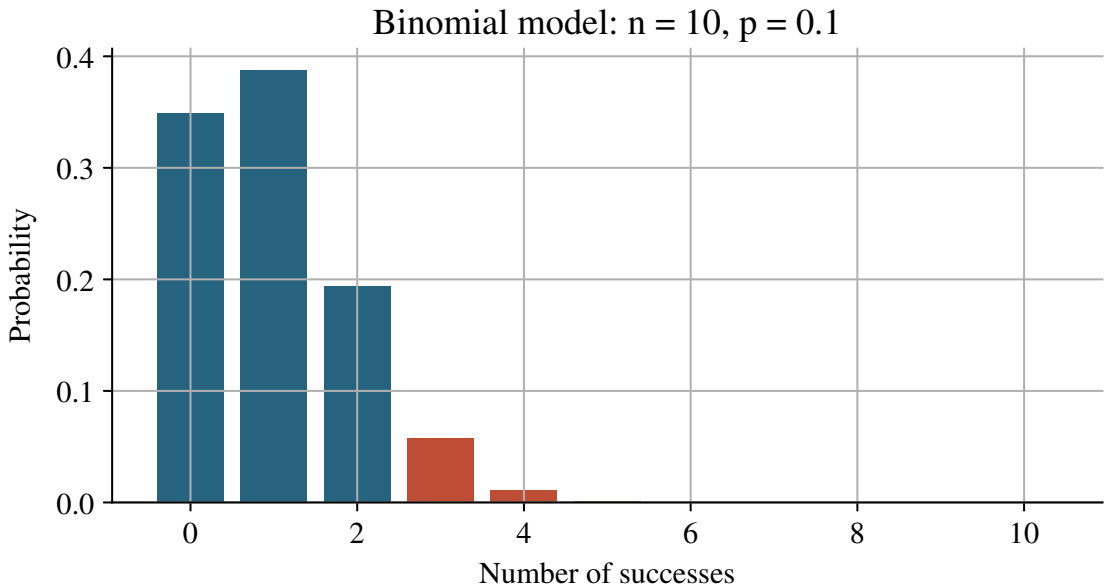
Work through a checked case

$$H_0 : p = p_0, \quad H_1 : p > p_0$$

- Check the result against the starting quantities. Substitute into the original relation, or compare the graph and numerical answer where appropriate.

Worked example

For $H_0:p=0.1$ and $H_1:p>0.1$ with $n=10$, observing at least 3 defects has probability $1-(0.9^{10}+10\times0.1\times0.9^9+45\times0.1^2\times0.9^8)\approx0.070191$. At 5%, this is insufficient evidence that the defect rate exceeds 10%.



Test a tempting shortcut

- Failing to reject H_0 is not proof that H_0 is true. Choose the tail from H_1 , not from whichever tail gives a small result. Statistical significance does not measure the practical size of an effect.
- When a shortcut fails, identify the assumption it breaks. Keep an exact value until the requested final rounding.

Watch out

Failing to reject a null hypothesis proves it is true. This claim is false. Explain which definition or assumption it violates.

Interpret a new situation

- Finish with a sentence about the population and the original claim. State the model's assumptions and consider whether the sampling procedure supports them.

- A complete solution gives the mathematical result and explains what it means. Check that it is possible in the stated context.
-

Use this in your course

- 7357 · A-level · O. Match the target tier and specification before assigning extensions.
- Give the method before the final answer, and use the paper's calculator and formula rules. Review a wrong answer by locating the first invalid step.

Key idea

The chosen probability threshold for rejecting a null hypothesis. Choose the relationship, show the method, check its assumptions and interpret the result.

P · Accuracy, bounds and compound measures

Is the printed measurement exact?

- A rectangular panel is labelled 8.0 cm by 5.0 cm, each to the nearest 0.1 cm. Its true area is not fixed at 40 cm².
 - This lesson studies **lower bound** : The smallest possible value consistent with a stated rounding rule.
-

Choose the mathematical structure

- A value rounded to the nearest unit u lies from stated value- $u/2$ up to, but usually not including, stated value+ $u/2$. For positive quantities, combine extremes according to the operation.
 - State the allowed inputs and units before calculating. An equation should express the relationship, not just record a calculator entry.
-

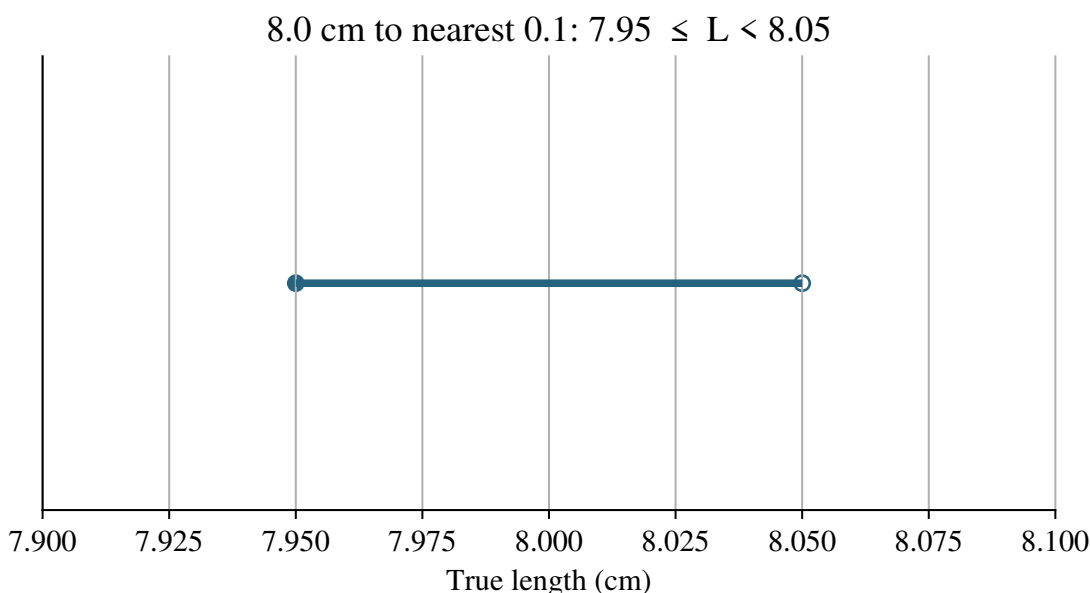
Work through a checked case

$$A = LW, \quad v = \frac{d}{t}$$

- Check the result against the starting quantities. Substitute into the original relation, or compare the graph and numerical answer where appropriate.

Worked example

The lengths satisfy $7.95 \leq L < 8.05$ and $4.95 \leq W < 5.05$. Since $A = LW$, $39.3525 \leq A < 40.6525$. For speed d/t , the largest speed uses the largest distance and smallest positive time.



Test a tempting shortcut

- An upper bound is not automatically achieved. Dividing upper distance by upper time does not give the largest speed. Keep enough digits in intermediate calculations.
- When a shortcut fails, identify the assumption it breaks. Keep an exact value until the requested final rounding.

Watch out

The greatest value of positive d/t uses the greatest d and greatest t . This claim is false. Explain which definition or assumption it violates.

Interpret a new situation

- Distinguish measurement uncertainty from arithmetic rounding. A sensible reported precision cannot be finer than the measurements justify.
 - A complete solution gives the mathematical result and explains what it means. Check that it is possible in the stated context.
-

Use this in your course

- 7357 · A-level · P. Match the target tier and specification before assigning extensions.
- Give the method before the final answer, and use the paper's calculator and formula rules. Review a wrong answer by locating the first invalid step.

Key idea

The smallest possible value consistent with a stated rounding rule. Choose the relationship, show the method, check its assumptions and interpret the result.

P · Kinematics and Newton's laws

Which force accelerates the trolley?

- A trolley accelerates while friction opposes its motion. The engine's force alone does not equal mass times acceleration.
 - This lesson studies **resultant force** : The vector sum of all external forces acting on the modelled object.
-

Choose the mathematical structure

- Choose a positive direction and draw a force diagram. Use resultant $F=ma$. For constant acceleration, $v=u+at$ and $s=ut+at^2/2$. A particle model ignores size; a smooth surface ignores friction.
 - State the allowed inputs and units before calculating. An equation should express the relationship, not just record a calculator entry.
-

Work through a checked case

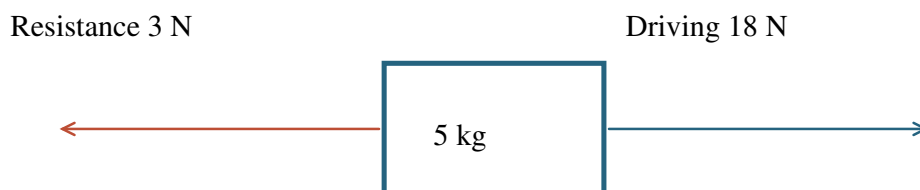
$$F_{\text{resultant}} = ma, \quad s = ut + \frac{1}{2}at^2$$

- Check the result against the starting quantities. Substitute into the original relation, or compare the graph and numerical answer where appropriate.

Worked example

A 5 kg trolley has driving force 18 N and resistance 3 N. The resultant is 15 N, so $a = 15/5 = 3 \text{ m/s}^2$. Starting from rest for 4 s gives $v = 0 + 3 \times 4 = 12 \text{ m/s}$ and $s = 0 \times 4 + 3 \times 4^2 / 2 = 24 \text{ m}$.

Identify the resultant before using $F = ma$



Test a tempting shortcut

- The normal reaction need not equal weight on a slope. Connected-particle models need one equation per particle and a consistent acceleration relation. The constant-acceleration formulae cannot be used for arbitrary variable acceleration.
- When a shortcut fails, identify the assumption it breaks. Keep an exact value until the requested final rounding.

Watch out

The constant-acceleration formulae apply unchanged when acceleration varies with time. This claim is false. Explain which definition or assumption it violates.

Interpret a new situation

- Explain what each idealisation permits and what it leaves out. For a velocity-time graph, gradient represents acceleration and signed area represents displacement.
 - A complete solution gives the mathematical result and explains what it means. Check that it is possible in the stated context.
-

Use this in your course

- 7357 · A-level · P. Match the target tier and specification before assigning extensions.
- Give the method before the final answer, and use the paper's calculator and formula rules. Review a wrong answer by locating the first invalid step.

Key idea

The vector sum of all external forces acting on the modelled object. Choose the relationship, show the method, check its assumptions and interpret the result.

Q · Kinematics and Newton's laws

Which force accelerates the trolley?

- A trolley accelerates while friction opposes its motion. The engine's force alone does not equal mass times acceleration.
 - This lesson studies **resultant force** : The vector sum of all external forces acting on the modelled object.
-

Choose the mathematical structure

- Choose a positive direction and draw a force diagram. Use resultant $F=ma$. For constant acceleration, $v=u+at$ and $s=ut+at^2/2$. A particle model ignores size; a smooth surface ignores friction.
 - State the allowed inputs and units before calculating. An equation should express the relationship, not just record a calculator entry.
-

Work through a checked case

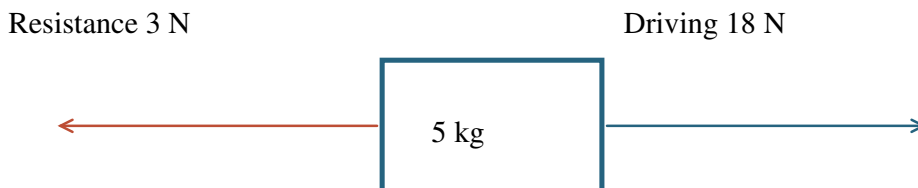
$$F_{\text{resultant}} = ma, \quad s = ut + \frac{1}{2}at^2$$

- Check the result against the starting quantities. Substitute into the original relation, or compare the graph and numerical answer where appropriate.

Worked example

A 5 kg trolley has driving force 18 N and resistance 3 N. The resultant is 15 N, so $a=15/5=3 \text{ m/s}^2$. Starting from rest for 4 s gives $v=0+3\times4=12 \text{ m/s}$ and $s=0\times4+3\times4^2/2=24 \text{ m}$.

Identify the resultant before using $F = ma$



Test a tempting shortcut

- The normal reaction need not equal weight on a slope. Connected-particle models need one equation per particle and a consistent acceleration relation. The constant-acceleration formulae cannot be used for arbitrary variable acceleration.
- When a shortcut fails, identify the assumption it breaks. Keep an exact value until the requested final rounding.

Watch out

The constant-acceleration formulae apply unchanged when acceleration varies with time. This claim is false. Explain which definition or assumption it violates.

Interpret a new situation

- Explain what each idealisation permits and what it leaves out. For a velocity-time graph, gradient represents acceleration and signed area represents displacement.
 - A complete solution gives the mathematical result and explains what it means. Check that it is possible in the stated context.
-

Use this in your course

- 7357 · A-level · Q. Match the target tier and specification before assigning extensions.
- Give the method before the final answer, and use the paper's calculator and formula rules. Review a wrong answer by locating the first invalid step.

Key idea

The vector sum of all external forces acting on the modelled object. Choose the relationship, show the method, check its assumptions and interpret the result.

Q · Integrals, area and accumulation

How much change has accumulated?

- A velocity graph tells us motion at an instant. How can we recover the displacement accumulated over time?

- This lesson studies **antiderivative** : A function whose derivative equals the given integrand.

Choose the mathematical structure

- For $n \neq -1$, the integral of ax^n is $ax^{n+1}/(n+1) + C$. A definite integral is signed accumulation. Split at sign changes when total area or distance is required.
- State the allowed inputs and units before calculating. An equation should express the relationship, not just record a calculator entry.

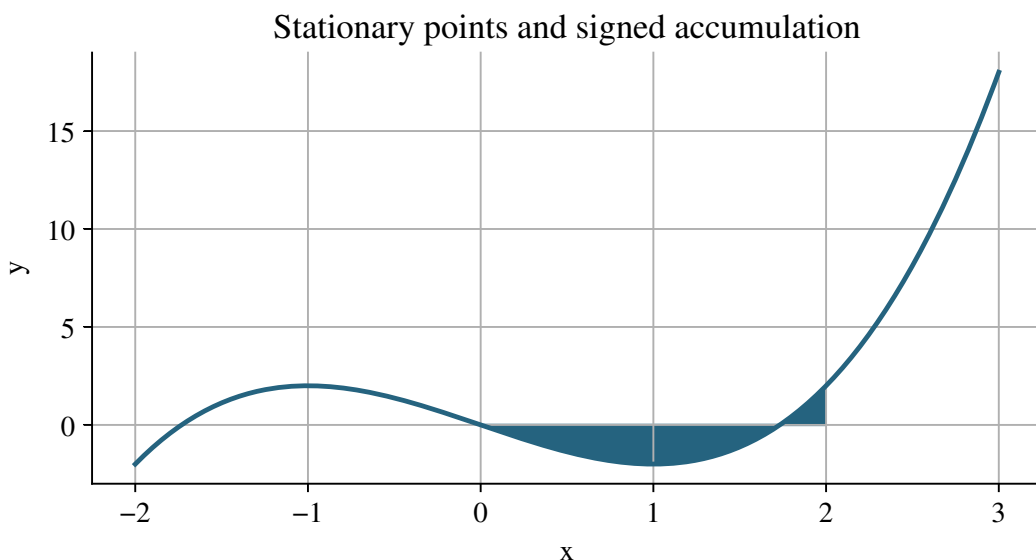
Work through a checked case

$$\int ax^n dx = \frac{ax^{n+1}}{n+1} + C \quad (n \neq -1)$$

- Check the result against the starting quantities. Substitute into the original relation, or compare the graph and numerical answer where appropriate.

Worked example

For $v(t) = 3t^2 - 3$ over $0 \leq t \leq 2$, displacement = $[t^3 - 3t]_0^2 = 2$. Since v changes sign at $t = 1$, distance = $-[t^3 - 3t]_0^1 + [t^3 - 3t]_1^2 = 2 + 4 = 6$. The constants cancel only for a definite integral.



Test a tempting shortcut

- The integral of $1/x$ is $\ln|x|+C$, not the power rule with $n=-1$. Signed area can be zero even when the total enclosed area is positive.
- When a shortcut fails, identify the assumption it breaks. Keep an exact value until the requested final rounding.

Watch out

A definite integral always equals the total positive area, even when the graph crosses the axis. This claim is false. Explain which definition or assumption it violates.

Interpret a new situation

- Identify what the integral means and include the correct units. A rate measured per second integrates to the underlying quantity, not to another rate. Differentiate an antiderivative to check it.
 - A complete solution gives the mathematical result and explains what it means. Check that it is possible in the stated context.
-

Use this in your course

- 7357 · A-level · Q. Match the target tier and specification before assigning extensions.
- Give the method before the final answer, and use the paper's calculator and formula rules. Review a wrong answer by locating the first invalid step.

Key idea

A function whose derivative equals the given integrand. Choose the relationship, show the method, check its assumptions and interpret the result.

R · Kinematics and Newton's laws

Which force accelerates the trolley?

- A trolley accelerates while friction opposes its motion. The engine's force alone does not equal mass times acceleration.
 - This lesson studies **resultant force** : The vector sum of all external forces acting on the modelled object.
-

Choose the mathematical structure

- Choose a positive direction and draw a force diagram. Use resultant $F=ma$. For constant acceleration, $v=u+at$ and $s=ut+at^2/2$. A particle model ignores size; a smooth surface ignores friction.
 - State the allowed inputs and units before calculating. An equation should express the relationship, not just record a calculator entry.
-

Work through a checked case

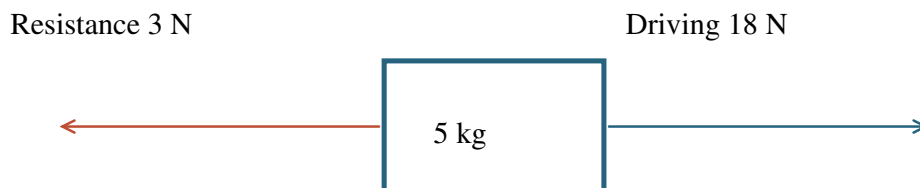
$$F_{\text{resultant}} = ma, \quad s = ut + \frac{1}{2}at^2$$

- Check the result against the starting quantities. Substitute into the original relation, or compare the graph and numerical answer where appropriate.

Worked example

A 5 kg trolley has driving force 18 N and resistance 3 N. The resultant is 15 N, so $a=15/5=3 \text{ m/s}^2$. Starting from rest for 4 s gives $v=0+3\times4=12 \text{ m/s}$ and $s=0\times4+3\times4^2/2=24 \text{ m}$.

Identify the resultant before using $F = ma$



Test a tempting shortcut

- The normal reaction need not equal weight on a slope. Connected-particle models need one equation per particle and a consistent acceleration relation. The constant-acceleration formulae cannot be used for arbitrary variable acceleration.
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Watch out

The constant-acceleration formulae apply unchanged when acceleration varies with time. This claim is false. Explain which definition or assumption it violates.

Interpret a new situation

- Explain what each idealisation permits and what it leaves out. For a velocity-time graph, gradient represents acceleration and signed area represents displacement.
 - A complete solution gives the mathematical result and explains what it means. Check that it is possible in the stated context.
-

Use this in your course

- 7357 · A-level · R. Match the target tier and specification before assigning extensions.
- Give the method before the final answer, and use the paper's calculator and formula rules. Review a wrong answer by locating the first invalid step.

Key idea

The vector sum of all external forces acting on the modelled object. Choose the relationship, show the method, check its assumptions and interpret the result.

S · Moments, equilibrium and centres of mass

Why does a longer spanner help?

- A long spanner turns a nut more easily. The turning effect depends on perpendicular distance as well as force.
 - This lesson studies **moment** : Force multiplied by the perpendicular distance from its line of action to a pivot.
-

Choose the mathematical structure

- For equilibrium, both the resultant force and resultant moment are zero. Take moments about a useful pivot to remove unknown forces through that point. A combined centre of mass is a mass-weighted mean position.
 - State the allowed inputs and units before calculating. An equation should express the relationship, not just record a calculator entry.
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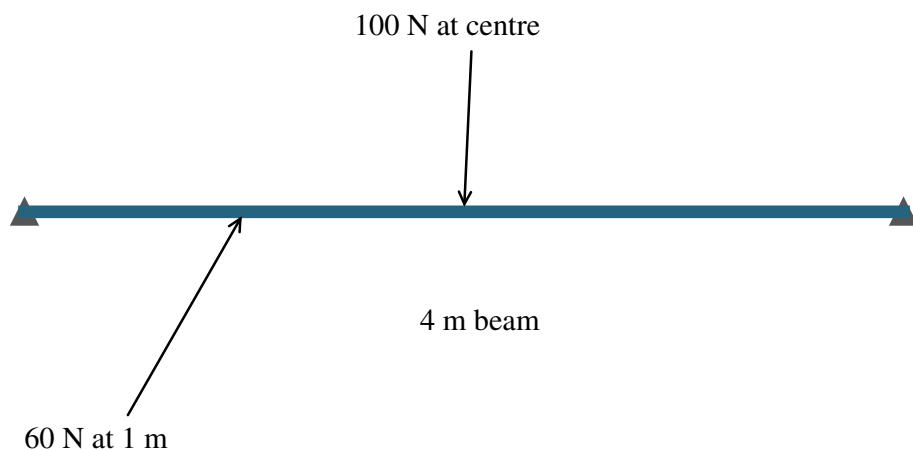
Work through a checked case

$$M = Fd_{\perp}, \quad \sum M = 0, \quad \sum \mathbf{F} = \mathbf{0}$$

- Check the result against the starting quantities. Substitute into the original relation, or compare the graph and numerical answer where appropriate.

Worked example

A uniform 4 m beam weighs 100 N and is supported at both ends. An extra 60 N acts 1 m from the left. Moments about the left give $4R_{\text{right}} = 100 \times 2 + 60 \times 1 = 260$, so $R_{\text{right}} = 65$ N. Vertical equilibrium gives $R_{\text{left}} = 95$ N.



Test a tempting shortcut

- Use the perpendicular lever arm, not a diagonal distance to the point of application. A zero resultant force alone does not prevent rotation. A support that would need a negative normal reaction may lose contact.
- When a shortcut fails, identify the assumption it breaks. Keep an exact value until the requested final rounding.

Watch out

Zero resultant force by itself is enough for a rigid body's equilibrium. This claim is false. Explain which definition or assumption it violates.

Interpret a new situation

- For a lamina made from parts, choose a common origin and tabulate each signed mass-area contribution. A removed region has negative mass in that calculation, not a new physical negative material.

- A complete solution gives the mathematical result and explains what it means. Check that it is possible in the stated context.
-

Use this in your course

- 7357 · A-level · S. Match the target tier and specification before assigning extensions.
- Give the method before the final answer, and use the paper's calculator and formula rules. Review a wrong answer by locating the first invalid step.

Key idea

Force multiplied by the perpendicular distance from its line of action to a pivot. Choose the relationship, show the method, check its assumptions and interpret the result.