Course notes

AQA · A-Level · Mathematics

7357: course teaching notes

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These are original teaching notes, and each lesson is labelled with its tier.

A · Deduction, contradiction and counterexamples

When does a pattern become a proof?

- Checking several integers can suggest a pattern. What turns the pattern into a proof for every integer?
- This lesson studies **counterexample** 反例: A single valid case that disproves a universal claim.

The method

- A deductive proof starts from stated definitions or assumptions.
- A counterexample refutes an all-values claim.
- For contradiction, assume the contrary and derive an impossibility.
- For exhaustion, check every case in a finite complete list.

A worked case

$$(2a+1)+(2b+1)=2(a+b+1)$$

■ Worked example

Odd integers 2a+1 and 2b+1 sum to 2(a+b+1), hence even. To disprove that n^2+n+41 is always prime, use n=41: the value is 41×43=1763. In a contradiction proof that $\sqrt{2}$ is irrational, assume $\sqrt{2}=p/q$ in lowest terms; $p^2=2q^2$ forces both p and q even, contradicting lowest terms.

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Odd + odd: (2a + 1) + (2b + 1)

Factor 2: 2(a + b + 1)

The bracket is an integer → the sum is even

Where it goes wrong

- Do not assume the conclusion while proving it.
- In contradiction, identify the impossible consequence and reject the initial contrary assumption.
- Induction is not compulsory in AQA 7357; it belongs in its own appropriate further-mathematics scope.

> ■ **Watch out**
>
> **Not true:** Checking the first ten integers proves a claim for every positive integer. A pattern can hold for the first ten cases and then fail; a proof must cover every case.

Applying it

- AQA proof includes deduction, exhaustion and contradiction.
- IAL P2 introduces exhaustion and counterexample, while P4 introduces contradiction.
- Keep the method matched to the named unit.

In your course

- AQA A-level Mathematics (7357), specification section A.
- Every paper allows a calculator, but method marks still need the expression or equation written before its value.

■ Key idea

- **Counterexample**: A single valid case that disproves a universal claim.
- A deductive proof starts from stated definitions or assumptions.
- Watch for: do not assume the conclusion while proving it.

A · Complete cases and counterexamples

When do three cases cover every integer?

- Testing a thousand integers may miss the next counterexample. A complete list of possible remainders can instead settle every integer.
- This lesson studies **proof by exhaustion** 穷举证明: A proof that checks every possibility in a finite, complete set of cases.

The method

- Write n=3k, 3k+1 or 3k+2 for integer k.
- These are all possible remainders on division by 3, including negative integers.
- Evaluate the expression in each case.
- One counterexample disproves a universal claim; examples supporting a claim do not prove it.

A worked case

$$n = 3k + r, \quad r \in \{0, 1, 2\}, \qquad n^2 \not\equiv 2 \pmod 3$$

■ Worked example

For n=3k, $n^2=9k^2$ is divisible by 3. For n=3k+1, $n^2=3(3k^2+2k)+1$. For n=3k+2, $n^2=3(3k^2+4k+1)+1$. Thus an integer square has remainder 0 or 1, never 2. The claim that n^2+n+41 is always prime fails at n=41: its value is $1763=41\times43$.

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All integer squares: remainder 0 or 1 modulo 3

$n = 3k$	$n^2 = 3(3k^2)$
$n = 3k + 1$	$n^2 = 3(3k^2 + 2k) + 1$
$n = 3k + 2$	$n^2 = 3(3k^2 + 4k + 1) + 1$

Where it goes wrong

- A finite sample is not exhaustion unless it contains every possible case.
- Checking n=0,1,2 alone is not enough: the expressions involving arbitrary integer k justify all integers.

> ■ **Watch out**
>
> **Not true:** If a claim works for the first hundred integers, it has been proved for all integers. Checking cases proves a claim only when every case is checked; integers go on forever.

Applying it

- Choose a complete case split, such as parity or remainders.
- State why no case is missing.
- For a false universal claim, give a valid input and verify the failure directly.

In your course

- AQA A-level Mathematics (7357), specification section A.
- Every paper allows a calculator, but method marks still need the expression or equation written before its value.

■ **Key idea**

- **Proof by exhaustion**: A proof that checks every possibility in a finite, complete set of cases.
- Write n=3k, 3k+1 or 3k+2 for integer k.
- Watch for: A finite sample is not exhaustion unless it contains every possible case.

A · Irrational roots and infinitely many primes

Why cannot √2 be an exact fraction?

- A calculator prints finitely many digits of √2. Those digits cannot decide whether an exact ratio of integers exists.
- This lesson studies **proof by contradiction** 反证法: A proof that assumes the negation of a claim and derives an impossibility.

The method

- To prove √2 irrational, assume √2=p/q for coprime integers p,q with q nonzero.
- Then $p^2=2q^2$ forces p even; writing p=2r forces q even too.
- This contradicts coprimality.
- To prove infinitely many primes, assume a complete finite prime list and construct a number with no listed prime divisor.

A worked case

$$p^2 = 2q^2, \quad p = 2r \implies q^2 = 2r^2$$

■ **Worked example**

From $p^2=2q^2$ and p=2r, obtain $q^2=2r^2$. Both p and q would have factor 2, contradicting lowest terms. For a supposed complete prime list $p_1,\ldots,p_k$, set $N=p_1\cdots p_k+1$. N>1 has a prime divisor, but every listed prime leaves remainder 1 when dividing N. That divisor is a new prime, contradicting completeness. N itself need not be prime: 2×3×5×7×11×13+1=30031=59×509.

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Assume a complete finite prime list: $p_1, \ldots, p_k$

Construct $N = p_1 \times \ldots \times p_k + 1$

Each listed prime leaves remainder 1

A prime divisor of N is missing → contradiction

Where it goes wrong

- The even-square step needs a reason: an odd integer $2r+1$ has odd square $4r(r+1)+1$.
- The prime proof needs a new prime divisor, not a claim that product-plus-one is always prime.
- A contradiction must attack the stated assumption.

> ■ **Watch out**
>
> **Not true:** The product of any finite list of primes plus one is prime. The product plus one need not be prime: $2\times3\times5\times7\times11\times13+1=30031=59\times509$. It only has a prime factor not in the list.

Applying it

- For an unfamiliar proof, state the contrary assumption precisely, preserve its conditions and name the exact contradiction.
- Decimal approximations or a large finite list do not settle these two claims.

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In your course

- AQA A-level Mathematics (7357), specification section A.
- Every paper allows a calculator, but method marks still need the expression or equation written before its value.

■ Key idea

- **Proof by contradiction**: A proof that assumes the negation of a claim and derives an impossibility.
- To prove √2 irrational, assume √2=p/q for coprime integers p,q with q nonzero.
- Watch for: the even-square step needs a reason: an odd integer 2r+1 has odd square 4r(r+1)+1.

B · Indices, surds and standard form

How small is a microscopic length?

- A microscope records a length of 0.000072 metres. A compact representation must keep its size correct.
- This lesson studies **index** 指数: The power to which a base is raised.

The method

- For the same positive base, multiplication adds indices and division subtracts them.
- A negative index means reciprocal; a fractional index represents a root.
- Standard form has 1≤a<10.

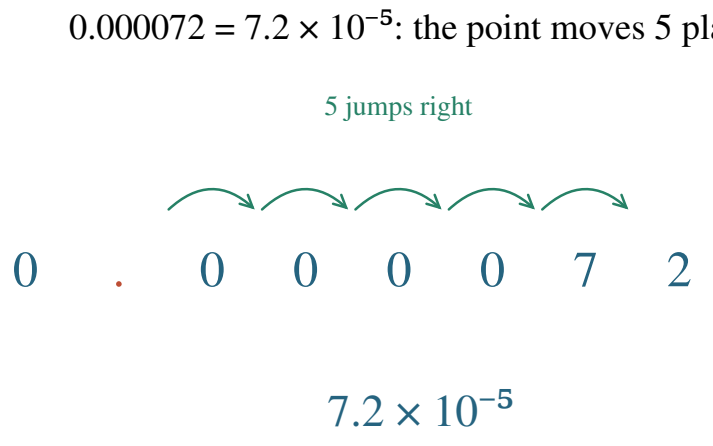
A worked case

$$a^m a^n = a^{m+n}, \quad a^{-n} = \frac{1}{a^n}, \quad a^{m/n} = \left(\sqrt[n]{a}\right)^m$$

■ Worked example

0.000072=7.2×10^(-5). Also 16^(3/4)=(16^(1/4))^3=2^3=8. Simplify √72=6√2, then rationalise 1/√2=√2/2.

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Where it goes wrong

- Index laws do not turn a sum into a single power: 2^3+2^4=24, not 2^7.
- Do not round a surd when an exact answer is requested.

> **Watch out**
>
> **Not true:** For every positive a, a^2+a^3 equals a^5. Index laws only add powers when the terms are multiplied: $2^2+2^3=12$, but $2^5=32$.

Applying it

- Check powers of ten against the original quantity.
- Use surds for exact geometry, and round only the final length when the question asks for a decimal.

In your course

- AQA A-level Mathematics (7357), specification section B.
- Every paper allows a calculator, but method marks still need the expression or equation written before its value.

■ **Key idea**

- **Index**: The power to which a base is raised.
- For the same positive base, multiplication adds indices and division subtracts them.
- Watch for: index laws do not turn a sum into a single power: 2^3+2^4=24, not 2^7.

B · Quadratics and inequalities

Which widths make enough space?

- A rectangular enclosure has area x(10-x). What widths give at least 21 square metres?
- This lesson studies **discriminant** 判别式: The quantity b^2-4ac that determines the real roots of ax^2+bx+c=0.

The method

- Factor where possible; otherwise complete the square or use the quadratic formula.
- A quadratic inequality needs the sign on intervals, not only the roots.
- The discriminant identifies repeated or missing real roots.

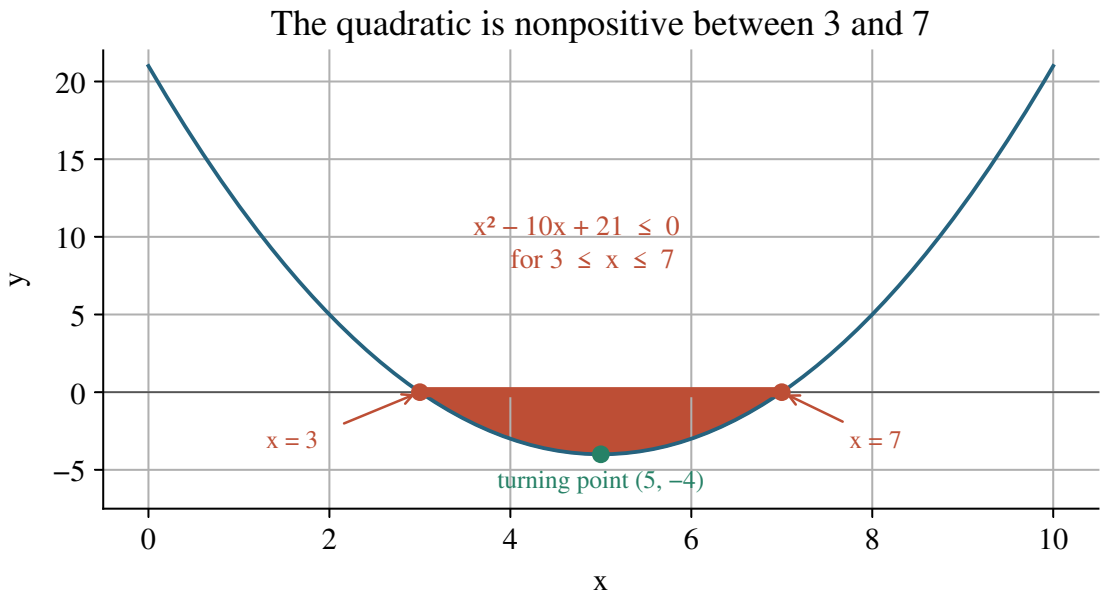
A worked case

$$ax^2 + bx + c = 0, \qquad \Delta = b^2 - 4ac$$

■ **Worked example**

The condition is x(10-x)≥21, so x^2-10x+21≤0. Factor (x-3)(x-7)≤0. The upward parabola is nonpositive between its roots, giving 3≤x≤7. The maximum area is 25 at x=5.

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Where it goes wrong

- Multiplying an inequality by a negative number reverses its direction.
- A sketch must show which side of each root satisfies the inequality.
- Geometry may restrict x further.

> **Watch out**
>
> **Not true:** A positive discriminant means that a quadratic has no real roots. A positive discriminant means two distinct real roots; a negative discriminant means no real roots.

Applying it

- Use the vertex to interpret an optimum.
- Check an endpoint and a point between the roots; the algebra and graph should tell the same story.

In your course

- AQA A-level Mathematics (7357), specification section B.
- Every paper allows a calculator, but method marks still need the expression or equation written before its value.

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■ **Key idea**

- **Discriminant**: The quantity b^2-4ac that determines the real roots of $ax^2+bx+c=0$.
- Factor where possible; otherwise complete the square or use the quadratic formula.
- Watch for: multiplying an inequality by a negative number reverses its direction.

B · Domains, inverses and composition

Which inputs are allowed?

- A square-root model returns a real output only for some inputs. Its formula alone does not specify a complete function.
- This lesson studies **domain** 定义域: The set of allowed inputs to a function.

The method

- State the domain and range.
- For an inverse, first ensure the function is one-to-one on its domain.
- Composition fg means apply g first, then f; the intermediate output must be an allowed input to f.

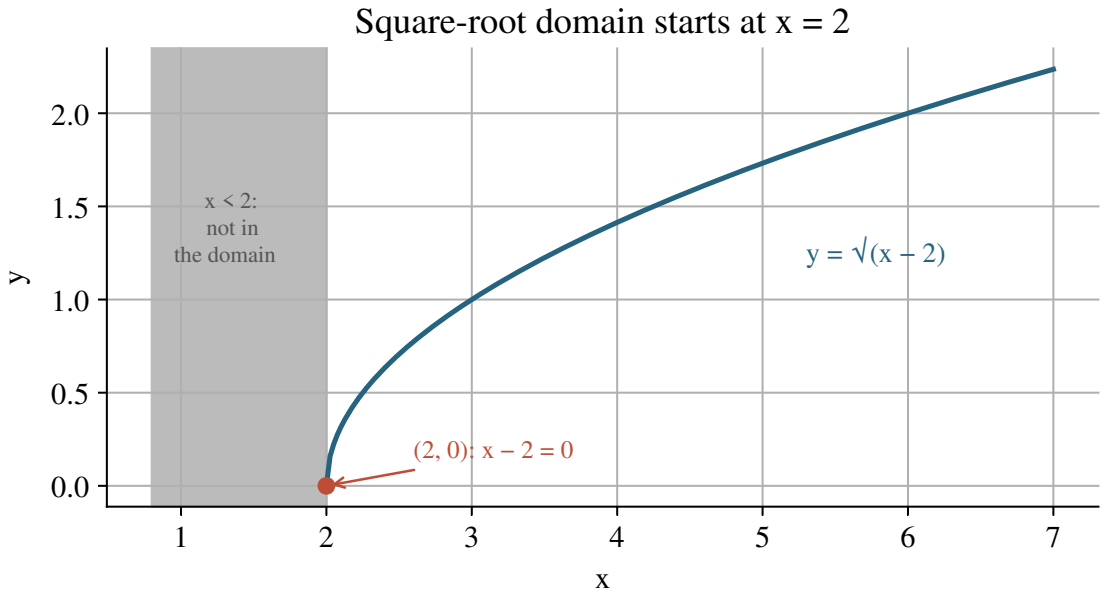
A worked case

$$f(g(x)) = (f \circ g)(x), \qquad f^{-1}(f(x)) = x$$

■ **Worked example**

For f(x)=√(x-2), x≥2 and the range is y≥0. From y=√(x-2), $x=y^2+2$. Thus f inverse(x)=x^2+2 with x≥0. For g(x)=x+3, fg(1)=f(4)=√2.

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Where it goes wrong

- Squaring can introduce extraneous solutions.
- Restricting a parabola's domain is essential before claiming an inverse.
- A horizontal translation inside f has the opposite sign to the graph's movement.

> ■ **Watch out**
>
> **Not true:** Every quadratic function on all real numbers has an inverse function. A quadratic on all real numbers is many-to-one (f(1)=f(−1) for x^2), so its domain must be restricted first.

Applying it

- Check f(f inverse(x))=x on the inverse domain.
- Use a sketch to test whether a horizontal line meets the original graph more than once.

In your course

- AQA A-level Mathematics (7357), specification section B.
- Every paper allows a calculator, but method marks still need the expression or equation written before its value.

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■ **Key idea**

- **Domain**: The set of allowed inputs to a function.
- State the domain and range.
- Watch for: squaring can introduce extraneous solutions.

B · Polynomial division, remainders and factors

How does one root unlock a cubic?

- A cubic graph crosses the axis at a known point. That root lets us reduce a cubic equation to a quadratic rather than guess the other roots.
- This lesson studies **factor theorem** 因式定理: The statement that x−a is a factor of polynomial P(x) exactly when P(a)=0.

The method

- Write P(x)=(x−a)Q(x)+R, with R constant.
- Substituting x=a gives R=P(a).
- Divide leading terms, multiply the whole divisor, subtract with brackets and repeat.
- Include zero coefficients for missing powers.

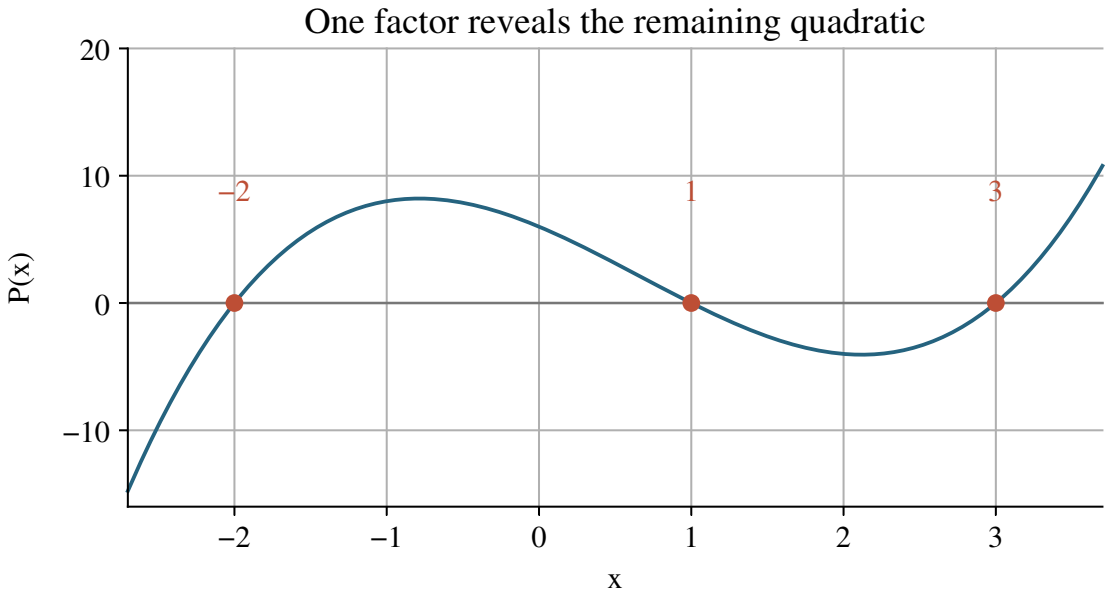
A worked case

$$P(x) = (x - a)Q(x) + P(a)$$

■ **Worked example**

For P(x)=x^3−$2x^2$−5x+6, division by x−1 gives Q(x)=x^2−x−6 and remainder 0. Indeed (x−1)(x^2−x−6)=P(x). Factor Q=(x−3)(x+2), so the roots are 1,3,−2. Dividing the same P by x−2 gives quotient x^2−5 and remainder −4: P=(x−2)(x^2−5)−4. Hence P/(x−2)=x^2−5−4/(x−2), for x≠2.

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Where it goes wrong

- The sign in x−a matters: for x+2 test P(−2).
- A nonzero remainder does not disappear.
- Subtract the entire product at each division step; omitting brackets changes signs.

> ■ **Watch out**
>
> **Not true:** If P(a) is nonzero, x−a is still a factor of P. By the factor theorem, x−a is a factor only when P(a)=0.

Applying it

- Check a quotient by expanding divisor times quotient plus remainder.
- Factorisation can solve P(x)=0, while division of a rational expression must retain every original excluded input.

In your course

- AQA A-level Mathematics (7357), specification section B.
- Every paper allows a calculator, but method marks still need the expression or equation written before its value.

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■ Key idea

- **Factor theorem**: The statement that x−a is a factor of polynomial P(x) exactly when P(a)=0.
- Write P(x)=(x−a)Q(x)+R, with R constant.
- Watch for: the sign in x−a matters: for x+2 test P(−2).

B · Algebraic fractions and excluded values

Cancelling a factor does not put an undefined input back into the original formula.

- A cancelled fraction can look like a straight line and still have one missing point.
- This lesson studies **excluded value** 排除值: An input for which an original denominator or divisor is zero.

The method

- Factor every numerator and denominator before cancelling a common factor.
- A term joined by addition is not a cancellable factor.
- Record exclusions from the original expression first.
- For addition or subtraction use a common denominator, retaining brackets around the entire numerator.
- Multiply factored numerators and denominators.
- To divide, multiply by the reciprocal and exclude inputs making the divisor zero as well as inputs making any original fraction undefined.

A worked case

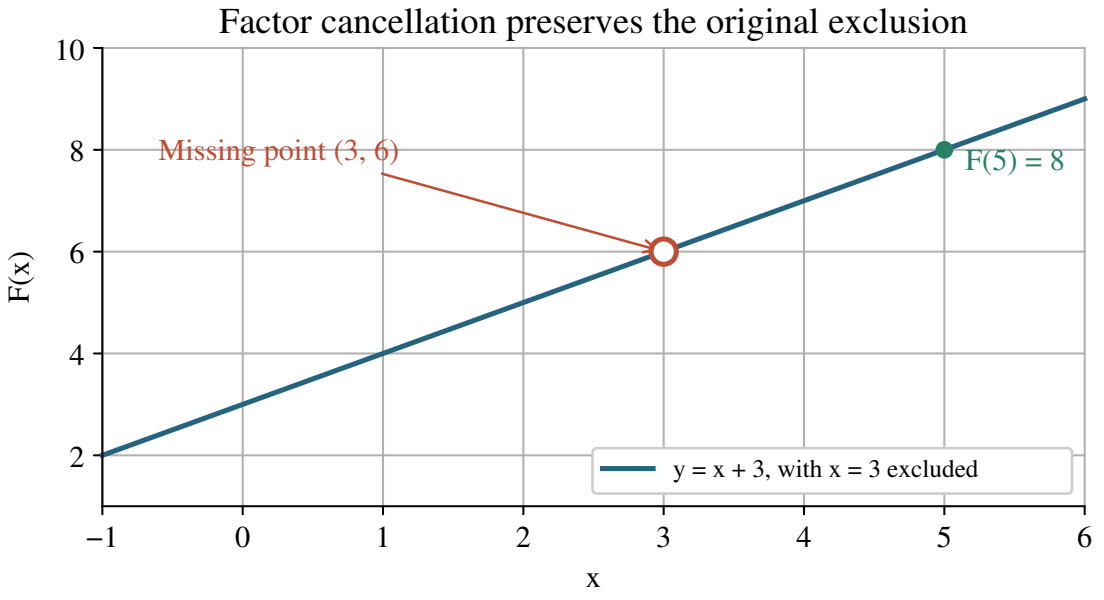
$$\frac{x^2-9}{x-3} = x+3 \quad (x \neq 3)$$

■ Worked example

For F(x)=(x²-9)/(x-3), factor x²-9=(x-3)(x+3). Thus F(x)=x+3 for x≠3; F(5)=8, but F(3) is undefined, not 6. For addition, 1/(x-1)+2/(x+1)=[(x+1)+2(x-1)]/[(x-1)(x+1)]=(3x-1)/(x²-1), with x≠±1. At x=3, both forms give1. For subtraction, the numerator is (x+1)-2(x-1)=3-x, so the minus sign acts on both terms. Multiplication [(x-1)/(x+2)]×[(x+2)/(x+1)] gives(x-1)/(x+1), but the original excludes x=-2 and x=-1. Division [(x²-1)/(x²+3x+2)]÷[(x-1)/(x+2)] gives1, yet x=-2,-1,1 are excluded: x=1 makes the divisor zero. Finally F(x)=6 reduces to x+3=6. Its only candidate

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x=3 is forbidden, so this equation has no solution. Check candidates in the original equation before reporting them.



Where it goes wrong

- Never cancel the x in (x+2)/x.
- A simplified denominator does not show all original restrictions.
- In a division task, check when the divisor is zero.
- Cross-multiplication can produce an excluded candidate; a formal root is not automatically a solution.

■ Watch out

Not true: Cancelling a denominator factor makes its excluded input valid. An input that makes the original denominator zero stays excluded, even after cancelling.

Applying it

- 7357 B6 covers rational-expression manipulation, cancellation and algebraic division by linear expressions.
- State the original restrictions, including zero-divisor exclusions, and check every equation candidate in the original domain.
- Combine this lesson with polynomial division for an improper rational expression.

In your course

- AQA A-level Mathematics (7357), specification section B.
- Every paper allows a calculator, but method marks still need the expression or equation written before its value.

Key idea

- **Excluded value**: An input for which an original denominator or divisor is zero.
- Factor every numerator and denominator before cancelling a common factor.
- Watch for: never cancel the x in (x+2)/x.

B · Partial fractions with distinct linear factors

Can one fraction become a sum of simpler fractions?

- A product of two linear denominators can hide two simpler contributions. Separating them makes later integration and equation work easier.
- This lesson studies **partial fraction** 部分分式: A simpler fraction in a sum that equals a rational expression on its original domain.

The method

- For proper rational expressions, use A/(x+a)+B/(x+b) for distinct linear factors.
- Clear the denominator to obtain a polynomial identity, then substitute convenient roots or compare coefficients.
- Divide first if the numerator degree is not smaller.

A worked case

$$\frac{3x+5}{(x+1)(x+2)} = \frac{2}{x+1} + \frac{1}{x+2}$$

Worked example

For (3x+5)/[(x+1)(x+2)], write 3x+5=A(x+2)+B(x+1). At x=−1, A=2; at x=−2, B=1. Thus the decomposition is 2/(x+1)+1/(x+2), with x≠−1,−2. With three factors, (3x+1)/[x(x+1)(x+2)]=1/(2x)+2/(x+1)−5/[2(x+2)]. Clearing denominators and using x=0,−1,−2 gives A=1/2, B=2, C=−5/2.

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$(3x + 5)/[(x + 1)(x + 2)]$

$3x + 5 = A(x + 2) + B(x + 1)$

$A = 2$, $B = 1$; exclude $x = -1, -2$

Where it goes wrong

- The roots used after clearing denominators are allowed in the resulting polynomial identity, not in the original fraction.
- Retain all excluded inputs.
- A coefficient can be negative or zero.

> ■ **Watch out**
>
> **Not true:** Once denominators are cleared, their excluded inputs become valid inputs of the original fraction. Clearing denominators does not change the domain; inputs that make a denominator zero stay excluded.

Applying it

- Check by recombining all terms and comparing coefficients, not just one numerical input.
- In 7357, numerators here are constant or linear and the decomposition has no more than three terms.

In your course

- AQA A-level Mathematics (7357), specification section B.
- Every paper allows a calculator, but method marks still need the expression or equation written before its value.

Key idea

- **Partial fraction**: A simpler fraction in a sum that equals a rational expression on its original domain.
- For proper rational expressions, use A/(x+a)+B/(x+b) for distinct linear factors.
- Watch for: the roots used after clearing denominators are allowed in the resulting polynomial identity, not in the original fraction.

B · Repeated linear factors in partial fractions

Why does a squared factor need two terms?

- A squared factor needs two separate terms. Keeping only the squared term can leave too few coefficients to reproduce the numerator.
- This lesson studies **repeated factor** 重因式: A denominator factor occurring more than once, requiring a term for each power.

The method

- For a repeated factor $(x+a)^2$, include $A/(x+a)+B/(x+a)^2$.
- With a distinct factor x+b, add C/(x+b).
- Clear denominators, find convenient coefficients and use another input or coefficient comparison for the remaining unknown.

A worked case

$$\frac{2x+3}{(x+1)^2(x+2)} = \frac{1}{x+1} + \frac{1}{(x+1)^2} - \frac{1}{x+2}$$

Worked example

For $(2x+3)/[(x+1)^2(x+2)]$, the identity is $2x+3=A(x+1)(x+2)+B(x+2)+C(x+1)^2$. At $x=-1$, $B=1$. At $x=-2$, $C=-1$. The x^2 coefficient gives $A+C=0$, hence $A=1$. The result is $1/(x+1)+1/(x+1)^2-1/(x+2)$, with $x\neq-1,-2$. At $x=0$ both sides are $3/2$; recombination gives numerator $2x+3$ for every allowed x.

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$(2x + 3)/[(x + 1)^2(x + 2)]$

$A/(x + 1) + B/(x + 1)^2 + C/(x + 2)$

$B = 1, C = -1, A = 1$

Both powers are needed; exclude $x = -1, -2$

Where it goes wrong

- Do not omit the first-power term or place Ax+B over each linear factor.
- A numerical agreement at one point is only a check, not an identity proof.
- Keep the square on its own denominator.

> ■ **Watch out**
>
> **Not true:** For every denominator $(x+1)^2(x+2)$, one term over $(x+1)^2$ and one over $x+2$ are enough. A repeated factor needs terms over both $(x+1)$ and $(x+1)^2$, plus one over $x+2$.

Applying it

- 7357 permits squared linear factors and at most three partial-fraction terms.
- Verify all numerator coefficients after recombination.
- Higher powers and irreducible quadratic factors are outside this lesson's specification scope.

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In your course

- AQA A-level Mathematics (7357), specification section B.
- Every paper allows a calculator, but method marks still need the expression or equation written before its value.

■ Key idea

- **Repeated factor**: A denominator factor occurring more than once, requiring a term for each power.
- For a repeated factor $(x+a)^2$, include $A/(x+a)+B/(x+a)^2$.
- Watch for: do not omit the first-power term or place $Ax+B$ over each linear factor.

B · Linear and quadratic simultaneous equations

How many places can a line meet a parabola?

- A line can cut a parabola twice, touch it once or miss it. An algebraic solution should give the same number of real intersection points as the sketch.
- This lesson studies **simultaneous solution** 联立解: An ordered pair that satisfies every equation in a system at the same time.

The method

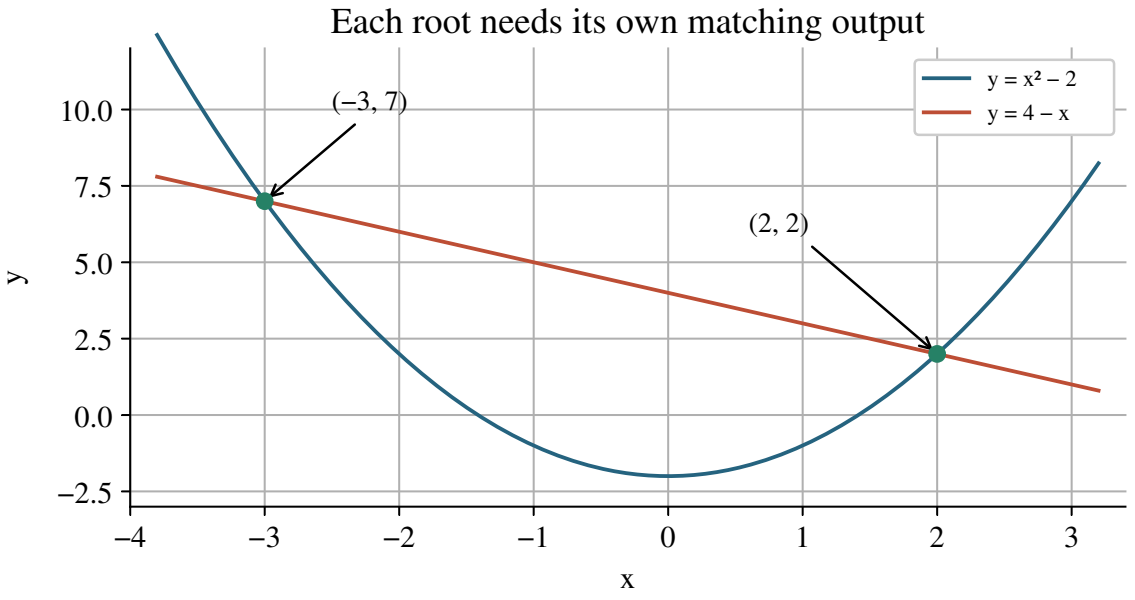
- For two linear equations, eliminate one unknown or substitute an expression for it.
- For one linear and one quadratic equation, express y from the line and substitute into the quadratic.
- Solve every resulting root, find its matching y and check both original equations.

A worked case

$$x + y = 4, \quad y = x^2 - 2 \implies (x - 2)(x + 3) = 0$$

■ Worked example

For $3x+2y=13$ and $x-y=1$, add twice the second equation to the first: $5x=15$, so $x=3$ and $y=2$. For $x+y=4$ and $y=x^2-2$, substitute $y=4-x$: $x^2+x-6=0=(x-2)(x+3)$. The solutions are $(2,2)$ and $(-3,7)$. The line $y=4x-6$ instead gives $(x-2)^2=0$, so only $(2,2)$: a tangent. The line $y=4x-7$ gives $x^2-4x+5=0$ with discriminant -4, so no real intersection.



Where it goes wrong

- A root for x is not a complete simultaneous solution.
- Pair each x with its own y; do not combine one root with another root's output.
- A repeated quadratic root gives one point, not two different points.

> ■ **Watch out**
>
> **Not true:** Every system of two equations has exactly one real simultaneous solution. Two equations can have no solution, one, or several; a line and a circle can meet twice or not at all.

Applying it

- Use the discriminant to distinguish two, one or no real intersections.
- The sketch helps predict the result, but substitutions into both original equations verify it.
- Preserve any restrictions if a more general system involves fractions.

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In your course

- AQA A-level Mathematics (7357), specification section B.
- Every paper allows a calculator, but method marks still need the expression or equation written before its value.

■ Key idea

- **Simultaneous solution**: An ordered pair that satisfies every equation in a system at the same time.
- For two linear equations, eliminate one unknown or substitute an expression for it.
- Watch for: A root for x is not a complete simultaneous solution.

B · Rational indices and real domains

Why does its edge grow by only a factor of 3?

- A cube grows to 27 times its original volume.
- This lesson studies **rational exponent** 有理指数: An exponent written as a fraction of integers.

The method

- For a positive base a, a^(p/q) is the qth root of a raised to p, with q positive.
- A negative exponent takes the reciprocal, so the base must be nonzero.
- Reduce p/q first.
- For a negative real base, a reduced odd denominator permits a real root; an even denominator does not.

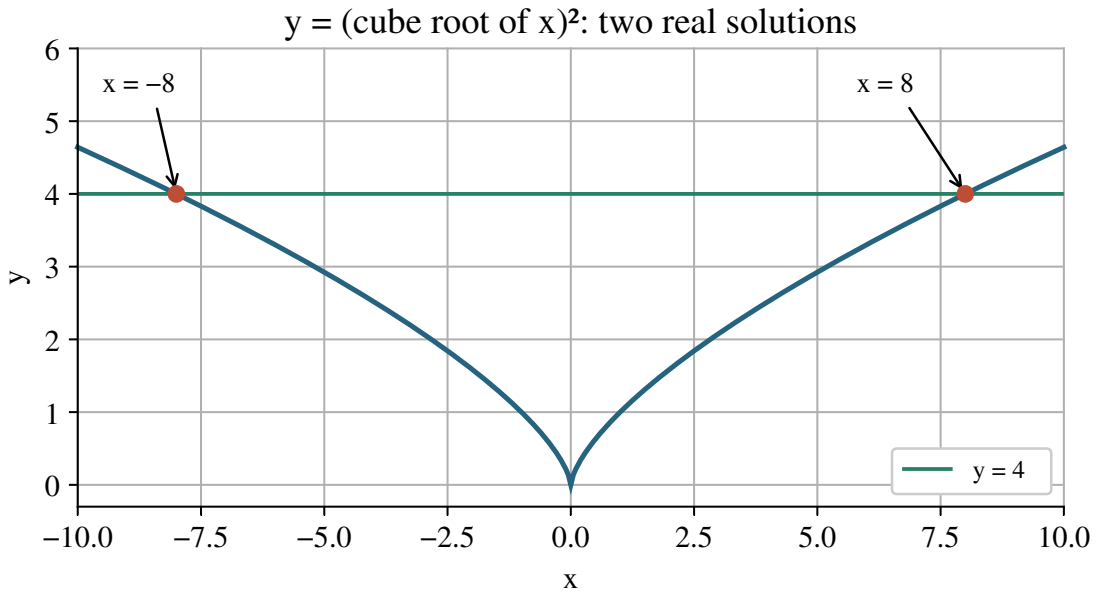
A worked case

$$a^{-p/q} = \frac{1}{(\sqrt[q]{a})^p} \quad (a > 0)$$

■ Worked example

27^(2/3)=(cube root of 27)^2=9, while 16^(−3/4)=1/(fourth root of 16)^3=1/8. For x^(2/3)=4 over the real numbers, let u be the cube root of x: u^2=4, so u=±2 and x=±8. Both inputs check. By contrast x^(1/2)=−2 has no real solution because the principal square root is nonnegative. For negative bases, (−8)^(1/3)=−2 and (−8)^(2/3)=4.

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Where it goes wrong

- Do not change a negative exponent into a negative answer.
- Do not discard a negative solution when an odd root is squared.
- General power-of-a-power manipulations can fail outside positive bases: $((-1)^2)^{(1/2)}=1$, whereas $(-1)^{(2\times1/2)}=-1$.

> ■ **Watch out**
>
> **Not true:** The equation $x^{(2/3)}=4$ has only the solution $x=8$. $x^{(2/3)}=4$ means $(x^{(1/3)})^2=4$, so the cube root is ±2 and $x=8$ or $x=-8$.

Applying it

- Use a positive scale factor for lengths and volumes: a volume ratio 27 gives edge ratio $27^{(1/3)}=3$ and area ratio $27^{(2/3)}=9$.
- In equations, state the real domain and substitute every candidate.
- Zero to a negative power is undefined; this lesson does not assign a value to 0^0.

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In your course

- AQA A-level Mathematics (7357), specification section B.
- Every paper allows a calculator, but method marks still need the expression or equation written before its value.

■ Key idea

- **Rational exponent**: An exponent written as a fraction of integers.
- For a positive base a, a^(p/q) is the qth root of a raised to p, with q positive.
- Watch for: do not change a negative exponent into a negative answer.

B · Surds and conjugate denominators

Can we remove the radical from its denominator without rounding?

- An exact length is $1/(\sqrt{5}-2)$.
- This lesson studies **conjugate** 共轭式: A paired expression with the sign between two terms reversed.

The method

- For nonnegative a and b, $\sqrt{a}\times\sqrt{b}=\sqrt{(ab)}$, but $\sqrt{(a+b)}$ is generally not $\sqrt{a}+\sqrt{b}$.
- Extract square factors to simplify surds.
- For a two-term denominator, multiply numerator and denominator by its conjugate: $(a+b\sqrt{c})(a-b\sqrt{c})=a^2-b^2c$.
- The original denominator and the conjugate multiplier must both be nonzero.

A worked case

$$(a + b\sqrt{c})(a - b\sqrt{c}) = a^2 - b^2c$$

■ Worked example

$\sqrt{72}-\sqrt{8}=6\sqrt{2}-2\sqrt{2}=4\sqrt{2}$. For $1/(\sqrt{5}-2)$, multiply by $(\sqrt{5}+2)/(\sqrt{5}+2)$: the denominator is $5-4=1$, giving $\sqrt{5}+2$. For $3/(2+\sqrt{3})$, multiply by $2-\sqrt{3}$ to get $3(2-\sqrt{3})/(4-3)=6-3\sqrt{3}$. Recombine with the original denominator to verify the exact result.

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The numerator changes too; the exact value stays the same

Original exact fraction

$$\frac{1}{\sqrt{5}-2}$$

Multiply by the conjugate

$$\frac{1}{\sqrt{5}-2}\times\frac{\sqrt{5}+2}{\sqrt{5}+2}$$

Difference of squares

$$\frac{\sqrt{5}+2}{5-4}=\sqrt{5}+2$$

Where it goes wrong

- The conjugate changes one sign, not every sign.
- Multiply the numerator too.
- Squaring a sum produces a cross term: $(2+\sqrt{3})^2=7+4\sqrt{3}$.
- For real x, $\sqrt{(x^2)}=|x|$, so $\sqrt{((-3)^2)}=3$.

Watch out

Not true: For every real x, $\sqrt{(x^2)}=x$. $\sqrt{(x^2)}=|x|$: for $x=-3$ it gives 3, not -3.

Applying it

- Keep exact surd expressions until a decimal is requested.
- Combine like surds only after simplification: $\sqrt{18}+\sqrt{8}=5\sqrt{2}$, but $\sqrt{2}+\sqrt{3}$ cannot be combined into one simple surd.
- If the conjugate is zero, this multiplier is invalid: simplify the original denominator directly, then check whether that denominator is zero.
- For example 1/(2+sqrt(4))=1/4 is valid even though its proposed conjugate is zero.

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In your course

- AQA A-level Mathematics (7357), specification section B.
- Every paper allows a calculator, but method marks still need the expression or equation written before its value.

■ Key idea

- **Conjugate**: A paired expression with the sign between two terms reversed.
- For nonnegative a and b, $\sqrt{a}\times\sqrt{b}=\sqrt{(ab)}$, but $\sqrt{(a+b)}$ is generally not $\sqrt{a}+\sqrt{b}$.
- Watch for: the conjugate changes one sign, not every sign.

B · Completing the square and quadratic structure

Which form tells us where it happens without drawing every point?

- A quadratic model reaches a lowest value.
- This lesson studies **completed square** 配方形式: A quadratic written as a multiple of a squared bracket plus a constant.

The method

- For x^2+bx, add and subtract $(b/2)^2$ to obtain $(x+b/2)^2-(b/2)^2$.
- If the leading coefficient a is not 1, factor a from the x^2 and x terms first.
- The form $a(x-h)^2+k$ gives vertex (h,k), axis x=h, and a minimum k if $a>0$ or a maximum k if $a<0$.

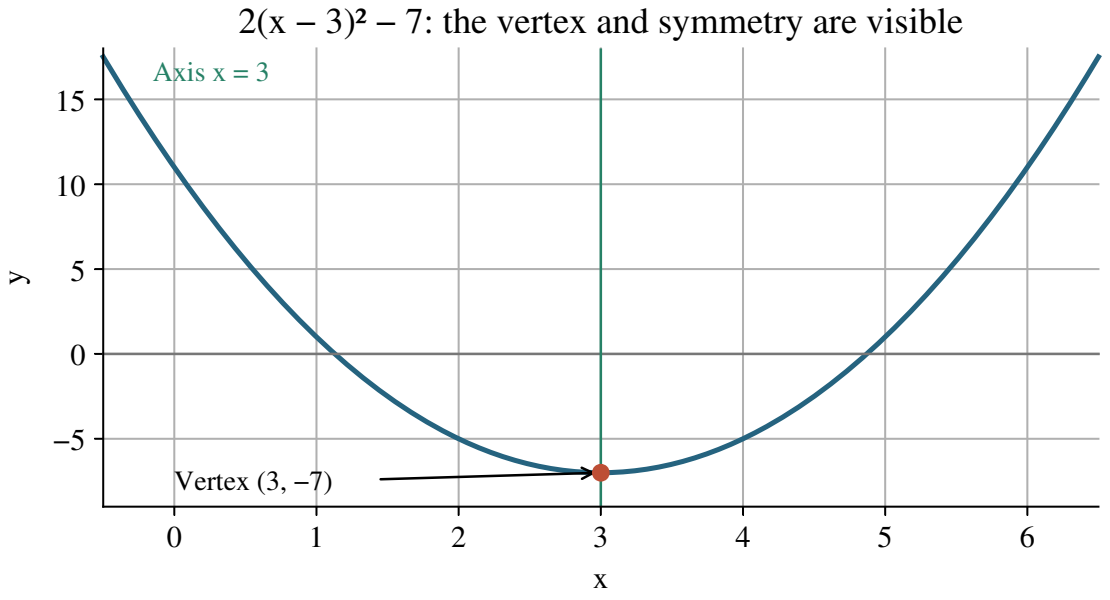
A worked case

$$2x^2 - 12x + 11 = 2(x - 3)^2 - 7$$

■ Worked example

$2x^2-12x+11=2(x^2-6x)+11=2[(x-3)^2-9]+11=2(x-3)^2-7$. Its minimum is -7 at x=3. To solve $2x^2-12x+11=0$, obtain $(x-3)^2=7/2$, so $x=3\pm\sqrt{(7/2)}$. The roots lie equally far from x=3. In general $ax^2+bx+c=0$ becomes $(x+b/(2a))^2=(b^2-4ac)/(4a^2)$, for $a\neq0$, leading to $x=(-b\pm\sqrt{(b^2-4ac)})/(2a)$. A negative discriminant means no real roots; zero gives one repeated root.

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Where it goes wrong

- Half the coefficient inside the factored bracket, not the original coefficient of x.
- Keep the outside multiplier when subtracting the added square.
- A positive squared bracket does not imply the whole quadratic is positive: its constant can be negative.

> ■ **Watch out**
>
> **Not true:** A completed-square expression represents a quadratic with a minimum. When the coefficient of x^2 is negative the parabola has a maximum, not a minimum.

Applying it

- For the same model, $y \leq 1$ becomes $2(x-3)^2 \leq 8$, giving $1 \leq x \leq 5$.
- Check both endpoints.
- A contextual restriction such as $x \geq 0$ must be combined with this interval.
- For $y = -2(x-3)^2 + 7$, the same vertex x-coordinate gives a maximum instead of a minimum.

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In your course

- AQA A-level Mathematics (7357), specification section B.
- Every paper allows a calculator, but method marks still need the expression or equation written before its value.

■ Key idea

- **Completed square**: A quadratic written as a multiple of a squared bracket plus a constant.
- For x^2+bx, add and subtract $(b/2)^2$ to obtain $(x+b/2)^2-(b/2)^2$.
- Watch for: half the coefficient inside the factored bracket, not the original coefficient of x.

B · Equations quadratic in another expression

Solving for the new variable is only half the job.

- A fourth-degree equation can hide an ordinary quadratic.
- This lesson studies **substitution** 代换: Replacing a repeated expression with a new variable to reveal a simpler equation.

The method

- Identify a repeated expression and write u for it, including any restrictions.
- Solve the resulting quadratic in u.
- Then solve the original expression equal to each allowed u value.
- Substitute every final candidate into the original equation; a solution for u is not yet a solution for x.

A worked case

$$x^4 - 5x^2 + 4 = 0 \quad \Longrightarrow \quad u^2 - 5u + 4 = 0, \quad u = x^2 \geq 0$$

■ Worked example

For $x^4-5x^2+4=0$, set $u=x^2$ with $u\geq 0$. Then $u^2-5u+4=(u-1)(u-4)=0$, so $x^2=1$ or 4 and $x=\pm 1,\pm 2$. For $x^4+x^2-2=0$, the u-equation is $(u-1)(u+2)=0$: $u=-2$ is impossible for real x, leaving $x=\pm 1$. For $(x+1)^2-5(x+1)+6=0$, set $u=x+1$: $u=2$ or 3 gives $x=1$ or 2. For $3^{\wedge}(2x)-4\times 3^{\wedge}x+3=0$, set $u=3^{\wedge}x>0$: $u=1$ or 3 gives $x=0$ or 1.

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Two allowed u values give four distinct real x values

$$x^4 - 5x^2 + 4 = 0$$

$$u = x^2 \geq 0 : \quad (u - 1)(u - 4) = 0$$

$$u = 1 \Rightarrow x = -1, 1$$

$$u = 4 \Rightarrow x = -2, 2$$

Where it goes wrong

- Do not report u=1 and u=4 as the x solutions of the quartic.
- Taking a square root needs both signs, but a principal square-root expression is nonnegative.
- A quadratic in u can have two real roots while one or both violate u's domain.

> ■ **Watch out**
>
> **Not true:** Two roots in the substituted variable give exactly two real solutions in x. Each root in the new variable must be checked: $u=x^2$ with $u<0$ gives no real x, and $u>0$ gives two.

Applying it

- Choose a substitution that repeats exactly: $3^{(2x)}=(3^x)^2$, while $3^{(x^2)}$ is a different expression.
- Count distinct real x solutions after returning to x, not before.
- Check by direct substitution: $x=-2$ gives $16-20+4=0$ in the first quartic.

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In your course

- AQA A-level Mathematics (7357), specification section B.
- Every paper allows a calculator, but method marks still need the expression or equation written before its value.

■ Key idea

- **Substitution**: Replacing a repeated expression with a new variable to reveal a simpler equation.
- Identify a repeated expression and write u for it, including any restrictions.
- Watch for: do not report u=1 and u=4 as the x solutions of the quartic.

B · Inequalities, fractions and solution sets

How can we keep the allowed interval without losing a boundary?

- A value must satisfy two conditions at once.
- This lesson studies **intersection** 交集: The values common to two solution sets, corresponding to AND.

The method

- Adding the same expression preserves an inequality; multiplying or dividing by a negative number reverses it.
- Clear constant denominators using a positive common multiple.
- Factor a quadratic and test the sign on the intervals separated by its roots.
- AND takes the intersection of sets; OR takes their union.
- Distinguish included and excluded endpoints.

A worked case

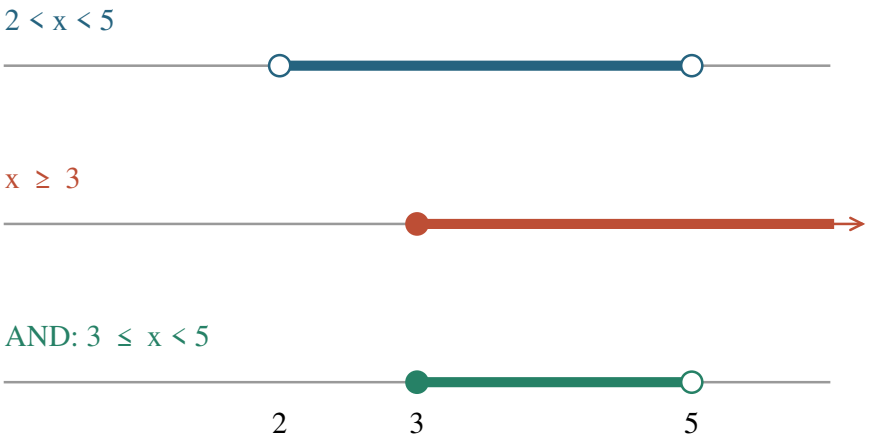
$$(x-2)(x-5) < 0 \ \wedge \ x \geq 3 \quad \Longrightarrow \quad x \in [3,5)$$

■ Worked example

For (2x−1)/3−(x+2)/2≥1/6, multiply by 6: 4x−2−3x−6≥1, so x≥9. For (x−2)(x−5)<0, the product is negative between its roots: 2<x<5. Combining with x≥3 gives 3≤x<5, or [3,5). Taking OR with x≤0 gives (−∞,0]∪[3,5). A denominator depending on x needs a separate sign/domain check: (x−1)/(x+2)≥0 has x<−2 or x≥1. The zero numerator at 1 is allowed; x=−2 is undefined.

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AND keeps the overlap; open endpoints stay excluded



Where it goes wrong

- Do not multiply by an expression of unknown sign as though it were positive.
- For $-2x<6$, dividing by -2 gives $x>-3$.
- A root can be included only for a non-strict inequality and only where the expression is defined.
- Infinity is never a real endpoint to include.

> ■ **Watch out**
>
> **Not true:** AND means combine every value from both sets. AND means the values in both sets at once, the intersection; OR combines them.

Applying it

- Use a sign table or test value on every interval, then write the whole set.
- The variable-denominator example extends the domain-checking method; B5's explicit core is linear/quadratic inequalities with brackets and fractions, and combined sets.
- Check $x=4$ and $x=6$ in the quadratic case to confirm which side is allowed.

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In your course

- AQA A-level Mathematics (7357), specification section B.
- Every paper allows a calculator, but method marks still need the expression or equation written before its value.

■ Key idea

- **Intersection**: The values common to two solution sets, corresponding to AND.
- Adding the same expression preserves an inequality; multiplying or dividing by a negative number reverses it.
- Watch for: do not multiply by an expression of unknown sign as though it were positive.

B · Regions between linear and quadratic boundaries

Which points satisfy both boundaries?

- A point can lie above a parabola but still fail a second condition.
- This lesson studies **boundary** 边界: The line or curve separating points that satisfy an inequality from those that do not.

The method

- Draw each equality boundary first.
- Use a solid line or curve for $\leq$ or $\geq$ and a dashed one for $<$ or $>$.
- For y greater than a function, shade above its graph; for y less than it, shade below.
- For AND, keep only the overlap.
- Test a point away from every boundary to confirm the chosen side.

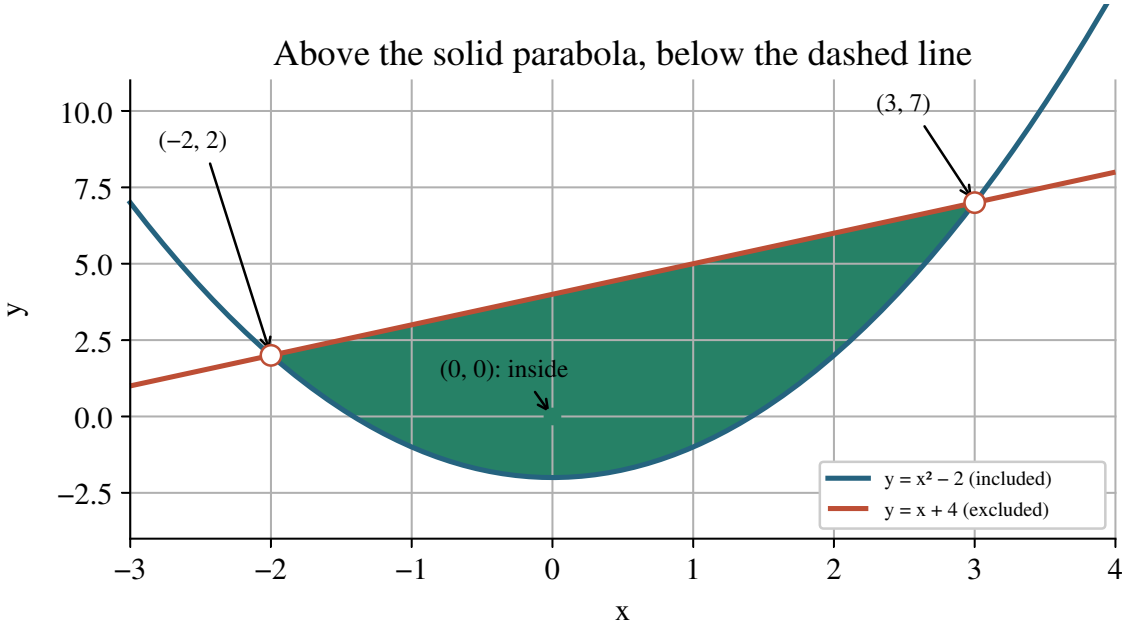
A worked case

$$x^2 - 2 \leq y < x + 4, \qquad -2 < x < 3$$

■ Worked example

For $y \geq x^2 - 2$ AND $y < x + 4$, draw the parabola solid and the line dashed. The boundaries meet when $x^2 - 2 = x + 4$: $(x-3)(x+2) = 0$, giving $(-2,2)$ and $(3,7)$. Between $-2 < x < 3$, the line is above the parabola. The allowed points satisfy $x^2 - 2 \leq y < x + 4$. At $x = 0$, the vertical interval is $-2 \leq y < 4$. The point $(0,0)$ is included, $(0,4)$ is excluded and $(0,-3)$ is excluded. The intersection points are excluded because the line inequality is strict.

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Where it goes wrong

- A dashed boundary is not part of the region, even where it meets a solid boundary.
- Solving for intersection x-values gives the horizontal extent, not all allowed coordinates.
- A region for OR is the union, which can be much larger than the overlap.

■ Watch out

Not true: Every intersection point is included if either boundary is solid. A point is included only if it satisfies every inequality, so a dashed boundary excludes it.

Applying it

- For a different inequality such as $y>x^2-2$, the permitted side is above the parabola and the curve itself is excluded.
- If a context also requires $x\geq 0$, clip the already combined region to that half-plane.
- Verify an interior point and a point outside each boundary separately.

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In your course

- AQA A-level Mathematics (7357), specification section B.
- Every paper allows a calculator, but method marks still need the expression or equation written before its value.

■ Key idea

- **Boundary**: The line or curve separating points that satisfy an inequality from those that do not.
- Draw each equality boundary first.
- Watch for: A dashed boundary is not part of the region, even where it meets a solid boundary.

B · Modulus graphs and piecewise solutions

Values on either side give the same distance, but their algebraic signs differ.

- A sensor measures distance from a target value.
- This lesson studies **modulus** 绝对值: The nonnegative distance of a real number from zero.

The method

- For a real expression u, $|u|=u$ when $u\geq 0$ and $|u|=-u$ when $u<0$.
- Find where a linear expression is zero to locate the corner of its V-shaped modulus graph.
- Solve equations by checking both branches and their domains; the right-hand side must be nonnegative.

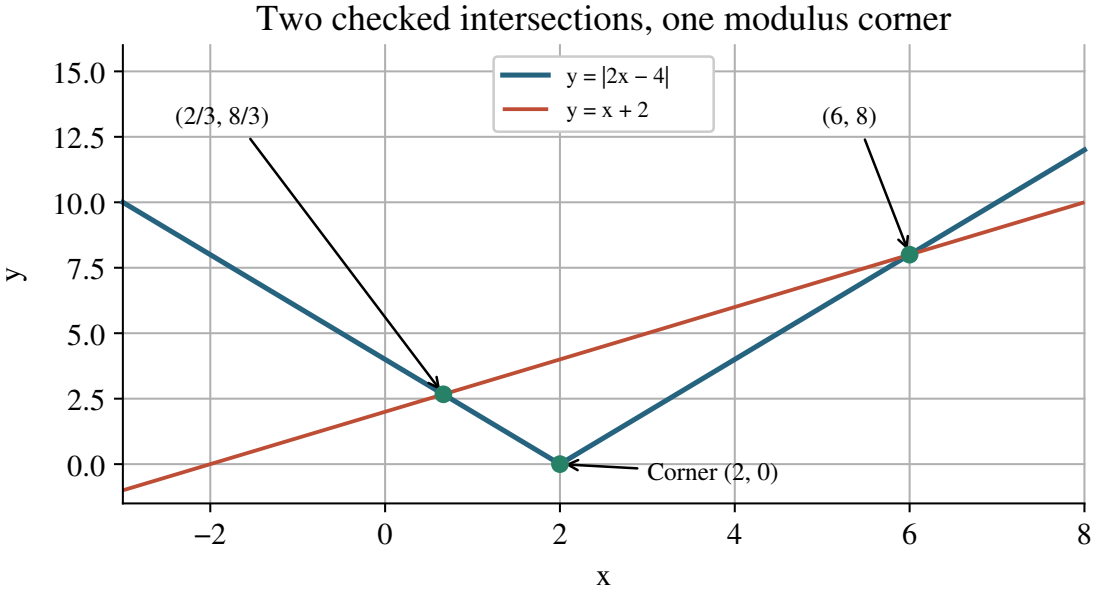
A worked case

$$|2x-4| = \begin{cases} 4-2x, & x<2 \\ 2x-4, & x\geq 2 \end{cases}$$

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■ Worked example

For $y=|2x-4|$, the corner is $(2,0)$. The left branch is $y=4-2x$ for $x<2$; the right is $y=2x-4$ for $x\geq 2$. To solve $|2x-4|=x+2$, require $x\geq -2$. The right branch gives $2x-4=x+2$, hence $x=6$. The left gives $4-2x=x+2$, hence $x=2/3$. Both satisfy their branch restrictions and the original equation. The graph meets the line at $(6,8)$ and $(2/3,8/3)$.



Where it goes wrong

- Do not replace $|u|$ by u on a negative branch.
- Squaring an equation can create candidates where its right-hand side is negative.
- The equation $|2x-4|=-1$ has no real solutions.
- The graph of $|2x-4|$ has one corner and its slopes change there.

■ Watch out

Not true: Every modulus equation has exactly two real solutions. A modulus equation can have no solution, one or two; $|x|=-1$ has none.

Applying it

- For $|2x-4|<6$, use $-6<2x-4<6$ to obtain $-1<x<5$.
- For $|2x-4|>6$, use $2x-4<-6$ OR $2x-4>6$: $x<-1$ or $x>5$.

- A bounded distance gives an inside interval; an excessive distance gives two outside intervals.
- Test both boundary and interior values.

In your course

- AQA A-level Mathematics (7357), specification section B.
- Every paper allows a calculator, but method marks still need the expression or equation written before its value.

■ Key idea

- **Modulus**: The nonnegative distance of a real number from zero.
- For a real expression u, |u|=u when u≥0 and |u|=−u when u<0.
- Watch for: do not replace |u| by u on a negative branch.

B · Reciprocal graphs and asymptotes

Zero is a missing input, not a point to join across.

- An inverse model grows very large as its input approaches zero.
- This lesson studies **asymptote** 渐近线: A line that a curve approaches in a limiting direction.

The method

- For nonzero a, y=a/x has domain x≠0 and range y≠0, with asymptotes x=0 and y=0.
- Positive a gives branches in quadrants I and III; negative a gives II and IV.
- For $y=a/x^2$ the domain is still x≠0, but the graph is symmetric about the y-axis; both branches are positive if a>0 and negative if a<0.

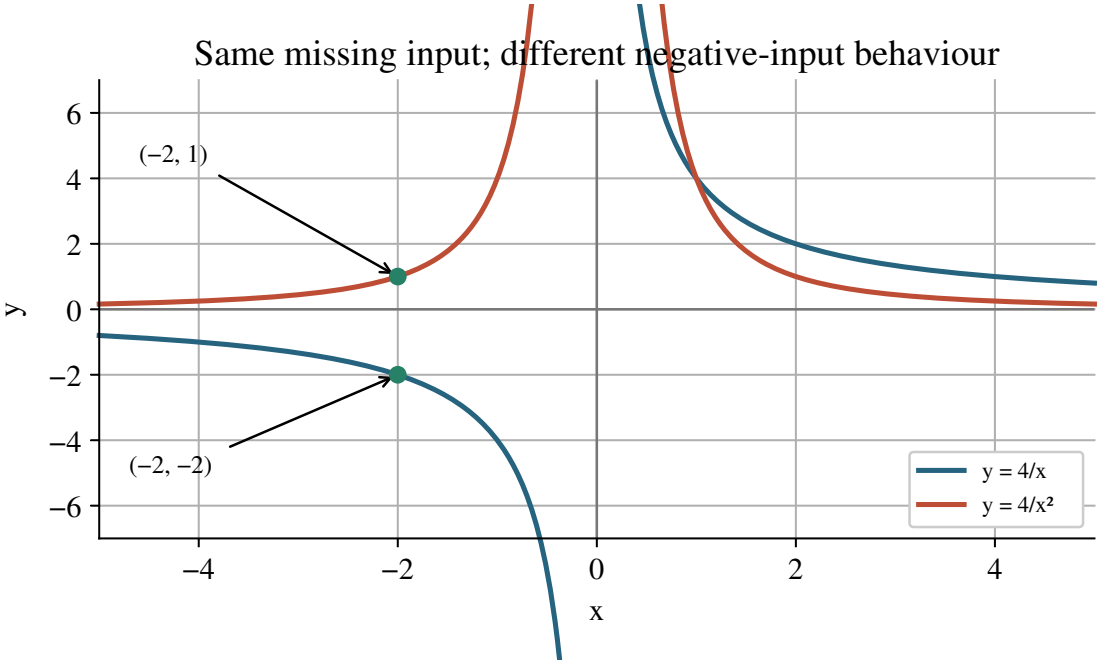
A worked case

$$y = \frac{4}{x} \quad \text{or} \quad y = \frac{4}{x^2}, \qquad x \neq 0$$

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Worked example

For y=4/x, checked points are (1,4), (2,2), (−1,−4), (−2,−2). As x approaches zero from the right, y grows positively; from the left it grows negatively. As |x| grows, y approaches zero. For $y=4/x^2$, (1,4) and (−1,4) have equal outputs, and both sides rise positively near zero. Intersecting y=4/x with y=x gives $x^2=4$, so (2,2) and (−2,−2). Intersecting $y=4/x^2$ with y=1 gives $x^2=4$, so (2,1) and (−2,1).



Where it goes wrong

- Never plot x=0 or draw a stroke joining the two branches through it.
- The reciprocal-square graph is not the reciprocal graph with the negative branch removed; the entire left branch is reflected above the x-axis for positive a.
- These particular curves do not meet y=0, but “no curve can ever cross any asymptote” is not a valid general rule.

Watch out

Not true: The graphs y=4/x and $y=4/x^2$ have the same outputs for every negative x. For negative x, 4/x is negative but $4/x^2$ is positive.

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Applying it

- For positive physical inputs, y=a/x can model inverse proportion and $y=a/x^2$ inverse-square proportion.
- If y=4/x represents a time and speed relation, the negative branch is mathematically valid but outside that model's physical domain.
- Doubling a positive x halves 4/x and quarters $4/x^2$.
- Check graph intersections in the original relation after clearing a nonzero denominator.

In your course

- AQA A-level Mathematics (7357), specification section B.
- Every paper allows a calculator, but method marks still need the expression or equation written before its value.

■ Key idea

- **Asymptote**: A line that a curve approaches in a limiting direction.
- For nonzero a, y=a/x has domain $x\neq0$ and range $y\neq0$, with asymptotes x=0 and y=0.
- Watch for: never plot x=0 or draw a stroke joining the two branches through it.

B · Inverse graphs and restricted branches

Which branch must we choose before reversing it?

- A square model can give the same output from two inputs.
- This lesson studies **one-to-one** 一一对应: A function for which different inputs always give different outputs.

The method

- An inverse reverses the input-output mapping, so the original function must be one-to-one on its stated domain.
- A horizontal line should meet its graph at most once.
- Swap x and y, solve for the new output and retain the correct branch.
- The inverse graph reflects the original in y=x: each point (a,b) becomes (b,a).
- Original domain and range exchange roles.

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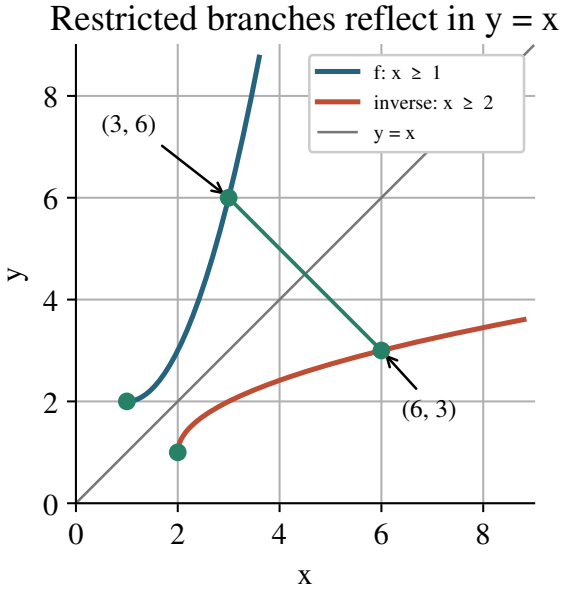
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A worked case

$$f(x) = (x-1)^2 + 2 \ (x \geq 1), \qquad f^{-1}(x) = 1 + \sqrt{x-2} \ (x \geq 2)$$

■ Worked example

Let f(x)=(x−1)²+2 for x≥1. Its range is y≥2. From y=(x−1)²+2, the domain selects x−1≥0, so x=1+√(y−2). Thus f inverse(x)=1+√(x−2), with domain x≥2 and range y≥1. The point (3,6) reflects to (6,3). Checking gives f(f inverse(x))=x for x≥2. In the other order, f inverse(f(x))=1+|x−1|=x for x≥1. Choosing the original domain x≤1 instead gives inverse 1−√(x−2), with range y≤1.



Where it goes wrong

- The same quadratic formula on all real inputs is not one-to-one.
- Do not choose the plus square root without using the original domain.
- An inverse function is not the reciprocal 1/f(x).
- A reflected graph needs its domain and range labels as well as its formula.

■ Watch out

Not true: Every quadratic on all real inputs has an inverse function. A quadratic is many-to-one on all real inputs, so restrict the domain before finding an inverse.

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Applying it

- On the branch $x \leq 1$, output 6 reverses to input -1, while on $x \geq 1$ it reverses to 3.
- Both are correct for their own original domains.
- Check both compositions on their actual domains; applying the inverse identity to an input outside the original domain can fail.

In your course

- AQA A-level Mathematics (7357), specification section B.
- Every paper allows a calculator, but method marks still need the expression or equation written before its value.

■ Key idea

- **One-to-one**: A function for which different inputs always give different outputs.
- An inverse reverses the input-output mapping, so the original function must be one-to-one on its stated domain.
- Watch for: the same quadratic formula on all real inputs is not one-to-one.

B · Composite functions and intermediate domains

An allowed starting input can still produce an invalid intermediate value.

- One machine's output becomes another machine's input.
- This lesson studies **composition** 复合函数: Applying one function to the output of another, in a specified order.

The method

- Here fg means f composed with g: $fg(x)=f(g(x))$, not multiplication.
- Apply g first.
- A composite input must belong to the inner function's domain, and its output must belong to the outer function's domain.
- In general fg and gf have different formulas and different domains.

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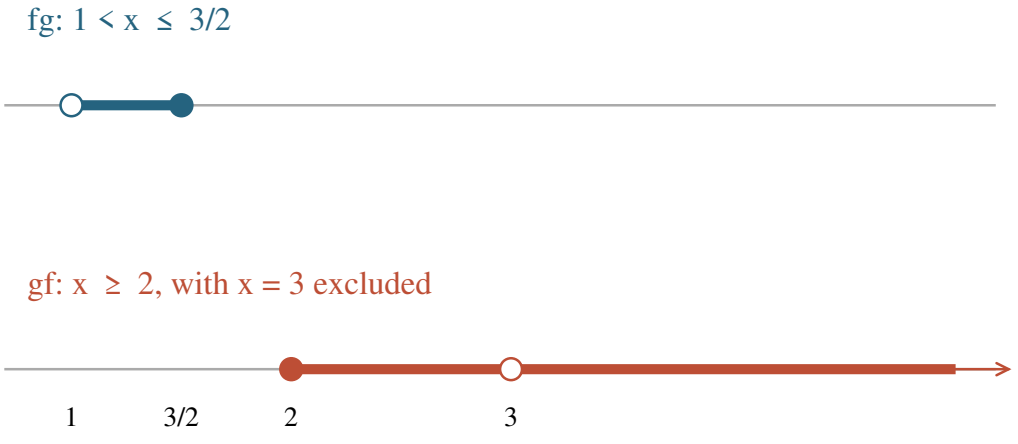
A worked case

$$(f \circ g)(x) = \sqrt{\frac{1}{x-1} - 2}, \qquad 1 < x \leq \frac{3}{2}$$

■ Worked example

Let f(x)=√(x−2), with x≥2, and g(x)=1/(x−1), with x≠1. Then fg(x)=√(1/(x−1)−2). It requires 1/(x−1)≥2. A negative denominator cannot meet this positive lower bound; for x>1, multiplying by x−1 preserves order and gives x≤3/2. Thus the domain is 1<x≤3/2. In the reverse order, gf(x)=1/(√(x−2)−1). It requires x≥2 and √(x−2)≠1, so x≠3. Its domain is [2,3)∪(3,∞). At x=6, gf(6)=1, while fg(6) is not real.

The intermediate output changes the allowed inputs



Where it goes wrong

- Checking only the inner domain is insufficient.
- Do not multiply the inequality by x−1 before deciding its sign.
- An excluded intermediate denominator remains excluded even if another form appears simpler.
- A formula and domain together define the composite function.

■ Watch out

Not true: A composite is defined wherever its inner function is defined. The inner output must also lie in the outer function's domain.

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Applying it

- Trace the intermediate value: gf(6) uses f(6)=2 then g(2)=1.
- For fg(5/4), g(5/4)=4 then f(4)=√2.
- At x=3, f(3)=1 is valid, but g(1) is undefined, so gf(3) is undefined.
- Mark every retained or excluded endpoint in interval notation.

In your course

- AQA A-level Mathematics (7357), specification section B.
- Every paper allows a calculator, but method marks still need the expression or equation written before its value.

■ Key idea

- **Composition**: Applying one function to the output of another, in a specified order.
- Here fg means f composed with g: fg(x)=f(g(x)), not multiplication.
- Watch for: checking only the inner domain is insufficient.

B · Combined graph transformations and order

Does moving it first give the same final graph?

- A compressed curve is moved to the right.
- This lesson studies **horizontal scale factor** 水平伸缩因子: The factor multiplying the original horizontal coordinates when a graph is stretched or compressed.

The method

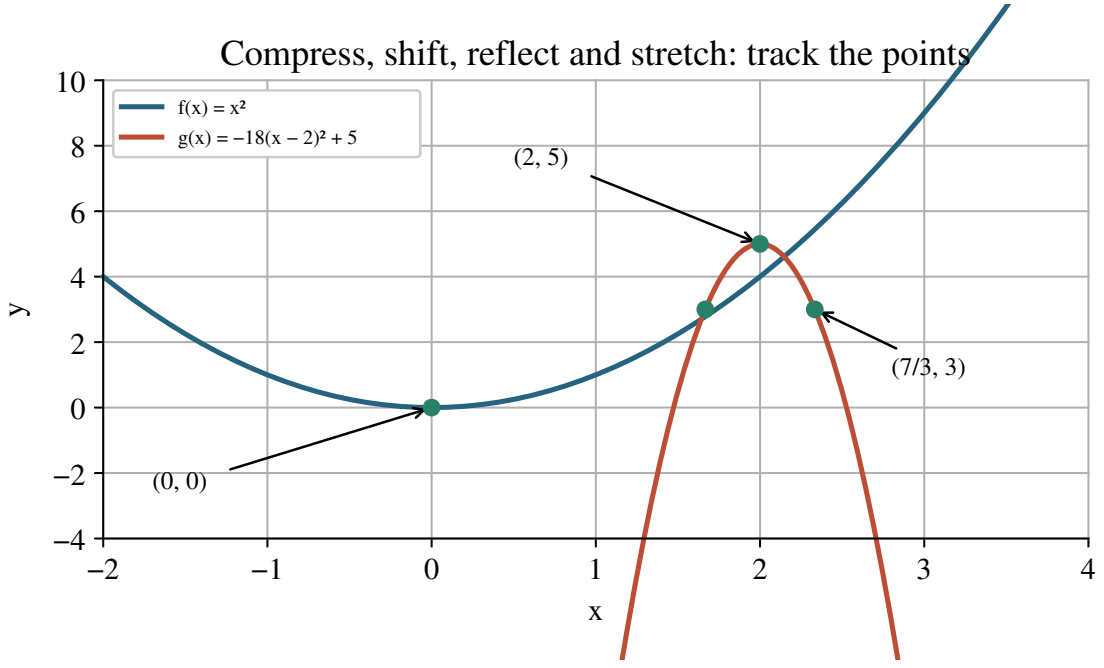
- For g(x)=a f(b(x−h))+k with b≠0, an original point (u,v) maps to (u/b+h,av+k).
- Horizontal scale is 1/|b|, with reflection in the y-axis if b<0.
- Vertical scale is |a|, with reflection in the x-axis if a<0.
- Read the factored inner expression before naming a horizontal translation.
- The transformed domain follows u=b(x−h).

A worked case

$$g(x) = af(b(x-h)) + k: \quad (u,v) \mapsto (u/b + h, av + k)$$

■ Worked example

With f(x)=x², g(x)=−2f(3x−6)+5=−2f(3(x−2))+5=−18(x−2)²+5. Compress horizontally by factor 1/3, then translate right 2; reflect vertically and stretch by factor 2, then translate up 5. The points (−1,1),(0,0),(1,1) become (5/3,3),(2,5),(7/3,3). Thus the vertex is (2,5) and the range is y≤5. Compressing first and then shifting right 2 gives f(3(x−2)); shifting first and then compressing gives f(3x−2), whose vertex is at x=2/3 instead.



Where it goes wrong

- The coefficient 3 inside f gives horizontal factor 1/3, not 3.
- In f(3x−6), the translation is 2, not 6.
- Translation and scaling generally do not commute.
- A symmetric example can hide a reflection: f(−x)=f(x) for x², but not for every f.

■ Watch out

Not true: A horizontal translation and horizontal scaling can be applied in either order. Order matters: y=f(2(x+1)) and y=f(2x+1) are different graphs.

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Applying it

- Use point mapping to check every combined transformation, and transfer any restricted domain.
- If the original f is defined only for $0 \le u \le 3$, the inner condition $0 \le 3(x-2) \le 3$ gives $2 \le x \le 3$.
- Changing b to a negative value reverses endpoint order when solving the domain inequality.
- Plot the transformed reference points before joining the curve.

In your course

- AQA A-level Mathematics (7357), specification section B.
- Every paper allows a calculator, but method marks still need the expression or equation written before its value.

■ Key idea

- **Horizontal scale factor**: The factor multiplying the original horizontal coordinates when a graph is stretched or compressed.
- For $g(x)=a\,f(b(x-h))+k$ with $b \neq 0$, an original point (u,v) maps to $(u/b+h, av+k)$.
- Watch for: the coefficient 3 inside f gives horizontal factor 1/3, not 3.

B · Fitting and refining a function model

Which extra observation can help choose between them?

- Two different curve models fit the same three measurements.
- This lesson studies **residual** 残差: An observed output minus the output predicted by a model.

The method

- State inputs, outputs, units and the modelling domain.
- Fit parameters using the stated observations, then compare predictions with independent measurements.
- An exact fit to the calibration points does not prove the chosen function is correct everywhere.
- A residual records observed minus predicted output; model refinement needs evidence and a reasoned change.

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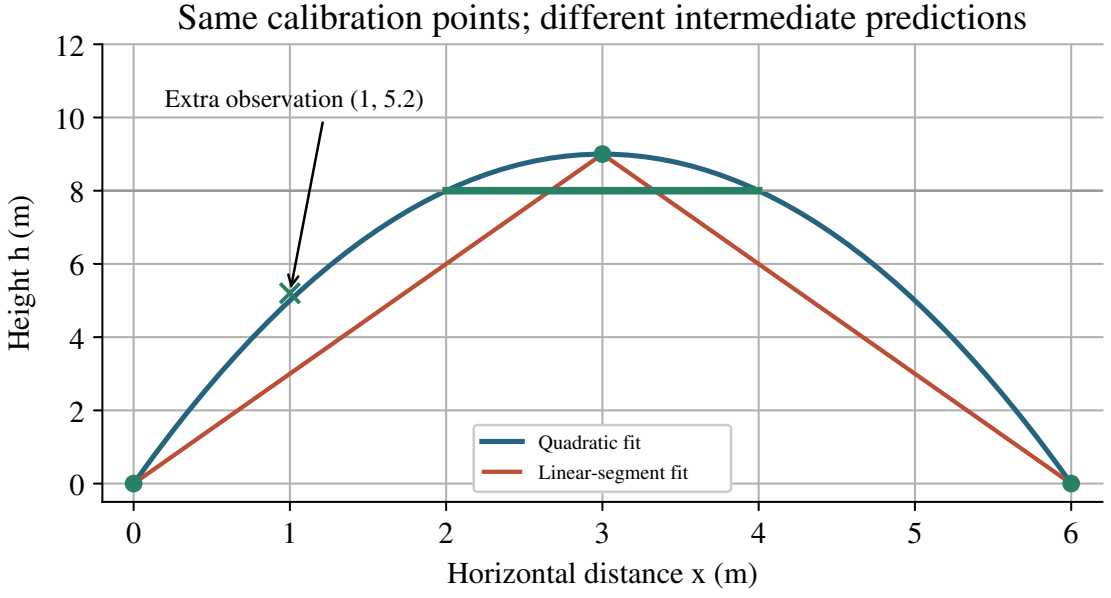
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A worked case

$$h(x) = -x^2 + 6x = 9 - (x-3)^2, \qquad 0 \le x \le 6$$

■ Worked example

For a proposed arch, x is horizontal distance and h its height, both in metres. Use the illustrative points (0,0),(3,9),(6,0), with $0 \le x \le 6$. Let $h = ax^2 + bx + c$. Then $c = 0$, $9a + 3b = 9$ and $36a + 6b = 0$ give $a = -1, b = 6$. Thus $h = -x^2 + 6x = 9 - (x-3)^2$, with maximum 9 m at $x = 3$. A piecewise-linear alternative $h = 3x$ for $0 \le x \le 3$ and $h = 18 - 3x$ for $3 \le x \le 6$ fits the same three points. At $x = 1$, an illustrative extra measurement 5.2 m gives residual 0.2 m for the quadratic but 2.2 m for the linear alternative. This favours the quadratic at that point, not a universal verdict.



Where it goes wrong

- Do not use a fitted calibration point as independent validation.
- Outside $0 \le x \le 6$, the quadratic can give negative height and no longer describes the proposed arch.
- A smaller residual at one point does not establish accuracy everywhere.
- The simpler model can be easier to interpret but still miss curvature.

■ Watch out

Not true: A function that fits three calibration points is proven correct at every input. Fitting three points does not prove a model; it may fail at other inputs.

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Applying it

- For height at least 8 m, $9-(x-3)^2 \geq 8$ gives $2 \leq x \leq 4$: modelled horizontal width 2 m.
- Collect more intermediate measurements and estimate measurement uncertainty before relying on this clearance.
- Thickness and changes in shape can require a refined domain or formula; explain which observation motivates the refinement.

In your course

- AQA A-level Mathematics (7357), specification section B.
- Every paper allows a calculator, but method marks still need the expression or equation written before its value.

■ Key idea

- **Residual**: An observed output minus the output predicted by a model.
- State inputs, outputs, units and the modelling domain.
- Watch for: do not use a fitted calibration point as independent validation.

B · Polynomial sketches, repeated roots and signs

Which one crosses the axis?

- Two roots can appear in a factorised formula but make very different marks on its graph.
- This lesson studies **multiplicity** 重数: The number of times a root's linear factor appears in a polynomial.

The method

- Find real roots and their multiplicities, the y-intercept and the leading term.
- A root of odd multiplicity changes the sign and crosses the x-axis; a root of even multiplicity keeps the sign and touches the axis.
- For a positive leading coefficient, an odd-degree curve goes down on the far left and up on the far right; an even-degree curve goes up at both ends.
- A negative leading coefficient reverses these directions.

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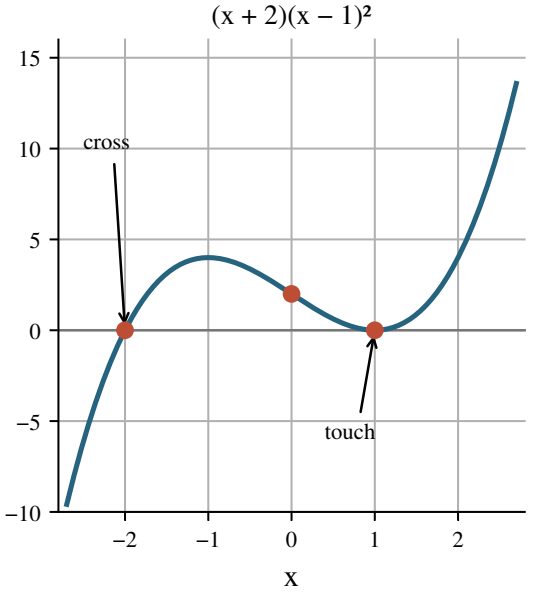
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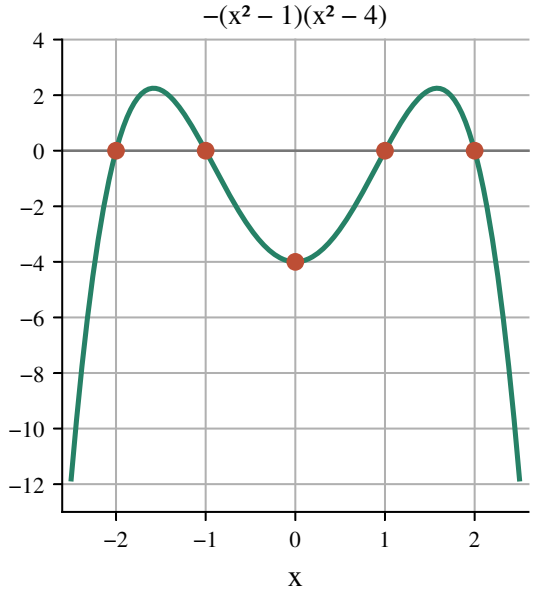
A worked case

$$P(x) = (x+2)(x-1)^2, \qquad Q(x) = -(x^2-1)(x^2-4)$$

Worked example

For P(x)=(x+2)(x−1)^2, roots are −2 with multiplicity 1 and 1 with multiplicity 2. The y-intercept is P(0)=2. The leading term is x^3, so the left tail falls and the right rises. P crosses at −2 and touches at 1. It is negative for x<−2, positive for −2<x<1 and x>1, and zero at both roots. For Q(x)=−(x^2−1)(x^2−4), the four simple roots are −2,−1,1,2, the y-intercept is −4 and the leading term is −x^4. Both tails fall. Q is positive on (−2,−1) and (1,2), and negative on the other three root-separated intervals.





Where it goes wrong

- The number of distinct roots need not equal the degree.
- A repeated root is one x-intercept even though its factor occurs more than once.
- Do not alternate the sign at an even-multiplicity root.
- A sketch shows structure; exact turning-point coordinates require further analysis, not guessed readings from the picture.

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■ **Watch out**

Not true: Every real root of a polynomial must change the sign of its values. A repeated root of even multiplicity touches the axis without changing sign, as $y=x^2$ does at 0.

Applying it

- Graphical solutions of $P(x)=k$ are intersections of the curve with $y=k$.
- For $k = 0$ the factored roots are exact; for another k a graph may only estimate solutions.
- Use the sign intervals to solve $P(x)\leq 0$: $x\leq -2$ or $x=1$.
- That isolated root matters even though its neighbouring values are positive.

In your course

- AQA A-level Mathematics (7357), specification section B.
- Every paper allows a calculator, but method marks still need the expression or equation written before its value.

■ **Key idea**

- **Multiplicity**: The number of times a root's linear factor appears in a polynomial.
- Find real roots and their multiplicities, the y-intercept and the leading term.
- Watch for: the number of distinct roots need not equal the degree.

C · Coordinate geometry and tangents

How does a path's slope become an equation?

- A path rises 6 metres over a horizontal distance of 3 metres. Its gradient connects a diagram to an equation.
- This lesson studies **gradient** 斜率: The change in y divided by the corresponding change in x.

The method

- A line through (x_1,y_1) with gradient m has $y-y_1=m(x-x_1)$.
- Parallel lines have equal gradients.
- Finite perpendicular gradients multiply to -1.

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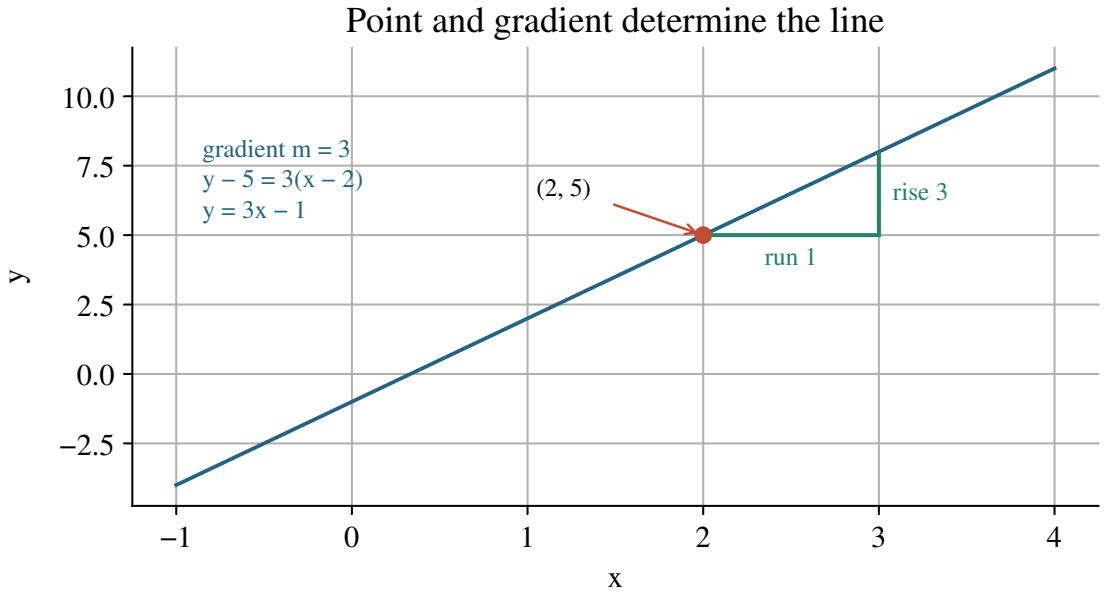
- A circle has $(x-a)^2+(y-b)^2=r^2$.

A worked case

$$y - y_1 = m(x - x_1), \qquad (x - a)^2 + (y - b)^2 = r^2$$

■ Worked example

Through (2,5) with gradient 3, y-5=3(x-2), so y=3x-1. A perpendicular through the same point has y-5=-(x-2)/3. The circle $(x-2)^2+(y+1)^2=25$ has centre (2,-1) and radius 5.



Where it goes wrong

- A vertical line has no finite gradient; do not force it into y=mx+c.
- Read the signs of a circle's centre carefully.
- The radius to a tangent is perpendicular to the tangent.

■ Watch out

Not true: Perpendicular nonvertical lines have equal gradients. Perpendicular gradients multiply to -1: a gradient of 2 needs $-1/2$. Parallel lines have equal gradients.

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Applying it

- Before solving a line-circle intersection, predict whether there are zero, one or two intersections.
- Substitution produces a quadratic whose discriminant checks the prediction.

In your course

- AQA A-level Mathematics (7357), specification section C.
- Every paper allows a calculator, but method marks still need the expression or equation written before its value.

■ Key idea

- **Gradient**: The change in y divided by the corresponding change in x.
- A line through (x_1,y_1) with gradient m has $y-y_1=m(x-x_1)$.
- Watch for: A vertical line has no finite gradient; do not force it into y=mx+c.

C · Parametric curves, domains and direction

What information disappears when time is eliminated?

- Two coordinates can change together as a third variable changes. Eliminating that variable can hide which part of the curve was traced.
- This lesson studies **parameter** 参数: A variable that determines both coordinates of a point.

The method

- For x=f(t), y=g(t), express t from one equation if possible and substitute into the other.
- Transfer the parameter interval to the Cartesian curve and mark the direction of increasing t.
- With trigonometric parameters, use identities and keep sign or branch restrictions.

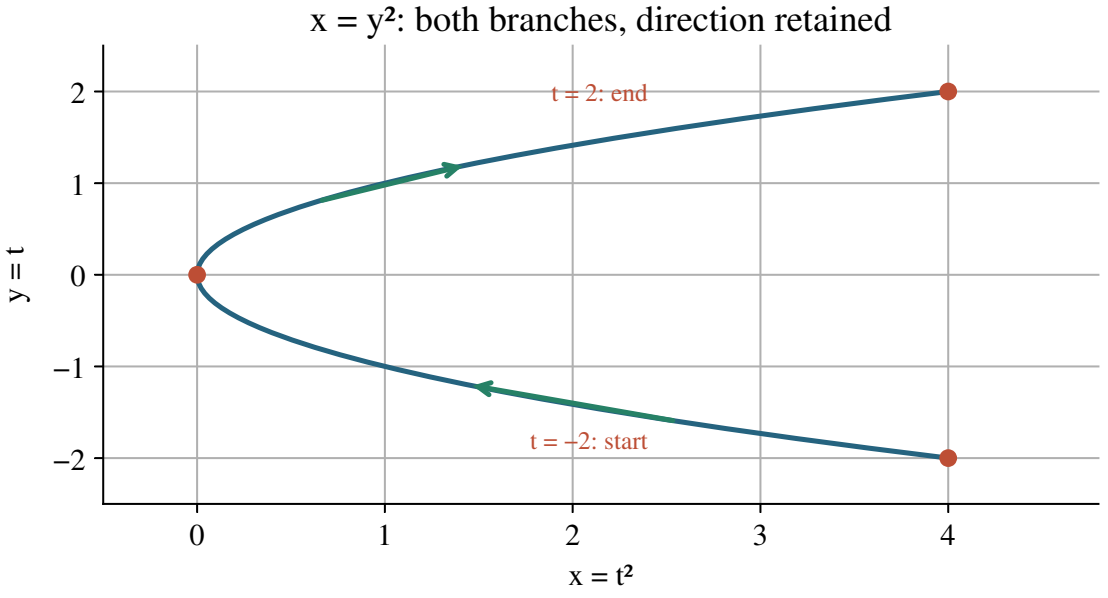
A worked case

$$x = 2t + 1, \quad y = t^2 - 2 \implies y = \frac{(x-1)^2}{4} - 2, \quad 1 \le x \le 7$$

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Worked example

For $x=2t+1$, $y=t^2-2$ and $0\le t\le 3$, $t=(x-1)/2$ gives $y=(x-1)^2/4-2$ with $1\le x\le 7$. The point moves from $(1,-2)$ to $(7,7)$. For $x=t^2$, $y=t$ and $-2\le t\le 2$, the relation is $x=y^2$ with $-2\le y\le 2$: both signs of y occur. For $x=3\cos\theta$, $y=2\sin\theta$ and $0\le\theta\le\pi$, $x^2/9+y^2/4=1$ with $y\ge 0$; increasing θ traces the upper half from $(3,0)$ to $(-3,0)$. Conversely, $y=x^2$ with $-1\le x\le 2$ can be parametrised by $x=t$, $y=t^2$ and $-1\le t\le 2$.



Where it goes wrong

- Writing $y=\sqrt{x}$ from $x=t^2$,$y=t$ loses the part where t is negative.
- A Cartesian equation alone does not record direction or the original parameter interval.
- Squaring can erase a sign restriction.

Watch out

Not true: Eliminating a parameter preserves the original domain and direction automatically. Elimination can lose the domain and direction, so state the restrictions from the parameter separately.

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Applying it

- Use a table of parameter values to sketch endpoints and intermediate points.
- A parameter can revisit an x value or a whole point.
- Check the original equations and interval when converting back to parametric form.

In your course

- AQA A-level Mathematics (7357), specification section C.
- Every paper allows a calculator, but method marks still need the expression or equation written before its value.

■ Key idea

- **Parameter**: A variable that determines both coordinates of a point.
- For x=f(t), y=g(t), express t from one equation if possible and substitute into the other.
- Watch for: writing y=√x from $x=t^2$,y=t loses the part where t is negative.

C · Parametric modelling in motion and design

When does a mathematical path stop describing a real flight?

- A ball's height and horizontal position depend on the same time. A Cartesian path cannot tell us when it is reached without the time model.
- This lesson studies **parametric model** 参数模型: A model expressing two quantities using a common variable and stated assumptions.

The method

- Name the parameter, its units and allowed interval.
- Evaluate both model quantities at the same parameter value.
- Eliminate the parameter to relate the quantities, then interpret endpoints and limitations.
- A mathematical curve beyond the allowed interval need not describe the physical situation.

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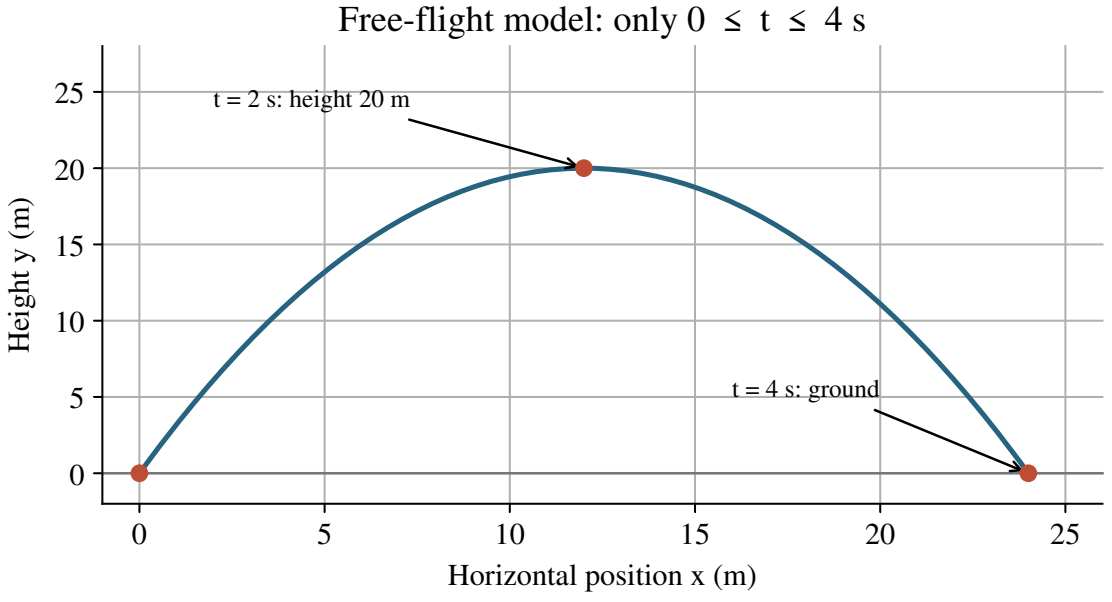
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A worked case

$$x = 6t, \quad y = 20t - 5t^2, \quad 0 \leq t \leq 4$$

■ Worked example

A simplified flight model uses x=6t and y=20t−5t² in metres, with t in seconds and 0≤t≤4. It assumes constant horizontal speed and downward acceleration 10 metres per second squared, ignoring air resistance. The noninitial ground time is t=4, giving x=24 m. Completing the square gives y=20−5(t−2)², so maximum height is 20 m at t=2. Eliminating t gives y=10x/3−5x²/36 for 0≤x≤24. For a rectangle of width t and length t+2 metres, P=4t+4 and A=t(t+2). With 0<t≤5, A=(P²−16)/16 and 4<P≤24.



Where it goes wrong

- The flight formula giving negative height after t=4 does not describe continued free flight below the ground.
- Do not use t measured in minutes with coefficients defined for seconds.
- In the rectangle model, t=0 is excluded because the width must be positive.

■ Watch out

Not true: The Cartesian flight curve is a physically valid model for every real x. The model applies only while the object is in flight, between launch and landing.

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Applying it

- Check a model against observed data and its assumptions.
- Wind or air resistance could require refinement of the flight model; a different length–width relation changes the rectangle model.
- The acceleration 10 here is a stated simplification, not a replacement for a paper's instructed value of g.

In your course

- AQA A-level Mathematics (7357), specification section C.
- Every paper allows a calculator, but method marks still need the expression or equation written before its value.

■ Key idea

- **Parametric model**: A model expressing two quantities using a common variable and stated assumptions.
- Name the parameter, its units and allowed interval.
- Watch for: the flight formula giving negative height after t=4 does not describe continued free flight below the ground.

C · General-form lines and contextual models

Which part of a delivery charge changes with distance?

- A straight-line cost model has a fixed charge and a rate per kilometre. Its intercept and gradient have different meanings and units.
- This lesson studies **general form** 一般式: The form ax+by+c=0 for a straight line, with a and b not both zero.

The method

- For a nonvertical line through two points, $m=(y_2-y_1)/(x_2-x_1)$ and $y-y_1=m(x-x_1)$.
- Rearrange into ax+by+c=0.
- Parallel nonvertical lines have equal gradients; perpendicular finite gradients multiply to -1.
- Treat vertical and horizontal lines separately.

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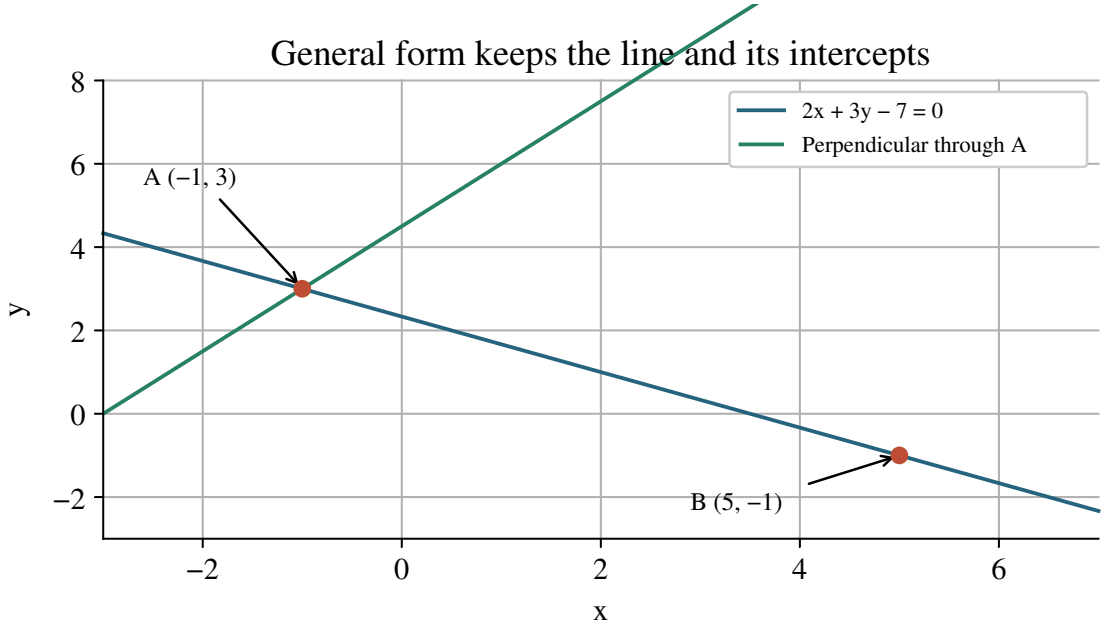
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A worked case

$$y - y_1 = m(x - x_1) \implies ax + by + c = 0$$

■ Worked example

Through A(−1,3) and B(5,−1), m=−4/6=−2/3. Thus y−3=−2(x+1)/3, giving 2x+3y−7=0. The perpendicular through A has gradient 3/2 and equation 3x−2y+9=0. A vertical line x=4 is perpendicular to a horizontal line y=3 without having a finite gradient. Delivery quotes C=8 at d=0 km and C=38 at d=12 km give rate (38−8)/12=2.5 currency units per kilometre and C=8+2.5d. At d=8, the predicted cost is 28.



Where it goes wrong

- For ax+by+c=0, the gradient is −a/b only if b≠0.
- Do not confuse the fixed charge with the price per kilometre.
- A line fitted to two quoted distances does not prove the same tariff applies at all distances.

■ Watch out

Not true: Every straight line has a finite gradient and can be written as y=mx+c. A vertical line x=k has an undefined gradient and cannot be written as y=mx+c.

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Applying it

- Interpret the cost intercept, gradient and nonnegative distance domain.
- Within $0 \leq d \leq 12$, interpolation uses the observed quote range; beyond it, extrapolation needs a tariff assumption.
- Check both original points after rearranging a geometric line.

In your course

- AQA A-level Mathematics (7357), specification section C.
- Every paper allows a calculator, but method marks still need the expression or equation written before its value.

■ Key idea

- **General form**: The form $ax+by+c=0$ for a straight line, with a and b not both zero.
- For a nonvertical line through two points, $m=(y_2-y_1)/(x_2-x_1)$ and $y-y_1=m(x-x_1)$.
- Watch for: for $ax+by+c=0$, the gradient is $-a/b$ only if $b \neq 0$.

C · Shifted circles, chords and tangents

Where is the centre hidden in an expanded circle equation?

- An expanded circle equation can hide its centre and radius. Completing the square exposes the geometry needed for a chord or tangent.
- This lesson studies **centre** 圆心: The point at equal distance from every point on a circle.

The method

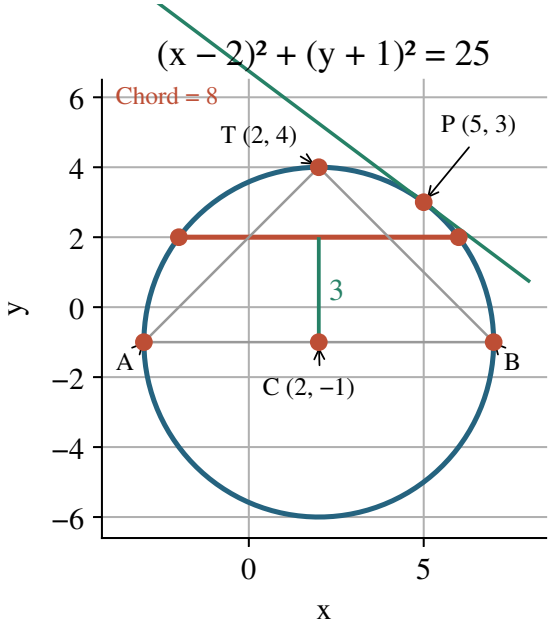
- Rewrite a circle as $(x-a)^2+(y-b)^2=r^2$ by completing both squares.
- A perpendicular from the centre bisects a chord, so half-chord length follows from a right triangle.
- A tangent is perpendicular to the radius at the contact point.
- An angle subtended by a diameter at the circumference is 90°.

A worked case

$$x^2 + y^2 - 4x + 2y - 20 = 0 \implies (x-2)^2 + (y+1)^2 = 25$$

■ Worked example

For $x^2+y^2-4x+2y-20=0$, completing squares gives $(x-2)^2+(y+1)^2=25$: centre $(2,-1)$, radius 5. The chord $y=2$ is 3 units above the centre; its half-length is $\sqrt{(25-9)}=4$, so endpoints are $(-2,2),(6,2)$ and length 8. At $P(5,3)$, the radius gradient is $4/3$, giving tangent $y-3=-3(x-5)/4$ or $3x+4y=27$. For diameter endpoints $A(-3,-1)$, $B(7,-1)$ and circumference point $T(2,4)$, gradients $AT=1$ and $BT=-1$ verify angle $ATB=90°$.



Where it goes wrong

- Completing a square adds a constant: adjust the other side too.
- The circle's centre has the opposite signs to the bracket constants.
- A proposed tangent point must lie on the circle; perpendicularity alone at an unrelated point is insufficient.

■ Watch out

Not true: A line perpendicular to any radius is a tangent, wherever it crosses the plane. A tangent must be perpendicular to the radius at the point where it touches the circle.

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Applying it

- Check radius distance for each contact point and both chord endpoints.
- A negative completed r^2 gives no real circle, while $r^2=0$ gives a single point rather than a nondegenerate circle.
- Handle a horizontal or vertical radius without dividing by zero.

In your course

- AQA A-level Mathematics (7357), specification section C.
- Every paper allows a calculator, but method marks still need the expression or equation written before its value.

■ Key idea

- **Centre**: The point at equal distance from every point on a circle.
- Rewrite a circle as $(x-a)^2+(y-b)^2=r^2$ by completing both squares.
- Watch for: completing a square adds a constant: adjust the other side too.

D · Sequences, series and recurrence

Does the change add or multiply?

- A saving plan adds ¥30 more each week; a population model grows by 5% each year. Equal differences and equal ratios need different models.
- This lesson studies **common ratio** 公比: The constant multiplier between consecutive terms of a geometric sequence.

The method

- For an arithmetic progression, u_n=a+(n-1)d and S_n=n(2a+(n-1)d)/2.
- For a geometric progression, u_n=ar^(n-1) and S_n=a(1-r^n)/(1-r).
- An infinite geometric sum exists only if |r|<1.

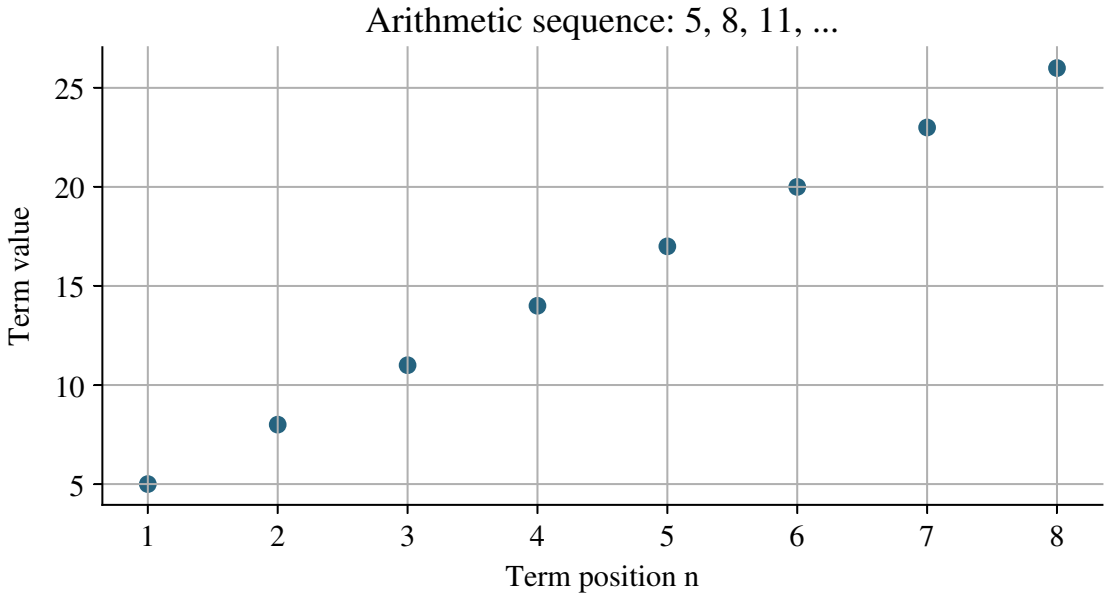
A worked case

$$u_n = a + (n-1)d, \qquad S_n = \frac{n}{2}\left[2a + (n-1)d\right]$$

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Worked example

For a=5,d=3,n=8, u_8=5+7×3=26 and S_8=8(10+21)/2=124. For a=12,r=1/2, S infinity=12/(1-1/2)=24. For u_(n+1)=2u_n+1 with u_1=1, the next terms are 3,7,15.



Where it goes wrong

- The first term has index 1, so the exponent is n-1.
- A sequence is a list; a series is a sum.
- A geometric sequence can alternate in sign and still converge.

Watch out

Not true: Every geometric series has a finite sum to infinity. A sum to infinity exists only when −1<r<1; for r=2 the terms grow without limit.

Applying it

- Explain whether the context justifies additive or multiplicative change.
- In finance, distinguish a single deposit from a stream of deposits before choosing a sum formula.

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In your course

- AQA A-level Mathematics (7357), specification section D.
- Every paper allows a calculator, but method marks still need the expression or equation written before its value.

■ Key idea

- **Common ratio**: The constant multiplier between consecutive terms of a geometric sequence.
- For an arithmetic progression, u_n=a+(n-1)d and S_n=n(2a+(n-1)d)/2.
- Watch for: the first term has index 1, so the exponent is n-1.

D · Binomial expansion and valid approximations

How does a small change affect a power?

- A small measurement change affects a power of a quantity. An expansion can show the size of first and second effects.
- This lesson studies **binomial coefficient** 二项式系数: The number of ways to choose a specified number of objects from a set.

The method

- For positive integer n, (a+b)^n is a finite binomial expansion.
- For noninteger n, expand (1+x)^n as 1+nx+n(n-1)x^2/2+... with |x|<1.
- Factor out constants before using this form.

A worked case

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2}x^2 + \cdots$$

■ Worked example

(1+2x)^5=1+10x+40x^2+80x^3+80x^4+32x^5. For (1+x)^(-1), the first three terms are 1-x+x^2; at x=0.1 this gives 0.91 versus the exact 1/1.1≈0.909091.

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$(1 + 2x)^5$: Pascal row × powers of 2

Pascal row 5	1	5	10	10	5	1
$\times 2^k$	1	2	4	8	16	32
coefficient	**1**	**10**	**40**	**80**	**80**	**32**
	1	x	x^2	x^3	x^4	x^5

Where it goes wrong

- An expansion in 2x requires $|2x|<1$ for the infinite series, not merely $|x|<1$.
- Positive-integer expansions are finite and do not have that convergence restriction.

> ■ **Watch out**
>
> **Not true:** Every binomial expansion is a finite polynomial, including powers that are not positive integers. For a negative or fractional power the expansion is an infinite series, valid only for $|x|<1$.

Applying it

- State the range of validity alongside an approximation.
- Retain enough terms to support the requested accuracy, and distinguish a coefficient from the whole term containing x.

In your course

- AQA A-level Mathematics (7357), specification section D.
- Every paper allows a calculator, but method marks still need the expression or equation written before its value.

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■ **Key idea**

- **Binomial coefficient**: The number of ways to choose a specified number of objects from a set.
- For positive integer n, (a+b)^n is a finite binomial expansion.
- Watch for: an expansion in 2x requires |2x|<1 for the infinite series, not merely |x|<1.

D · Factorials, combinations and binomial coefficients

Why is the x^2 coefficient not just the middle number in Pascal's triangle?

- A binomial coefficient counts which factors contribute an x term.
- This lesson studies **combination** 组合: A selection in which order does not distinguish different choices.

The method

- For a nonnegative integer n, n!=n(n−1)…1 and 0!=1.
- For 0≤r≤n, nCr=n!/[r!(n−r)!] counts unordered selections.
- In (a+bx)^n with positive integer n, the x^r coefficient is nCr × a^(n−r) × b^r.
- The term number is r+1 when the expansion is written in ascending powers of x.

A worked case

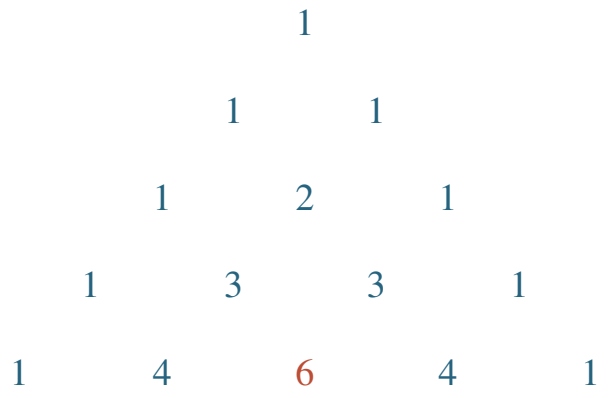
$$\binom{n}{r} = \frac{n!}{r!(n-r)!}, \qquad [x^r](a+bx)^n = \binom{n}{r} a^{n-r} b^r$$

■ **Worked example**

4!=24 and 4C2=24/(2×2)=6. The coefficients for n=4 are 1,4,6,4,1. In (2+3x)^4, the x^2 term is $4C2\times 2^2\times(3x)^2=216x^2$; the constant is $2^4=16$ and the x term is $4\times 2^3\times 3x=96x$. For four independent trials each with success probability 1/3, exactly two successes have probability $4C2\times(1/3)^2\times(2/3)^2=8/27$. The coefficient 6 counts the possible success positions; each arrangement has the same probability.

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Choose positions, then include every factor's power



x^2 coefficient: $6 \times 2^2 \times 3^2 = 216$

Where it goes wrong

- Do not omit the powers of a or b when using a combination coefficient.
- The x^2 term is the third term, not the second.
- The probability expression needs a fixed number of trials, independent binary outcomes and the same success probability; a changing probability invalidates this simple model.

> **Watch out**
>
> **Not true:** The x^2 coefficient of (a+bx)^n is just nC2. The x^2 term is nC2 · a^(n−2) · b^2x^2, so a and b contribute too.

Applying it

- Use symmetry nCr=nC(n−r) and cancel factorial factors before calculating.
- To find the x^3 coefficient in (2+3x)^4, use $4C3 \times 2 \times 3^3 = 216$.
- The finite polynomial formula here is for positive integer n; a rational exponent instead gives the separate general binomial series with its stated validity interval.

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In your course

- AQA A-level Mathematics (7357), specification section D.
- Every paper allows a calculator, but method marks still need the expression or equation written before its value.

Key idea

- **Combination**: A selection in which order does not distinguish different choices.
- For a nonnegative integer n, n!=n(n−1)…1 and 0!=1.
- Watch for: do not omit the powers of a or b when using a combination coefficient.

D · Increasing, decreasing and periodic recurrences

What decides its behaviour?

- The same update rule can increase one starting value and decrease another.
- This lesson studies **periodic sequence** 周期数列: A sequence whose terms repeat after a fixed positive number of steps.

The method

- A recurrence needs an initial term as well as a rule.
- Compute each new term from the previous one, preserving exact values where useful.
- Compare u_(n+1)−u_n and check bounds before claiming increasing or decreasing behaviour.
- A periodic sequence repeats after a fixed number of steps; its smallest positive repeating step is its period.

A worked case

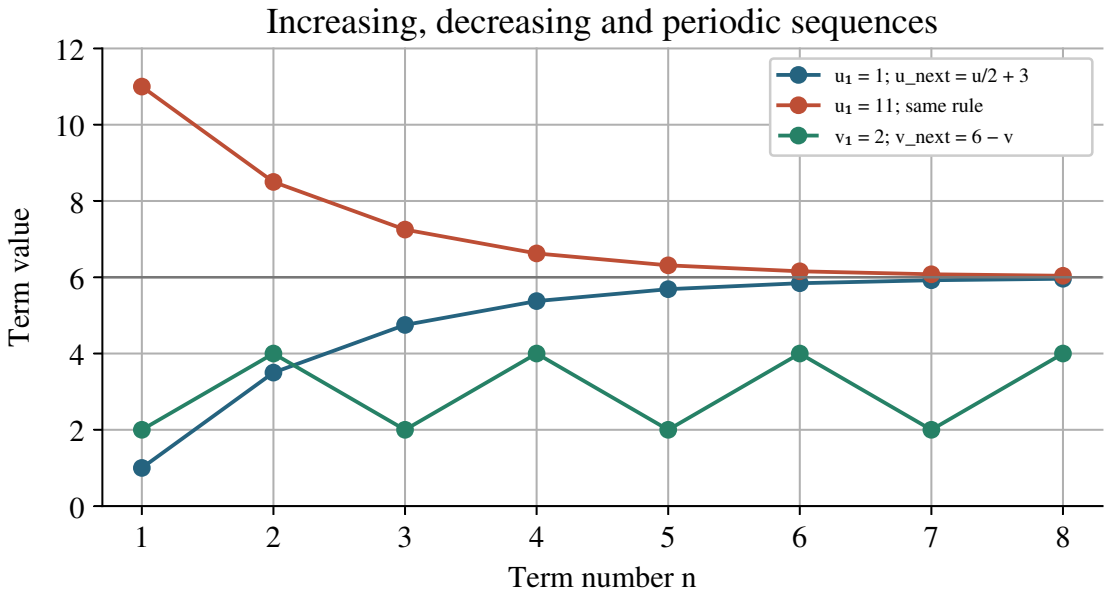
$$u_{n+1} = \frac{1}{2}u_n + 3, \qquad u_n = 6 + \frac{u_1 - 6}{2^{n-1}}$$

Worked example

For u_1=1 and u_(n+1)=u_n/2+3, the first terms are 1,7/2,19/4,43/8. The fixed value solves L=L/2+3, giving L=6. If u_n<6, the next term remains below 6 and u_(n+1)−u_n=(6−u_n)/2>0. Thus this sequence increases toward 6. Starting instead at 11 gives 11,17/2,29/4,…, decreasing while staying above 6. The exact form is u_n=6+(u_1−6)/2^(n−1). Separately, v_1=2 and v_(n+1)=6−v_n gives 2,4,2,4,… with period 2 because applying the rule twice returns the original term. Starting that

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second rule at 3 gives a constant sequence with smallest period 1.



Where it goes wrong

- Early numerical terms suggest behaviour but do not prove it for every n.
- A fixed-point equation alone does not prove convergence: the update must keep approaching it.
- Alternating values need not be periodic; (−2)^n alternates in sign while its magnitude grows.
- State whether "increasing" means strictly increasing or merely nondecreasing.

■ Watch out

Not true: Every recurrence with a fixed value must converge to it from every starting value. A fixed point can be unstable; iterations from many starting values move away from it.

Applying it

- Verify the explicit form by substituting it into the recurrence and checking the initial term.
- The distance from 6 halves at every update, explaining convergence in this example.
- A different initial term changes the direction or gives a constant sequence, even though the recurrence formula stays the same.

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In your course

- AQA A-level Mathematics (7357), specification section D.
- Every paper allows a calculator, but method marks still need the expression or equation written before its value.

■ **Key idea**

- **Periodic sequence**: A sequence whose terms repeat after a fixed positive number of steps.
- A recurrence needs an initial term as well as a rule.
- Watch for: early numerical terms suggest behaviour but do not prove it for every n.

D · Sigma notation, indices and changed limits

How many terms does it contain, and what changes if we rename or shift its index?

- A sum starts at the third term.
- This lesson studies **summation index** 求和指标: The variable that takes each integer value between a sum's stated limits.

The method

- In a sum from k=a to b, substitute each integer a,a+1,…,b into the summand and add the results.
- There are b−a+1 terms when a≤b.
- The summation index is a dummy variable, but other parameters keep their values.
- Renaming an index preserves the sum only when its bound occurrences are renamed consistently.

A worked case

$$\sum_{k=3}^{7}(2k-1)=\sum_{j=1}^{5}(2j+3)=45$$

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Worked example

The sum from k=3 to 7 of (2k−1) is 5+7+9+11+13=45, with five terms. It also equals the sum from k=1 to 7 minus the sum from k=1 to 2: 49−4=45. With j=k−2, the lower limit becomes 1, the upper becomes 5 and 2k−1 becomes 2j+3, again giving 45. By linearity, the original sum is 2 times the sum of k from 3 to 7, minus five copies of 1: 2×25−5=45.

Five included indices; the shifted sum is unchanged

$$\sum_{k=3}^{7} (2k-1)$$

k: 3 4 5 6 7

term: 5 7 9 11 13

$$j=k-2: \quad \sum_{j=1}^{5} (2j+3) = 45$$

Where it goes wrong

- An inclusive sum from 3 to 7 has five terms, not four or seven.
- Subtract the prefix through a−1, not through a, when changing the lower limit.
- A constant is added once per index value: the sum of c from a to b is c(b−a+1).
- Shifting the index changes both bounds and the summand.

Watch out

Not true: Renaming or shifting an index never requires changing the summand. Shifting the index by k means replacing r by r+k in the summand as well.

Applying it

- For the sum of (2k−1) from k=1 to n, the index k changes while n is the fixed upper bound; the result is n^2 for positive integer n.
- Thus a tail from k=m to n is $n^2-(m-1)^2$ for $1 \le m \le n$.

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- Expand a few terms to check a symbolic manipulation before applying a sum formula.

In your course

- AQA A-level Mathematics (7357), specification section D.
- Every paper allows a calculator, but method marks still need the expression or equation written before its value.

Key idea

- **Summation index**: The variable that takes each integer value between a sum's stated limits.
- In a sum from k=a to b, substitute each integer a,a+1,…,b into the summand and add the results.
- Watch for: an inclusive sum from 3 to 7 has five terms, not four or seven.

D · Arithmetic series and inverse problems

Can a stated total tell us exactly how many terms were added?

- Two known terms reveal an arithmetic sequence.
- This lesson studies **common difference** 公差: The fixed amount added to each term to obtain the next.

The method

- For first term a and common difference d, u_n=a+(n−1)d.
- The sum of n terms is S_n=n[2a+(n−1)d]/2=n(a+u_n)/2.
- Pairing the first and last terms gives the same pair total throughout.
- An inverse problem may require solving simultaneous equations for a,d or a quadratic for n; a term count must be a positive integer.

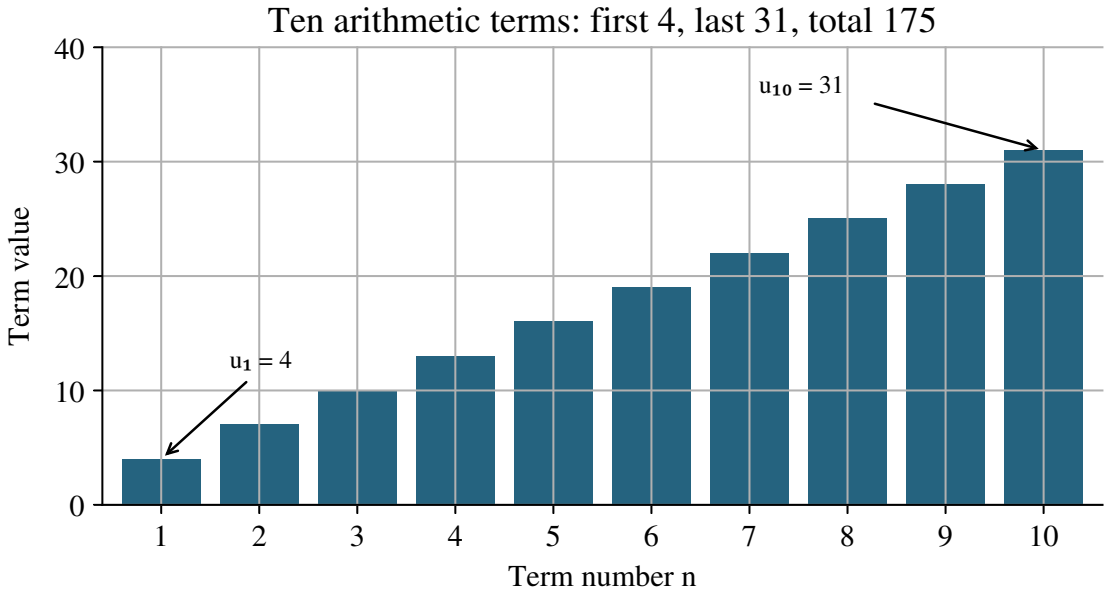
A worked case

$$u_n = a + (n-1)d, \qquad S_n = \frac{n}{2}(2a + (n-1)d)$$

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Worked example

Given $u_3=10$ and $u_7=22$, write $a+2d=10$ and $a+6d=22$. Subtraction gives $d=3$ and $a=4$, so $u_n=3n+1$ and $S_n=n(3n+5)/2$. To find a total of 175, solve $3n^2+5n-350=0=(n-10)(3n+35)$. The roots are 10 and $-35/3$; only $n=10$ is a positive integer. Check $u_{10}=31$ and $S_{10}=10(4+31)/2=175$. A total of exactly 200 has no integer n: $S_{10}=175$ and $S_{11}=209$, and these positive terms make the sums strictly increasing.



Where it goes wrong

- The nth term and sum of n terms are different quantities.
- The nth term contains $n-1$ differences, not n.
- Do not round a noninteger solution for an exact number of terms.
- If the question asks for the first total at least a target, check the neighbouring integer totals.
- Not every arithmetic series has increasing partial sums.

Watch out

Not true: A noninteger solution for an exact term count can be rounded to the nearest integer. A number of terms must be a whole number; a noninteger solution means no term takes that value.

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Applying it

- For this sequence, the least n giving S_n≥200 is 11, since the previous total is 175.
- State whether a problem asks for an exact total or a threshold.
- The sum formula still works for negative common differences, but any positivity or monotonicity assumption must be checked from the actual terms.

In your course

- AQA A-level Mathematics (7357), specification section D.
- Every paper allows a calculator, but method marks still need the expression or equation written before its value.

■ Key idea

- **Common difference**: The fixed amount added to each term to obtain the next.
- For first term a and common difference d, u_n=a+(n−1)d.
- Watch for: the nth term and sum of n terms are different quantities.

D · Geometric sums, alternating ratios and convergence

Why does a negative ratio not automatically prevent convergence?

- Alternating payments can have a stable accumulated total.
- This lesson studies **common ratio** 公比: The fixed factor multiplying each term to obtain the next.

The method

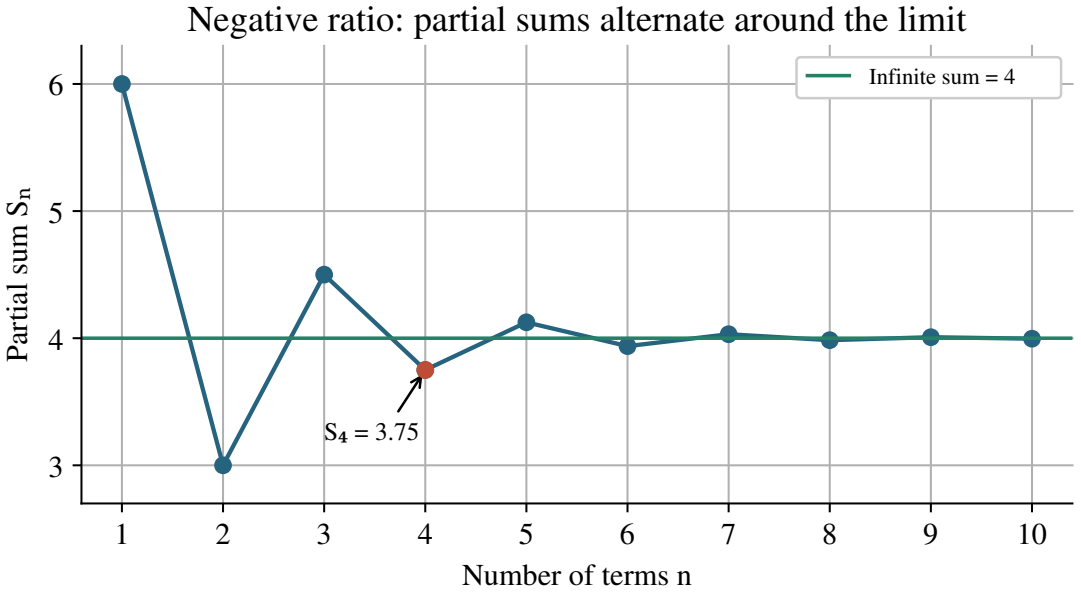
- For first term a and nonzero ratio r, u_n=ar^(n−1).
- If r=0, the terms are a,0,0,…, avoiding a 0^0 expression at the first term.
- For r≠1, the finite sum is S_n=a(1−r^n)/(1−r); for r=1 it is na.
- For nonzero a, an infinite geometric sum exists when |r|<1 and equals a/(1−r).
- A negative ratio alternates signs; convergence depends on its magnitude, not its sign alone.

A worked case

$$S_n = \frac{a(1-r^n)}{1-r}, \qquad S_\infty = \frac{a}{1-r} \quad (|r|<1)$$

■ Worked example

For a=6 and r=−1/2, the terms are 6,−3,3/2,−3/4,… . The first four sum to 15/4=3.75. The finite formula gives S_4=6[1−(−1/2)^4]/(1+1/2)=15/4. Since |r|=1/2<1, S infinity=6/(1+1/2)=4. The signed remainder is 4−S_n=4(−1/2)^n, so odd partial sums are above 4 and even partial sums below 4. For error below 0.01, require 4/2^n<0.01. At n=8 the error is 1/64>0.01; at n=9 it is 1/128<0.01, so 9 terms first meet the condition.



Where it goes wrong

- Do not replace r=−1/2 by r=1/2 in the finite formula: the parity of n matters.
- The infinite sum formula is not justified at r=1 or r=−1 for nonzero a; their terms do not tend to zero.
- A finite sum can exist even when its infinite continuation diverges.

■ Watch out

Not true: Every geometric series with a negative ratio diverges. A geometric series converges when |r|<1, and negative ratios such as −1/2 are included.

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Applying it

- Check the first few terms and partial sums directly before using a formula.
- If |r|>1 and a is nonzero, term magnitudes grow instead of vanishing.
- At r=−1, partial sums alternate between a and 0 rather than approaching one value.
- The zero sequence a=0 is a degenerate exception; identify it rather than applying a nonzero-first-term convergence claim.

In your course

- AQA A-level Mathematics (7357), specification section D.
- Every paper allows a calculator, but method marks still need the expression or equation written before its value.

■ Key idea

- **Common ratio**: The fixed factor multiplying each term to obtain the next.
- For first term a and nonzero ratio r, u_n=ar^(n−1).
- Watch for: do not replace r=−1/2 by r=1/2 in the finite formula: the parity of n matters.

D · Repeated-deposit sequence models and timing

Which timeline does the sum describe?

- Two equal payments can earn different amounts of interest if one is paid earlier.
- This lesson studies **deposit** 存款: An amount added to an account at a stated time.

The method

- Define the balance immediately after a stated event and give the initial balance.
- With growth factor r and an end-of-period deposit d, B_0=0 and B_n=rB_(n−1)+d.
- The nth balance is d(1+r+…+r^(n−1))=d(r^n−1)/(r−1) for r≠1.
- For r=1, use B_n=nd.
- The period count n is a nonnegative integer.

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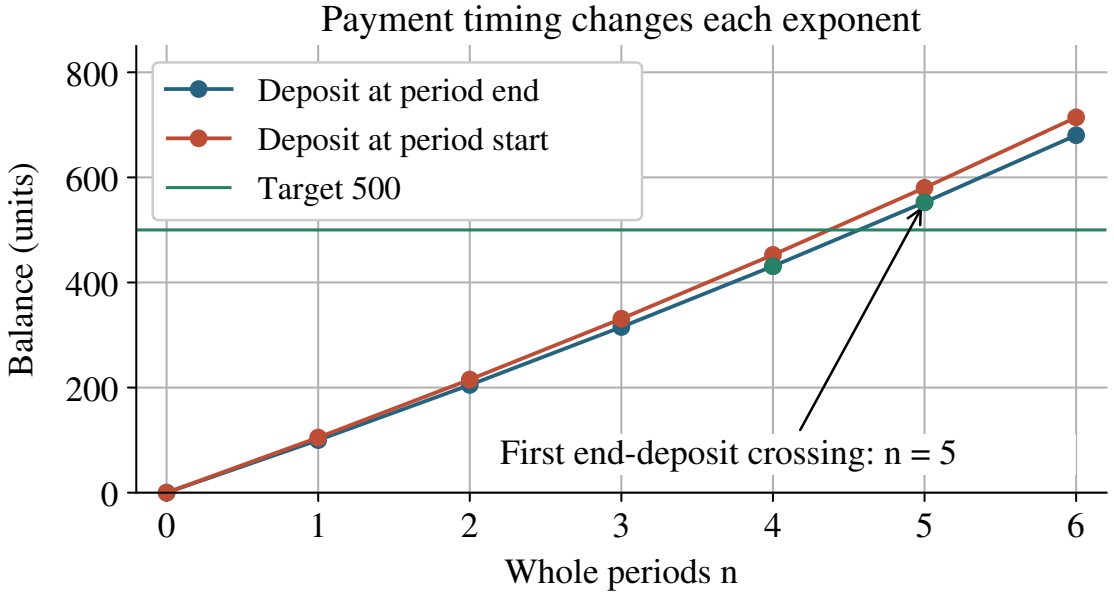
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A worked case

$$B_0 = 0, \quad B_n = rB_{n-1} + d, \quad B_n = d\frac{r^n - 1}{r - 1} \ (r \neq 1)$$

■ Worked example

Use an illustrative growth rate of 5% per period and deposit 100 units at each period end. B_1=100, B_2=1.05×100+100=205 and B_3=1.05×205+100=315.25. The first deposit earns interest for n−1 periods, while the final one earns none at the time B_n is measured. Thus B_n=100(1.05^n−1)/0.05. For a target of at least 500, B_4=431.0125<500 and B_5=552.563125≥500, so 5 periods first suffice. The balance exceeds the five deposits totalling 500 by 52.563125 units. If the same deposits are made at each period start and the balance is measured at the period end, C_n=1.05(B_n); hence C_3=331.0125, not 315.25.



Where it goes wrong

- Do not give the final end-of-period deposit an extra interest period.
- A balance includes both deposits and growth; it is not the interest alone.
- An exact target and an at-least target need different integer checks.
- A quoted annual rate cannot simply be used as a per-month factor without an appropriate conversion.

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■ **Watch out**

Not true: Beginning- and end-of-period deposits give the same final balance. A deposit at the start of a period earns one more period of interest, so the final balance is larger.

Applying it

- The illustration assumes a fixed per-period rate, equal deposits, no fees or withdrawals and the stated compounding/payment times.
- If these change, refine the recurrence: B_n=(1+i_n)B_(n−1)+d_n−w_n for end-of-period deposits/withdrawals.
- Keep a separate ledger of payments to identify growth.
- Compare adjacent period balances before claiming the first target crossing; here all balances are nonnegative and B_n−B_(n−1)=0.05B_(n−1)+100>0.

In your course

- AQA A-level Mathematics (7357), specification section D.
- Every paper allows a calculator, but method marks still need the expression or equation written before its value.

■ **Key idea**

- **Deposit**: An amount added to an account at a stated time.
- Define the balance immediately after a stated event and give the initial balance.
- Watch for: do not give the final end-of-period deposit an extra interest period.

E · Right triangles and non-right triangles

Which side does the ladder need?

- A ladder reaches a height of 4 m while its foot is 3 m from a wall. Which sides are known, and which angle do we need?
- This lesson studies **hypotenuse** 斜边: The side opposite the right angle in a right-angled triangle.

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The method

- In a right triangle $a^2+b^2=c^2$; $\sin\theta$=opposite/hypotenuse, $\cos\theta$=adjacent/hypotenuse and $\tan\theta$=opposite/adjacent.
- For other triangles, use the sine or cosine rule, or area=ab sin C/2.

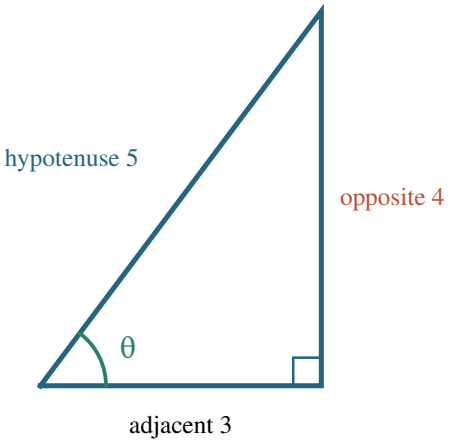
A worked case

$$a^2 + b^2 = c^2, \qquad \tan\theta = \frac{\text{opposite}}{\text{adjacent}}$$

Worked example

The ladder length is $c=\sqrt{(3^2+4^2)}=5$ m. Its angle to the ground satisfies $\tan\theta=4/3$, so $\theta\approx53.1°$. With two sides 6 and 8 enclosing 60°, $c^2=6^2+8^2-2\times6\times8\cos60°=52$.

Label the known sides before choosing a rule



opposite: across from θ
adjacent: next to θ
hypotenuse: across from the right angle

tan θ = 4/3 sin θ = 4/5
cos θ = 3/5

Where it goes wrong

- Label sides relative to the chosen angle.
- Pythagoras needs a right angle.
- A calculator angle mode error can produce a plausible but wrong result.
- Keep unrounded values for later steps.

■ **Watch out**

Not true: Pythagoras applies to every triangle, including triangles without a right angle. Pythagoras needs a right angle; for other triangles use the cosine rule.

Applying it

- Use a plan or elevation for a three-dimensional problem before applying a triangle rule.
- Explain why the chosen triangle contains the required length or angle.

In your course

- AQA A-level Mathematics (7357), specification section E.
- Every paper allows a calculator, but method marks still need the expression or equation written before its value.

■ **Key idea**

- **Hypotenuse**: The side opposite the right angle in a right-angled triangle.
- In a right triangle $a^2+b^2=c^2$; $\sin\theta$=opposite/hypotenuse, $\cos\theta$=adjacent/hypotenuse and $\tan\theta$=opposite/adjacent.
- Watch for: label sides relative to the chosen angle.

E · Radians, identities and trigonometric equations

How far does the rim travel?

- A rotating wheel sweeps an arc. Radians let its angle connect directly to the distance travelled along the rim.
- This lesson studies **radian** 弧度: The angle subtended by an arc whose length equals the radius.

The method

- For θ in radians, arc length $s=r\theta$ and sector area $A=r^2\theta/2$.
- Use $\sin^2\theta+\cos^2\theta=1$ and angle identities to simplify or solve.
- List every solution in the stated interval.

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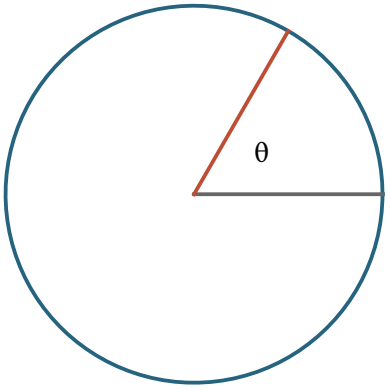
A worked case

$$s = r\theta, \qquad A = \frac{1}{2}r^2\theta, \qquad \sin^2\theta + \cos^2\theta = 1$$

■ Worked example

For r=4 and $\theta=\pi/3$, s=$4\pi/3$ and A=$8\pi/3$. Solving sin x=1/2 on $0\leq x<2\pi$ gives x=$\pi/6$ and $5\pi/6$. For sin2x=1/2, solve first on $0\leq 2x<4\pi$ before dividing by 2.

A radius fixes distance from the centre



radius × angle gives arc length

Where it goes wrong

- A calculator's inverse sine gives one principal value.
- It does not give every solution.
- Dividing by sin x can discard solutions where sin x=0; check that case first.

■ Watch out

Not true: The principal inverse sine value is the only solution in a full turn. Sine takes most values twice in a full turn: sin x=0.5 gives 30° and 150°.

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Applying it

- Choose an identity that reduces the equation to one trigonometric function.
- Keep exact familiar angles where possible, and check each solution against the original equation.

In your course

- AQA A-level Mathematics (7357), specification section E.
- Every paper allows a calculator, but method marks still need the expression or equation written before its value.

■ Key idea

- **Radian**: The angle subtended by an arc whose length equals the radius.
- For θ in radians, arc length $s=r\theta$ and sector area $A=r^2\theta/2$.
- Watch for: A calculator's inverse sine gives one principal value.

E · Radian measure, arcs, sectors and segments

How can we find the glass area without counting the triangle below the arc?

- A curved window has a straight lower edge.
- This lesson studies **sector** 扇形: The region bounded by two radii and their circular arc.

The method

- One radian is the central angle with arc length equal to the radius.
- A full turn is 2π radians$=360°$, so radians$=$degrees$\times\pi/180$.
- For $r>0$ and a nonnegative swept angle θ in radians, arc length $s=r\theta$ and sector area $A=r^2\theta/2$.
- A sector perimeter is $2r+s$, not s alone.
- Specify minor or major region.

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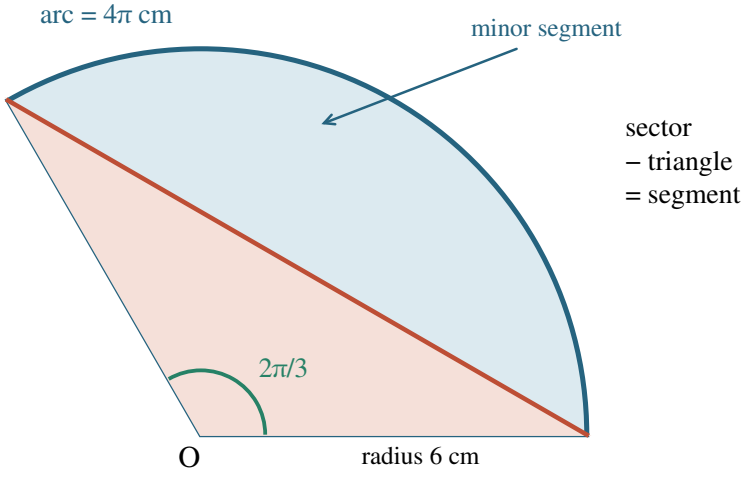
A worked case

$$s = r\theta, \quad A_{\text{sector}} = \frac{1}{2}r^2\theta, \quad A_{\text{minor segment}} = \frac{1}{2}r^2(\theta - \sin\theta)$$

■ Worked example

With r=6 cm and θ=120°=2π/3, s=4π cm and sector area=12π cm^2. Bisect the isosceles triangle: chord=2r sin(θ/2)=12 sin(π/3)=6$\sqrt{3}$ cm. The triangle area is r^2 sinθ/2=9$\sqrt{3}$ cm^2. The minor segment above the chord has area 12π−9$\sqrt{3}$ cm^2. Its boundary length is arc+chord=4π+6$\sqrt{3}$ cm. The complementary major segment has area 36π−(12π−9$\sqrt{3}$)=24π+9$\sqrt{3}$ cm^2. In an inverse task, radius 5 cm and arc 8 cm give θ=8/5=1.6 radians and sector area=5×8/2=20 cm^2.

A chord separates the triangle from the minor segment



Where it goes wrong

- Never put 120 directly into rθ: first convert degrees to radians.
- A chord is straight and generally shorter than its minor arc.
- Sector, segment and triangle have different boundaries.
- The minor-segment subtraction shown uses 0≤θ≤π; for a major segment subtract the complementary minor region from the disk.

■ Watch out

Not true: The perimeter of a sector consists only of its circular arc. The perimeter also includes the two radii: arc length plus 2r.

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Applying it

- Draw the radii, chord and required boundary first.
- Recover θ from s/r or $2A/r^2$ when needed.
- Keep π and surds exact; attach length or area units only after choosing the correct formula.
- In the inverse example the sector perimeter is 10+8=18 cm.
- The checked inverse example stays radius 5 cm and arc 8 cm.
- The later problems use other lengths and angles.
- Decide sector or segment before you substitute.
- A major segment is the disk minus the minor segment on the same chord.

In your course

- AQA A-level Mathematics (7357), specification section E.
- Every paper allows a calculator, but method marks still need the expression or equation written before its value.

■ Key idea

- **Sector**: The region bounded by two radii and their circular arc.
- One radian is the central angle with arc length equal to the radius.
- Watch for: never put 120 directly into rθ: first convert degrees to radians.

E · Small-angle approximations and their limits

When is replacing a trigonometric function by the angle itself accurate enough?

- A small turn moves a pointer almost sideways.
- This lesson studies **approximation** 近似: A nearby simpler value used with a stated accuracy limit.

The method

- When θ is close to zero and measured in radians, $\sin\theta\approx\theta$, $\tan\theta\approx\theta$ and $\cos\theta\approx1-\theta^2/2$.
- These are approximations, not identities.
- The discarded sine/tangent terms begin at order θ^3; the discarded cosine term begins at order θ^4.
- A smaller absolute angle usually improves these local approximations.

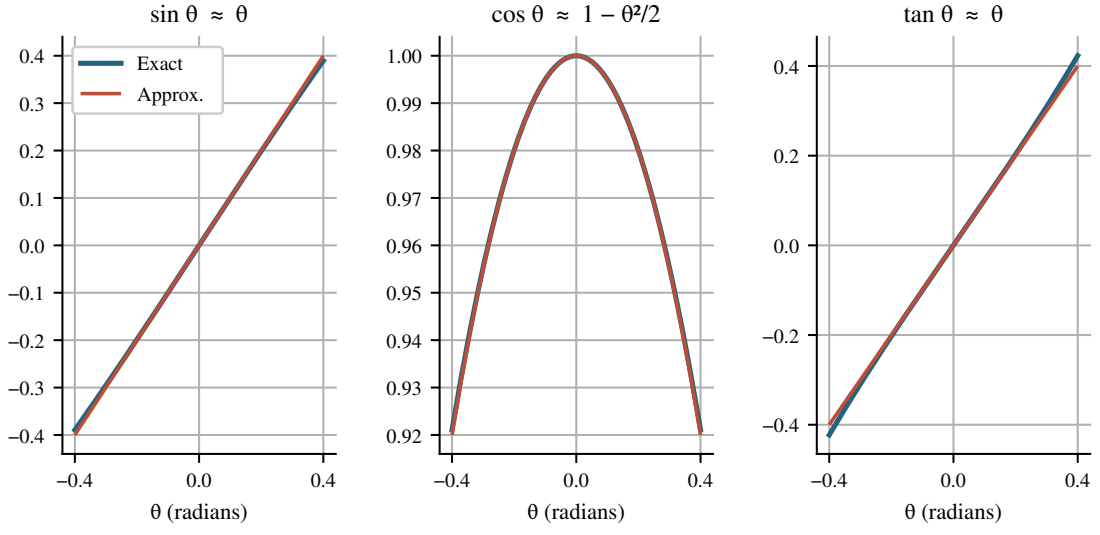
A worked case

$$\sin\theta \approx \theta, \quad \tan\theta \approx \theta, \quad \cos\theta \approx 1 - \frac{\theta^2}{2} \quad (\theta \text{ in radians})$$

Worked example

At θ=0.1 radians, sinθ≈0.1 (actual 0.0998334), tanθ≈0.1 (actual 0.100335) and cosθ≈0.995 (actual 0.995004). Absolute errors are about 0.000167, 0.000335 and 0.00000417. For 6(1−cos x)=0.03, approximate 1−cos x by $x^2/2$: $3x^2$≈0.03, so x≈±0.1 radians. Check the small-angle assumption after solving; substitution at x=0.1 gives 0.0299750 rather than exactly 0.03. Also sin(2x)/x≈2 for small nonzero x, while 1−cos(2x)≈$2x^2$.

Local agreement improves as the angle approaches zero



Where it goes wrong

- Degrees do not work in these formulas: sin(1°)≈π/180, not 1.
- Write ≈ rather than =.
- Subtraction can make relative error more important; preserve the θ^2 term in 1−cosθ.
- A ratio at θ=0 may be undefined even when its nearby limit exists.

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■ **Watch out**

Not true: The formula $\sin\theta \approx \theta$ is equally valid for an angle entered in degrees. $\sin\theta \approx \theta$ is valid only with θ in radians and small.

Applying it

- Convert to radians first and identify the argument of each trig function: $\sin(3x) \approx 3x$, not x.
- For a model, compare the approximation error with the precision required.
- If a solution is not small, use the original trigonometric equation; these formulas have no universal accuracy cutoff.

In your course

- AQA A-level Mathematics (7357), specification section E.
- Every paper allows a calculator, but method marks still need the expression or equation written before its value.

■ **Key idea**

- **Approximation**: A nearby simpler value used with a stated accuracy limit.
- When θ is close to zero and measured in radians, $\sin\theta \approx \theta$, $\tan\theta \approx \theta$ and $\cos\theta \approx 1 - \theta^2/2$.
- Watch for: degrees do not work in these formulas: $\sin(1°) \approx \pi/180$, not 1.

E · All-angle definitions, exact values and periodic graphs

Which coordinate gives its vertical position after more than half a turn?

- A wheel turns beyond a right angle.
- This lesson studies **unit circle** 单位圆: The circle of radius one centred at the coordinate origin.

The method

- For a signed angle θ from the positive x-axis, the unit-circle point is $(\cos\theta, \sin\theta)$; $\tan\theta = \sin\theta/\cos\theta$ where $\cos\theta \neq 0$.
- Sine and cosine have period 2π and range $[-1, 1]$.

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- Tangent has period π, every real output and vertical asymptotes $\theta{=}\pi/2{+}k\pi$ for integer k.
- Sine/tangent are odd functions; cosine is even.

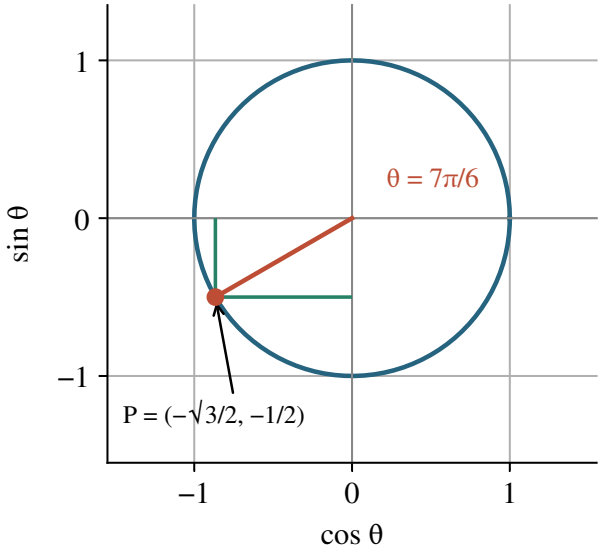
A worked case

$$(x, y) = (\cos\theta, \sin\theta), \qquad \tan\theta = \frac{\sin\theta}{\cos\theta}$$

■ Worked example

Special triangles give (sinθ,cosθ): at 0 use (0,1); $\pi/6$ gives (1/2,$\sqrt{3}/2$); $\pi/4$ gives ($\sqrt{2}/2$,$\sqrt{2}/2$); $\pi/3$ gives ($\sqrt{3}/2$,1/2); $\pi/2$ gives (1,0); π gives (0,-1). Divide to obtain tangent, excluding $\pi/2$. Reference angles and quadrant signs give $\sin(7\pi/6){=}-1/2$, $\cos(4\pi/3){=}-1/2$ and $\tan(3\pi/4){=}-1$. Subtract complete turns before finding the quadrant: $\sin(19\pi/6){=}\sin(7\pi/6)$. For y=2 sin(3x$-\pi$)+1, amplitude=2, period=$2\pi/3$, centre line y=1 and range [-1,3]. Writing 3x$-\pi$=3(x$-\pi/3$) gives a right shift $\pi/3$.

Coordinates define sine and cosine beyond acute angles



Where it goes wrong

- A negative angle is clockwise; do not automatically make its sine positive.
- Tan($\pi/2$) is undefined rather than a very large finite number.
- Multiply the input frequency to shorten the period; adding a vertical shift does not change the period.

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- The sine graph crosses its centre line rather than always crossing y=0.

Watch out

Not true: Sine and cosine repeat after π radians for every input angle. Sine and cosine have period 2π; after π radians they change sign.

Applying it

- Build a graph from a full cycle of key points, its symmetry and period.
- For $\cos\theta$ use reflection symmetry about $\theta=0$; for $\sin\theta/\tan\theta$ use origin symmetry.
- Keep exact reference values, then use a calculator only when an unfamiliar angle needs a decimal.
- State angle mode and the domain interval.

In your course

- AQA A-level Mathematics (7357), specification section E.

Key idea

- **Unit circle**: The circle of radius one centred at the coordinate origin.
- For a signed angle θ from the positive x-axis, the unit-circle point is $(\cos\theta,\sin\theta)$; $\tan\theta=\sin\theta/\cos\theta$ where $\cos\theta\neq0$.
- Watch for: A negative angle is clockwise; do not automatically make its sine positive.

E · Reciprocal trigonometric functions and excluded angles

What happens as that coordinate approaches zero?

- A formula divides by a rotating point's horizontal coordinate.
- This lesson studies **secant** 正割: The reciprocal of cosine wherever cosine is nonzero.

The method

- $\sec\theta=1/\cos\theta$ and $\operatorname{cosec}\theta=1/\sin\theta$.
- Define $\cot\theta=\cos\theta/\sin\theta$, so it exists when $\sin\theta\neq0$.
- Secant excludes $\theta=\pi/2+k\pi$; cosecant and cotangent exclude $\theta=k\pi$, for integer k.
- Secant/cosecant have period 2π and range $(-\infty,-1]\cup[1,\infty)$.

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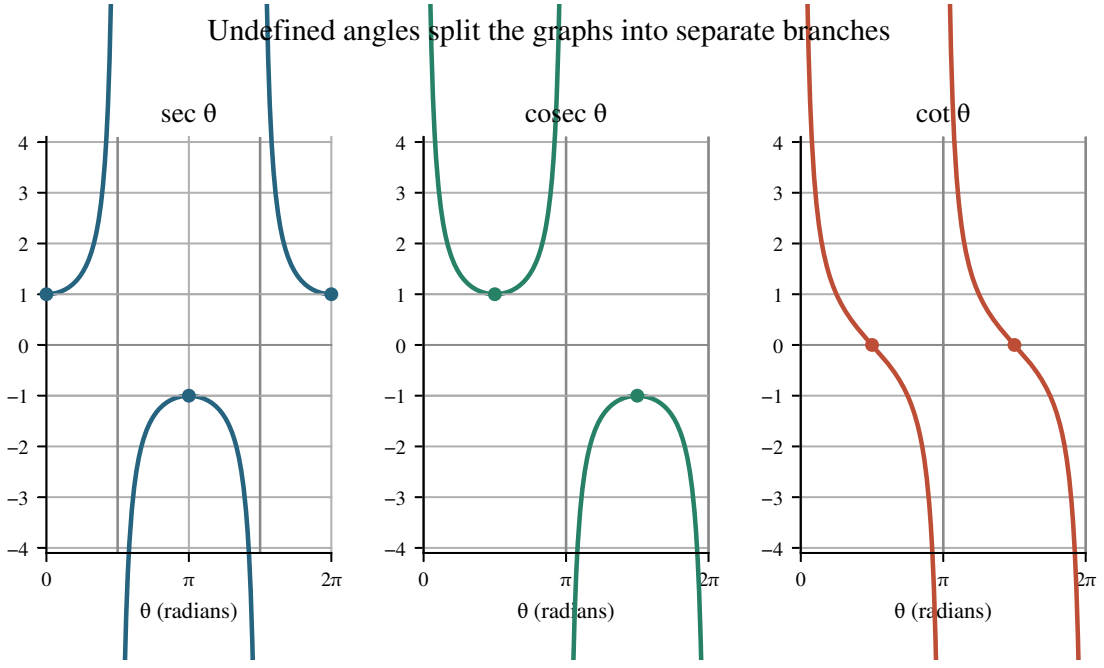
- Cotangent has period π and every real output.
- Secant is even; cosecant/cotangent are odd.

A worked case

$$\sec\theta = \frac{1}{\cos\theta}, \quad \operatorname{cosec}\theta = \frac{1}{\sin\theta}, \quad \cot\theta = \frac{\cos\theta}{\sin\theta}$$

Worked example

At $\theta{=}2\pi/3$, $\cos\theta{=}{-}1/2$ and $\sin\theta{=}\sqrt{3}/2$, so $\sec\theta{=}{-}2$, $\operatorname{cosec}\theta{=}2\sqrt{3}/3$ and $\cot\theta{=}{-}\sqrt{3}/3$. At $\theta{=}\pi/2$, $\cot\theta{=}0/1{=}0$ and $\operatorname{cosec}\theta{=}1$, but $\sec\theta$ is undefined. Draw secant branches outside the strip $-1<y<1$: $y{=}1$ at 0, $y{=}{-}1$ at π, with vertical asymptotes at $\pi/2$ and $3\pi/2$. Cosecant reaches 1 at $\pi/2$ and -1 at $3\pi/2$, with asymptotes at 0, π and 2π. Cotangent decreases from positive to negative infinity on $(0,\pi)$, crossing zero at $\pi/2$; repeat after π.



Where it goes wrong

- $\cot\theta{=}1/\tan\theta$ is usable only where both expressions exist and $\tan\theta{\neq}0$.
- It cannot define $\cot(\pi/2)$, because $\tan(\pi/2)$ is undefined while $\cot(\pi/2){=}0$.
- The exponent -1 in $\sin^{-1}$ often denotes inverse sine, not the reciprocal; write $\operatorname{cosec}\theta$ when you mean $1/\sin\theta$.

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■ **Watch out**

Not true: Cotangent is undefined at every angle where tangent is undefined. cot θ=cos θ/sin θ: where tan is undefined, cos θ=0, so cot θ=0.

Applying it

- Sketch the base sine/cosine zeros first: these locate the reciprocal asymptotes.
- Values 1 and −1 are unchanged by taking a reciprocal.
- Split the domain at each excluded angle and never join branches through an asymptote.
- State the function's domain before rearranging an equation involving its denominator.

In your course

- AQA A-level Mathematics (7357), specification section E.
- Every paper allows a calculator, but method marks still need the expression or equation written before its value.

■ **Key idea**

- **Secant**: The reciprocal of cosine wherever cosine is nonzero.
- secθ=1/cosθ and cosecθ=1/sinθ.
- Watch for: cotθ=1/tanθ is usable only where both expressions exist and tan$\theta\neq$0.

E · Inverse trigonometric branches and principal values

How does a calculator choose just one of them?

- A height ratio gives many possible wheel angles.
- This lesson studies **principal value** 主值: The single inverse output chosen from a stated restricted interval.

The method

- Restrict sine to $[-\pi/2,\pi/2]$ to define arcsin with domain $[-1,1]$ and range $[-\pi/2,\pi/2]$.
- Restrict cosine to $[0,\pi]$ for arccos, with domain $[-1,1]$ and range $[0,\pi]$.

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- Restrict tangent to $(-\pi/2,\pi/2)$ for arctan, with every real input and range $(-\pi/2,\pi/2)$.
- These inverse graphs reflect the restricted base graphs in y=x.

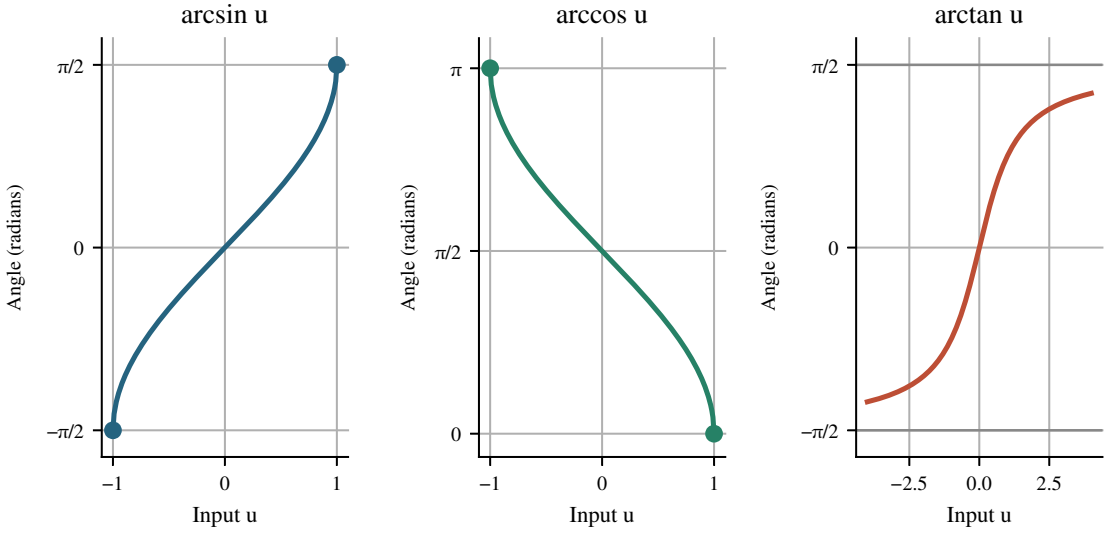
A worked case

$$\arcsin : [-1,1] \to [-\pi/2,\pi/2], \quad \arccos : [-1,1] \to [0,\pi]$$

Worked example

$\arcsin(1/2)=\pi/6$, $\arccos(-1/2)=2\pi/3$ and $\arctan(-1)=-\pi/4$. Although $\sin(5\pi/6)=1/2$, $\arcsin(\sin(5\pi/6))=\pi/6$, not $5\pi/6$. Similarly $\arccos(\cos(4\pi/3))=2\pi/3$ and $\arctan(\tan(3\pi/4))=-\pi/4$. For sin x=1/2 on $[0,2\pi)$, the principal value $\alpha=\pi/6$ generates both α and $\pi-\alpha=5\pi/6$. For cos x=−1/2 the solutions are $2\pi/3$ and $4\pi/3$; tangent repeats its principal value after π.

Principal inverse branches have different output intervals



Where it goes wrong

- $\sin(\arcsin u)=u$ only for $-1\le u\le 1$; $\arcsin(\sin x)=x$ only on the restricted sine branch.
- Arccos is decreasing, so $\arccos(-1)=\pi$ and $\arccos(1)=0$.
- Arctan approaches $\pm\pi/2$ as input grows in magnitude but never attains those limits.
- A principal value alone is not a full interval solution set.

■ **Watch out**

Not true: Applying inverse sine after sine returns the original angle for every real angle. arcsin returns an angle between $-\pi/2$ and $\pi/2$, so angles outside that range are not returned.

Applying it

- Mark domain endpoints and output intervals before composing functions.
- On the inverse graphs arcsin runs from $(-1,-\pi/2)$ to $(1,\pi/2)$, arccos from $(-1,\pi)$ to $(1,0)$, and arctan passes through $(0,0)$ between horizontal asymptotes.
- Keep answers in radians unless degrees are requested, and check all equation candidates in the original interval.

In your course

- AQA A-level Mathematics (7357), specification section E.
- Every paper allows a calculator, but method marks still need the expression or equation written before its value.

■ **Key idea**

- **Principal value**: The single inverse output chosen from a stated restricted interval.
- Restrict sine to $[-\pi/2,\pi/2]$ to define arcsin with domain $[-1,1]$ and range $[-\pi/2,\pi/2]$.
- Watch for: $\sin(\arcsin u)=u$ only for $-1\leq u\leq 1$; $\arcsin(\sin x)=x$ only on the restricted sine branch.

E · Reciprocal identities and one-sided proof chains

Does that prove they agree for every angle where they are defined?

- Two formulas agree at one angle.
- This lesson studies **identity** 恒等式: An equality true for every input in its stated common domain.

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The method

- The unit-circle equation gives $\sin^2\theta+\cos^2\theta=1$.
- Divide by $\cos^2\theta$ when $\cos\theta\neq0$ to obtain $\tan^2\theta+1=\sec^2\theta$.
- Divide by $\sin^2\theta$ when $\sin\theta\neq0$ to obtain $1+\cot^2\theta=\operatorname{cosec}^2\theta$.
- Record exclusions before cancelling or dividing.
- A proof starts from one side and reaches the other through valid equalities; numerical checks alone are not a proof.

A worked case

$$\sec^2\theta = 1+\tan^2\theta, \qquad \operatorname{cosec}^2\theta = 1+\cot^2\theta$$

■ Worked example

Prove $\tan^2\theta-\sin^2\theta=\tan^2\theta\ \sin^2\theta$ for $\cos\theta\neq0$. The left side is $\sin^2\theta(1/\cos^2\theta-1)=\sin^2\theta(1-\cos^2\theta)/\cos^2\theta=\sin^4\theta/\cos^2\theta=\tan^2\theta\ \sin^2\theta$. No division by $\sin\theta$ is needed, so $\sin\theta=0$ remains allowed. For $\sin\theta\neq0$, $(1-\cos\theta)/\sin\theta$ becomes $(1-\cos\theta)(1+\cos\theta)/[\sin\theta(1+\cos\theta)]=\sin^2\theta/[\sin\theta(1+\cos\theta)]=\sin\theta/(1+\cos\theta)$. The stated exclusion also makes $1+\cos\theta$ nonzero. If $\sec\theta=2$ with θ acute, $\tan^2\theta=4-1=3$, $\cos\theta=1/2$, $\sin^2\theta=3/4$, $\operatorname{cosec}^2\theta=4/3$ and $\cot^2\theta=1/3$.

A proof chain keeps its common domain

$$\tan^2\theta-\sin^2\theta$$

$$=\sin^2\theta\left(\frac{1}{\cos^2\theta}-1\right)$$

$$=\frac{\sin^2\theta(1-\cos^2\theta)}{\cos^2\theta}$$

$$=\frac{\sin^4\theta}{\cos^2\theta}=\tan^2\theta\sin^2\theta$$

$\cos\theta \neq 0$
$\sin\theta = 0$ allowed

Where it goes wrong

- Proving an identity does not mean assuming both sides equal and manipulating that assumption without justification.
- A squared identity does not fix a function's sign: the quadrant does.
- The last expression $\sin\theta/(1+\cos\theta)$ is defined at $\theta=0$, but the original $(1-\cos\theta)/\sin\theta$ is not; simplification does not repair the original domain.

■ Watch out

Not true: Checking a proposed identity at one angle proves it for all allowed angles. An identity must hold for every allowed angle; one test can agree by chance.

Applying it

- Choose the more complicated side, replace reciprocal functions by sine/cosine, use a common denominator and apply $\sin^2+\cos^2=1$.
- Annotate every cancelled factor.
- To disprove a claimed identity, one valid counterexample suffices; to prove it, explain why the chain works for all angles in the common domain.

In your course

- AQA A-level Mathematics (7357), specification section E.
- Every paper allows a calculator, but method marks still need the expression or equation written before its value.

■ Key idea

- **Identity**: An equality true for every input in its stated common domain.
- The unit-circle equation gives $\sin^2\theta+\cos^2\theta=1$.
- Watch for: proving an identity does not mean assuming both sides equal and manipulating that assumption without justification.

E · Compound-angle formulae and geometric proofs

Can its length explain why cosine of a difference is not the difference of two cosines?

- Two unit-radius points define a chord.
- This lesson studies **compound angle** 复合角: An angle expressed as the sum or difference of two angles.

The method

- sin(A±B)=sinA cosB±cosA sinB. cos(A±B)=cosA cosB∓sinA sinB: the cosine sign changes. tan(A±B)=(tanA±tanB)/(1∓tanA tanB) where the tangents and denominator are defined.
- Derive tangent by dividing the corresponding sine and cosine formulas; do not extend a quotient through a zero denominator.

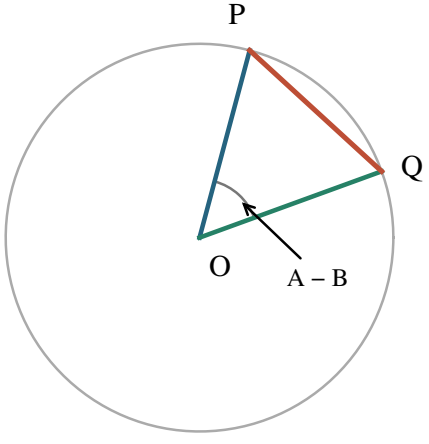
A worked case

$$\cos(A-B)=\cos A\cos B+\sin A\sin B,\quad \sin(A+B)=\sin A\cos B+\cos A\sin B$$

Worked example

Let P=(cosA,sinA) and Q=(cosB,sinB) on the unit circle. Coordinate distance gives PQ^2=2−2(cosA cosB+sinA sinB). The cosine rule on the two unit radii gives PQ^2=2−2cos(A−B), hence the cosine difference formula. Replace B by −B, using cosine even/sine odd, to obtain cosine addition. Apply cosine difference to (π/2−A) and B to obtain sin(A+B); then change B's sign for sine subtraction. Exact evaluations: sin75°=sin(45°+30°)=(√6+√2)/4; cos75°=(√6−√2)/4; sin15°=(√6−√2)/4, cos15°=(√6+√2)/4 and tan15°=2−√3.

Measure the same chord in two ways



Coordinate distance

$$PQ^2 = 2 - 2(\cos A\cos B + \sin A\sin B)$$

Cosine rule, radii both 1

$$PQ^2 = 2 - 2\cos(A - B)$$

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Where it goes wrong

- sin(A+B) is not sinA+sinB.
- The cosine rule chord uses the smaller included angle when necessary, but its cosine is still cos(A−B).
- Coordinate distance and the resulting identities hold for all signed angles; the initial diagram is one illustrative case. tan(45°+45°) is undefined because 1−tan45°tan45°=0.

■ Watch out

Not true: Cosine of a sum equals the sum of the two cosine values. cos(A+B)=cos A cos B−sin A sin B, which is not cos A+cos B.

Applying it

- Choose familiar angles whose sum or difference is the target.
- Keep exact roots and rationalise a tangent quotient if needed.
- In a geometric proof, state both expressions for the same chord rather than assuming the desired identity.
- Check the denominator before applying a tangent addition/subtraction formula.

In your course

- AQA A-level Mathematics (7357), specification section E.
- Every paper allows a calculator, but method marks still need the expression or equation written before its value.

■ Key idea

- **Compound angle**: An angle expressed as the sum or difference of two angles.
- sin(A±B)=sinA cosB±cosA sinB. cos(A±B)=cosA cosB∓sinA sinB: the cosine sign changes. tan(A±B)=(tanA±tanB)/(1∓tanA tanB) where the tangents and denominator are defined.
- Watch for: sin(A+B) is not sinA+sinB.

E · Double-angle identities, quadrant signs and squared forms

Which signs survive the calculation?

- Doubling a wheel angle can move its point into another quadrant.
- This lesson studies **double angle** 倍角: An angle equal to twice the original angle.

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The method

- Set B=A=θ in the compound formulas: $\sin 2\theta = 2\sin\theta\cos\theta$; $\cos 2\theta = \cos^2\theta - \sin^2\theta = 1 - 2\sin^2\theta = 2\cos^2\theta - 1$.
- Where $\tan\theta$ and $\tan 2\theta$ exist, $\tan 2\theta = 2\tan\theta/(1-\tan^2\theta)$.
- Rearranging gives $\sin^2\theta = (1-\cos 2\theta)/2$ and $\cos^2\theta = (1+\cos 2\theta)/2$.
- These identities depend on the full angle 2θ, not on squaring the function.

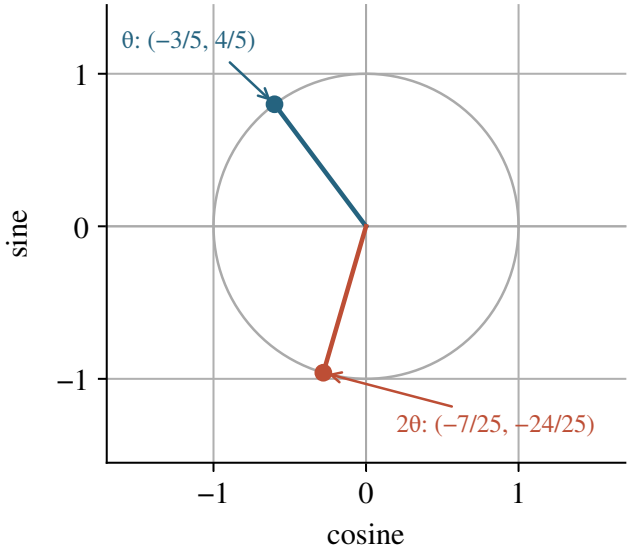
A worked case

$$\sin 2\theta = 2\sin\theta\cos\theta, \qquad \cos 2\theta = 1 - 2\sin^2\theta = 2\cos^2\theta - 1$$

■ Worked example

If $\cos\theta = -3/5$ and θ is in quadrant II, $\sin\theta = +4/5$. Then $\sin 2\theta = 2(4/5)(-3/5) = -24/25$ and $\cos 2\theta = 9/25 - 16/25 = -7/25$. Both are negative, so the doubled angle lies in quadrant III modulo a turn; $\tan 2\theta = 24/7$. The tangent formula gives the same answer from $\tan\theta = -4/3$. Recover $\sin^2\theta$ from $\cos 2\theta$: $[1-(-7/25)]/2 = 16/25$. At $\theta = \pi/4$, the sine/cosine double-angle formulas still work, but $\tan 2\theta$ is undefined. At $\theta = \pi/2$, $\tan\theta$ is undefined while $\tan 2\theta = 0$; use sine/cosine rather than the tangent quotient.

A doubled angle can move into a different quadrant



Where it goes wrong

- $\sin 2\theta$ means $\sin(2\theta)$; $\sin^2\theta$ means $(\sin\theta)^2$.
- A squared value gives a magnitude but not a sign; use the stated quadrant before taking a root.
- Never infer the doubled angle's quadrant from the original one without checking signs.
- Simplified identities do not remove the original tangent exclusions.

■ Watch out

Not true: The functions $\sin 2\theta$ and $\sin^2\theta$ have the same value for every angle. $\sin 2\theta = 2\sin\theta\cos\theta$, while $\sin^2\theta$ means $(\sin\theta)^2$: at $\theta = 90°$ they are 0 and 1.

Applying it

- Choose the cosine form that matches the known quantity: $2\cos^2\theta - 1$ when cosine is given, or $1 - 2\sin^2\theta$ when sine is given.
- Use the squared forms to replace a trig square with a constant and a double-angle term.
- Verify a derived sign against the unit circle and retain exact fractions until a decimal is requested.

In your course

- AQA A-level Mathematics (7357), specification section E.
- Every paper allows a calculator, but method marks still need the expression or equation written before its value.

■ Key idea

- **Double angle**: An angle equal to twice the original angle.
- Set $B = A = \theta$ in the compound formulas: $\sin 2\theta = 2\sin\theta\cos\theta$; $\cos 2\theta = \cos^2\theta - \sin^2\theta = 1 - 2\sin^2\theta = 2\cos^2\theta - 1$.
- Watch for: $\sin 2\theta$ means $\sin(2\theta)$; $\sin^2\theta$ means $(\sin\theta)^2$.

E · Auxiliary-angle forms, ranges and interval solutions

Can one shifted curve show their combined maximum and minimum?

- Two periodic effects act together.

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- This lesson studies **amplitude** 振幅: The nonnegative size of an oscillation about its centre line.

The method

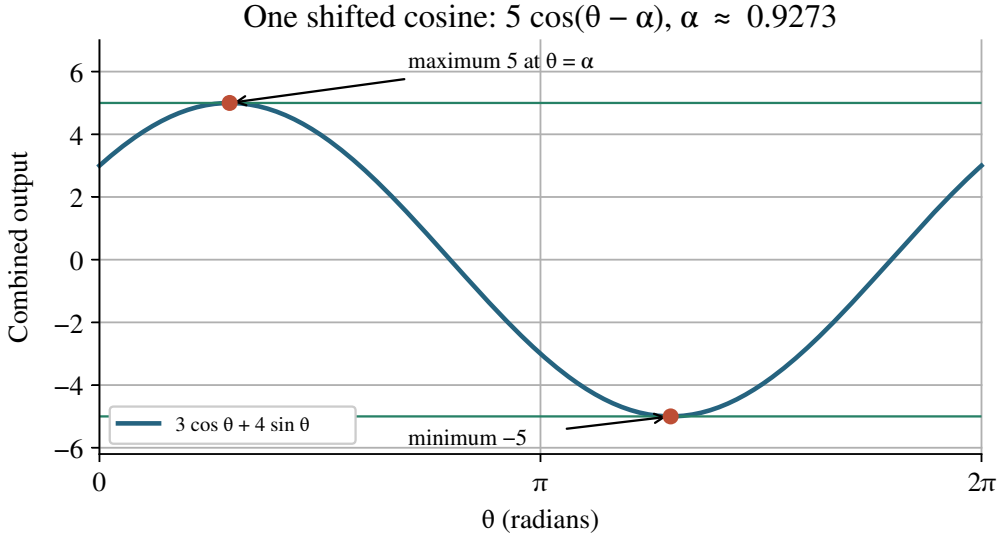
- For a and b not both zero, write a cosθ+b sinθ=R cos(θ−α), where R=√(a²+b²)>0, cosα=a/R and sinα=b/R.
- Choose α in the correct quadrant.
- The equivalent sine form R sin(θ+β) has sinβ=a/R and cosβ=b/R.
- A constant c shifts the range to [c−R,c+R].
- If a=b=0, the expression is zero and no unique phase is defined.

A worked case

$$a\cos\theta + b\sin\theta = R\cos(\theta - \alpha), \quad R = \sqrt{a^2 + b^2}$$

Worked example

3cosθ+4sinθ=5cos(θ−α), with cosα=3/5, sinα=4/5 and α≈0.927295 radians. Its sine form is 5sin(θ+β), where β≈0.643501 radians. Thus 2+3cosθ+4sinθ ranges from −3 to 7. For 3cosθ−4sinθ, use 5cos(θ+α). For −3cosθ+4sinθ, the cosine phase is in quadrant II, not the negative principal arctangent. To solve 3cosθ+4sinθ=2 on [0,2π), put δ=arccos(2/5)≈1.159279. Then θ=α±δ+2kπ; the two allowed values are about 2.086575 and 6.051201 radians. A right side of 6 would give no solution, since the amplitude is only 5.



Where it goes wrong

- A plain arctan(b/a) can select the wrong quadrant and fails when a=0.
- Match both sine and cosine coefficients.
- The amplitude is $\sqrt{(a^2+b^2)}$, not $|a|+|b|$.
- Do not treat the principal inverse cosine value as the only solution.
- The angle-mode unit must match the stated interval.

■ Watch out

Not true: The amplitude of $a\cos\theta+b\sin\theta$ is $|a|+|b|$. The amplitude is $R=\sqrt{(a^2+b^2)}$: for $3\cos\theta+4\sin\theta$ it is 5, not 7.

Applying it

- Expand your proposed $R\cos(\theta-\alpha)$ or $R\sin(\theta+\beta)$ to verify every coefficient and sign.
- For equations, compare the target with the range first, generate both phase branches and translate each by full turns into the original interval.
- At the upper or lower range endpoint the two branches coincide modulo 2π.

In your course

- AQA A-level Mathematics (7357), specification section E.
- Every paper allows a calculator, but method marks still need the expression or equation written before its value.

■ Key idea

- **Amplitude**: The nonnegative size of an oscillation about its centre line.
- For a and b not both zero, write $a\cos\theta+b\sin\theta=R\cos(\theta-\alpha)$, where $R=\sqrt{(a^2+b^2)}>0$, $\cos\alpha=a/R$ and $\sin\alpha=b/R$.
- Watch for: A plain arctan(b/a) can select the wrong quadrant and fails when a=0.

E · Quadratic trigonometric equations and complete root sets

How many angles do they create in one full turn?

- An equation gives two possible sine values.

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- This lesson studies **candidate** 候选解: A possible solution that still needs checking in the original equation and interval.

The method

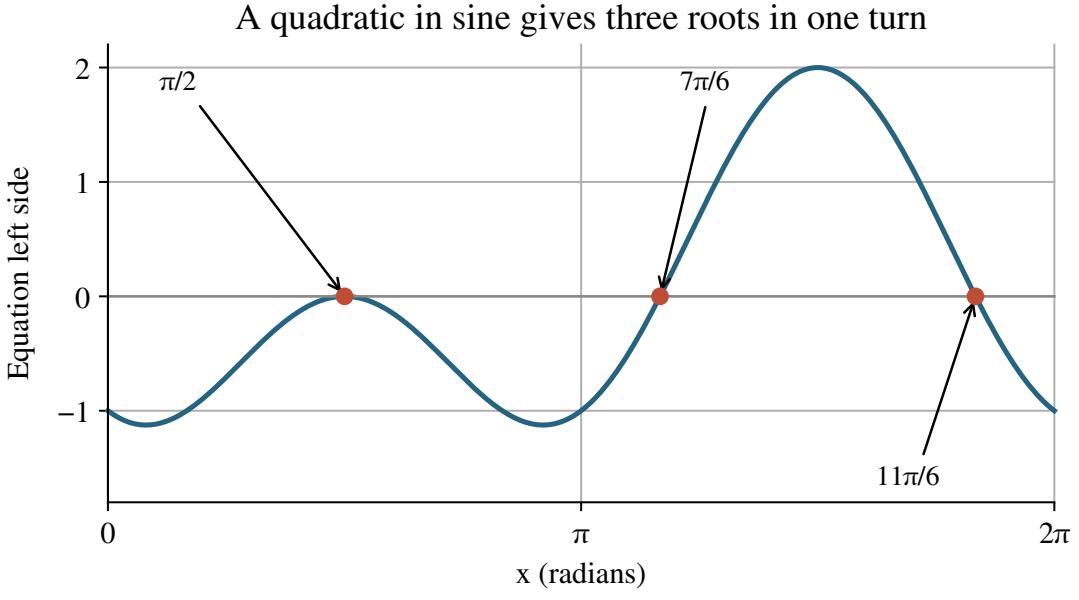
- Replace one trig function by u, solve the algebraic equation, then solve each valid trig value on the stated interval.
- For sine/cosine, discard u outside [−1,1].
- Tangent accepts every finite real u but excludes angles where cosine is zero.
- Factor before dividing by a trig function, so zero cases are retained.

A worked case

$$2\sin^2 x - \sin x - 1 = (2\sin x + 1)(\sin x - 1)$$

Worked example

On $0 \le x < 2\pi$, solve $2\sin^2 x - \sin x - 1 = 0$. Put $u = \sin x$: $(2u+1)(u-1)=0$, so $u = -1/2$ or 1. The solutions are $7\pi/6$, $11\pi/6$ and $\pi/2$; sort them as $\pi/2$, $7\pi/6$, $11\pi/6$. For $2\cos^2 x - 5\cos x + 2 = 0$, the algebraic values are $1/2$ and 2; discard 2, leaving $x = \pi/3, 5\pi/3$. For $\tan^2 x - 3\tan x + 2 = 0$, $\tan x = 1$ or 2. Thus $x = \pi/4, 5\pi/4, \arctan 2, \pi + \arctan 2$, with all four checked in the original equation. For $\sin x \cos x = \sin x$, factor $\sin x(\cos x - 1) = 0$: $x = 0$ or π. Cancelling $\sin x$ would lose π.



Where it goes wrong

- Count angles rather than only algebraic roots: a sine value 1 has one angle per turn, while most interior values have two.
- Reject impossible sine/cosine outputs before taking inverse functions.
- Do not include 2π in a half-open interval, and do not list a repeated endpoint twice.

Watch out

Not true: Every real root of a quadratic in cosx is an allowed cosine value. Cosine lies between -1 and 1, so a root such as cos x=2 must be rejected.

Applying it

- Use an identity first if both sine and cosine occur: $2\sin^2x+\cos x-2=0$ becomes $\cos x(1-2\cos x)=0$, giving $x=\pi/2,3\pi/2,\pi/3,5\pi/3$ on $[0,2\pi)$.
- A factorised product is zero when either factor is zero.
- Verify every candidate against the original domain and interval, especially after division or squaring.

In your course

- AQA A-level Mathematics (7357), specification section E.
- Every paper allows a calculator, but method marks still need the expression or equation written before its value.

Key idea

- **Candidate**: A possible solution that still needs checking in the original equation and interval.
- Replace one trig function by u, solve the algebraic equation, then solve each valid trig value on the stated interval.
- Watch for: count angles rather than only algebraic roots: a sine value 1 has one angle per turn, while most interior values have two.

E · Multiple-angle equations and transformed intervals

How can we avoid losing half the solutions?

- Doubling an angle doubles the number of cycles searched.
- This lesson studies **transformed interval** 变换后的区间: The interval for a substituted angle after applying the same input change to its endpoints.

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The method

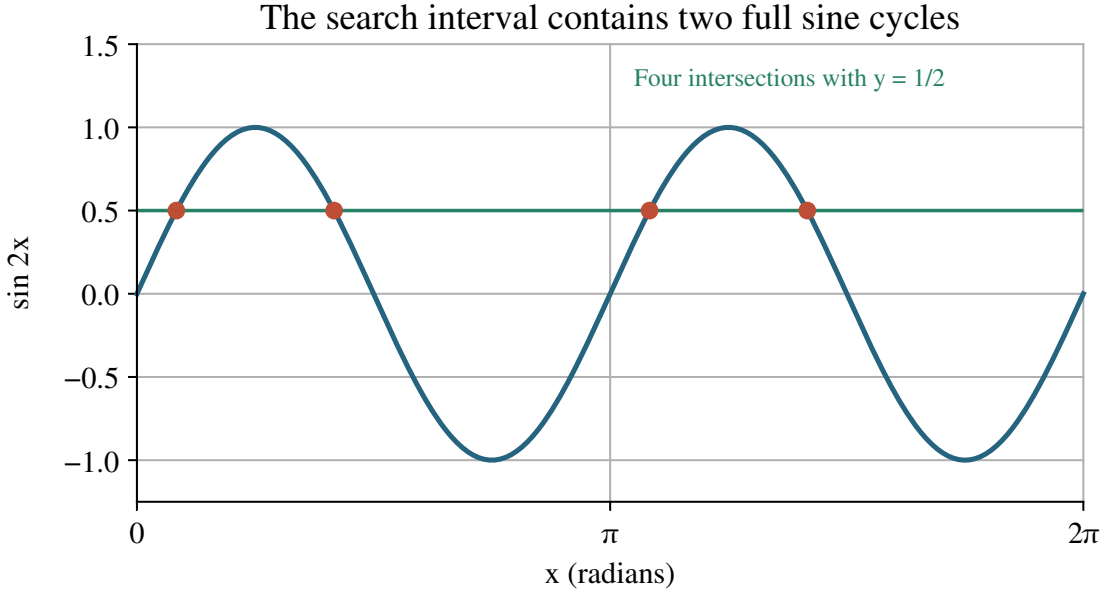
- For an argument u=kx+c, transform the whole interval for x into the corresponding interval for u.
- If k is negative, reverse endpoint order and preserve whether each endpoint is included.
- Solve in u using all repeated cycles, then recover x=(u−c)/k and check the original interval.
- Sine/cosine repeat after 2π; tangent repeats after π.

A worked case

$$u = kx + c, \qquad x = \frac{u - c}{k} \quad (k \neq 0)$$

Worked example

Solve sin2x=1/2 on $0 \leq x < 2\pi$. Then $0 \leq u = 2x < 4\pi$, giving $u = \pi/6, 5\pi/6, 13\pi/6, 17\pi/6$. Divide by 2: $x = \pi/12, 5\pi/12, 13\pi/12, 17\pi/12$. For cos(2x−π/3)=1/2 on $[0, 2\pi)$, the new interval is $[-\pi/3, 11\pi/3)$. Its allowed u values are $-\pi/3, \pi/3, 5\pi/3, 7\pi/3$, hence $x = 0, \pi/3, \pi, 4\pi/3$. For tan3x=1 on $[0, \pi)$, u ranges over $[0, 3\pi)$, so $x = \pi/12, 5\pi/12, 3\pi/4$. For sin(−2x)=0 on $[0, \pi)$, the transformed interval is $(-2\pi, 0]$; its roots $u = -\pi, 0$ give $x = \pi/2, 0$.



Where it goes wrong

- Solving only one cycle of u misses roots.
- An excluded upper endpoint for x need not be the upper endpoint for u when $k<0$.
- Dividing by a trig factor can remove zero roots; squaring can introduce extra roots.
- Substitute into the original equation, whose reciprocal/tangent denominators may still exclude a candidate.

■ Watch out

Not true: Solving sin2x on one full turn of 2x finds every x solution on $[0,2\pi)$. If x runs over $[0,2\pi)$, then 2x runs over $[0,4\pi)$, so solve for 2x over two full turns.

Applying it

- Use general families such as $u=\alpha+2j\pi$ or $\pi-\alpha+2j\pi$ for sine, $u=\pm\arccos a+2j\pi$ for cosine, and $u=\arctan a+j\pi$ for tangent, with integer j.
- Keep only values in the transformed interval.
- At sine/cosine extrema, deduplicate the families.
- Return the answer in the requested angle unit and original variable.

In your course

- AQA A-level Mathematics (7357), specification section E.
- Every paper allows a calculator, but method marks still need the expression or equation written before its value.

■ Key idea

- **Transformed interval**: The interval for a substituted angle after applying the same input change to its endpoints.
- For an argument $u=kx+c$, transform the whole interval for x into the corresponding interval for u.
- Watch for: solving only one cycle of u misses roots.

E · Trigonometric components and motion models

Which reference angle and axis decides the formula?

- A speed, force and height can each use sine or cosine.

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- This lesson studies **component** 分量: The signed projection of a vector along a chosen coordinate direction.

The method

- A vector of magnitude R at angle θ anticlockwise from the positive horizontal axis has components (R$\cos\theta$,R$\sin\theta$).
- A bearing β measured clockwise from north instead gives east R$\sin\beta$ and north R$\cos\beta$.
- Preserve units and signs.
- For a periodic height c+a $\sin(\omega t+\varphi)$, a full cycle has duration $2\pi/|\omega|$ when $\omega\neq0$; interpret the time interval and physical assumptions.

A worked case

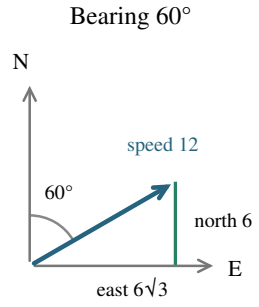
$$(v_x, v_y) = (R\cos\theta, R\sin\theta), \qquad T = \frac{2\pi}{|\omega|}$$

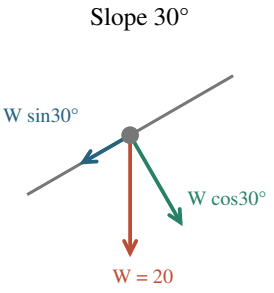
■ Worked example

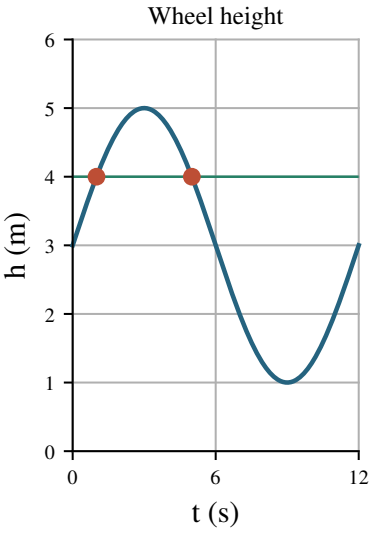
A velocity of 12 metres per second at bearing 60° has east component $6\sqrt{3}$ and north component 6 metres per second; its horizontal direction angle is 30°. For a 2 kg block on an ideal frictionless 30° slope, take g=10 metres per second squared: weight=20 N, downslope component=$20\sin30°=10$ N and normal component=$20\cos30°=10\sqrt{3}$ N. If there are no other normal forces, the reaction has that normal magnitude. For a uniformly rotating wheel, centre height 3 m and radius 2 m give $h(t)=3+2\sin(\pi t/6)$, with t in seconds and phase zero at t=0. Its period is 12 s and range 1–5 m. On $0\leq t<12$, height 4 m occurs at t=1 and 5 s; the first is 1 s. Differentiation gives vertical velocity $(\pi/3)\cos(\pi t/6)$, zero at the highest point t=3 s.

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Define the reference angle, axes and physical interval







Where it goes wrong

- A bearing is not an angle from the horizontal.
- The downslope component of weight uses sine of the slope angle; the normal uses cosine.
- A reaction does not always equal the weight's normal component if other forces act normally.
- A position or height function is not itself a velocity.
- Negative components show direction, not negative magnitude.

■ Watch out

Not true: For every bearing, the north component is the magnitude multiplied by the sine of the bearing. For a bearing measured from north, the north component uses cosine: magnitude × cos(bearing).

Applying it

- Draw and label axes, angle origin and the interval before projecting or solving.
- Recover magnitude with $\sqrt{(u^2+v^2)}$ and choose the direction quadrant from both signed components.
- State the ideal assumptions: fixed rotation rate for the wheel; no friction or extra normal forces for the block.
- Restrict mathematical roots to physically allowed times and check dimensions.

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In your course

- AQA A-level Mathematics (7357), specification section E.
- Every paper allows a calculator, but method marks still need the expression or equation written before its value.

■ Key idea

- **Component**: The signed projection of a vector along a chosen coordinate direction.
- A vector of magnitude R at angle θ anticlockwise from the positive horizontal axis has components (R$\cos\theta$,R$\sin\theta$).
- Watch for: A bearing is not an angle from the horizontal.

E · Sine rule, cosine rule and triangle area

Two known sides and their included angle can determine the remaining side without inventing a perpendicular side length.

- A triangular plot has no right angle.
- This lesson studies **included angle** 夹角: The angle between the two named sides.

The method

- Sine rule pairs opposite sides/angles: a/sinA=b/sinB=c/sinC.
- Cosine rule a^2=b^2+c^2-2bc cosA uses the angle opposite a.
- Area is ab sinC/2 when C lies between a and b.
- Choose the rule from the known information.
- An inverse sine may give an acute angle and an obtuse supplement; check the angle sum and supplied sides before accepting either.

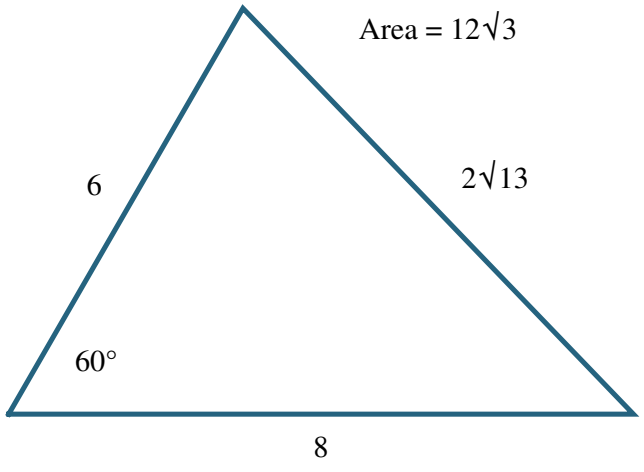
A worked case

$$a^2 = b^2 + c^2 - 2bc\cos A, \quad A_{\triangle} = \frac{1}{2}ab\sin C$$

Worked example

With sides 6 and 8 enclosing 60°, $c^2=36+64-96\times(1/2)=52$, hence $c=2\sqrt{13}$. Its area is $(1/2)\times6\times8\times\sin60°=12\sqrt{3}$. For $a=4$ opposite $A=30°$ and $B=45°$, $b=4\sin45°/\sin30°=4\sqrt{2}$. If sides $a=7, b=5, c=6$, $\cos A=(25+36-49)/(2\times5\times6)=1/5$, so $A\approx78.5°$. To find an angle from area 12 with enclosing sides 6 and 8, $\sin C=24/48=1/2$; C could be 30° or 150° until the remaining data selects a shape. Label opposite pairs and check triangle inequalities to reject impossible side combinations.

Two sides and their included angle



Where it goes wrong

- Do not pair a side with its adjacent angle in the sine rule.
- The area angle must be included.
- A calculator's first inverse-sine answer need not be the only possible triangle.

Watch out

Not true: The sine rule pairs each side with any angle in the triangle. Each side is paired with the angle opposite it: $a/\sin A=b/\sin B$.

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Applying it

- AQA 7357 E1 includes sine/cosine rules and general-triangle area.
- Choose the rule from the known opposite pairs or included angle; preserve exact values and test all feasible triangle configurations.
- The dedicated ambiguous-case lesson develops the non-included-angle case.

In your course

- AQA A-level Mathematics (7357), specification section E.
- Every paper allows a calculator, but method marks still need the expression or equation written before its value.

■ Key idea

- **Included angle**: The angle between the two named sides.
- Sine rule pairs opposite sides/angles: a/sinA=b/sinB=c/sinC.
- Watch for: do not pair a side with its adjacent angle in the sine rule.

E · The ambiguous sine-rule case and valid triangle counts

How do we check whether both triangles exist?

- Two sides and a non-included angle can describe two different plots of land.
- This lesson studies **ambiguous case** 歧义情形: Given triangle data that can produce two different valid triangles.

The method

- Let A be opposite a, with b another known side.
- The sine rule gives sinB=b sinA/a.
- If this ratio is above 1 there is no triangle.
- Otherwise test both B=arcsin(b sinA/a) and its supplement 180°−B.
- Retain only positive C=180°−A−B; when B=90° the two branches coincide.
- Every side must be positive.

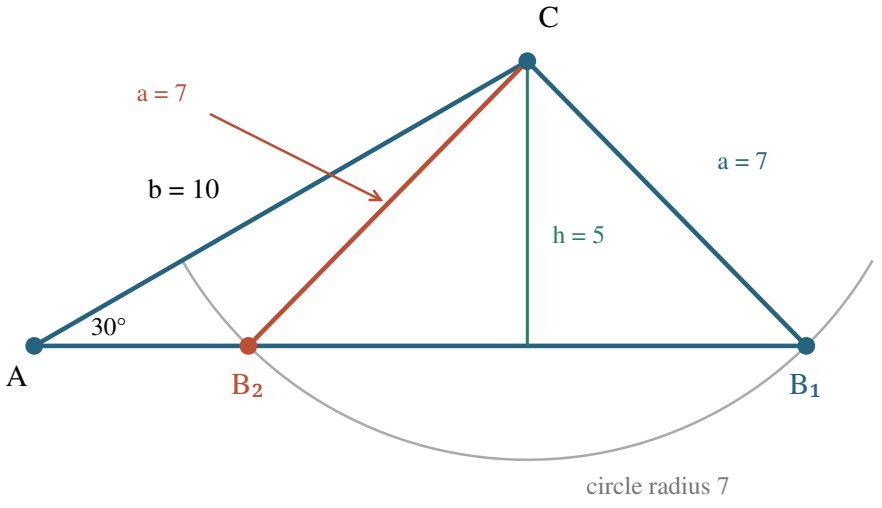
A worked case

$$\sin B = \frac{b \sin A}{a}, \qquad C = 180^\circ - A - B > 0$$

■ Worked example

Take A=30°, a=7 and b=10. Then sinB=5/7, so B≈45.5847° or 134.4153°. Both give positive C, about 104.4153° or 15.5847°. The corresponding third sides are c=5√3+2√6≈13.5592 and c=5√3−2√6≈3.7613. To derive them, place A at the origin and C=(10cos30°,10sin30°)=(5√3,5); B=(c,0). The condition BC=7 gives $(c-5\sqrt{3})^2+25=49$. Both intersections lie on the positive baseline. Their areas are bc sinA/2=25√3/2±5√6, so the same data does not determine one area.

The same a, b and A permit two positive triangles



Where it goes wrong

- A second inverse-sine value is only a candidate: it may make the remaining angle zero or negative.
- For acute A and h=b sinA, the cases are a<h: none; a=h: one right triangle; h<a<b: two; a≥b: one.
- At a=b the extra baseline intersection is c=0, a degenerate shape.
- For right or obtuse A, a must exceed b for one triangle; otherwise none.

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■ **Watch out**

Not true: Two inverse-sine values give two valid triangles. The second angle is valid only if it leaves a positive third angle; sometimes only one triangle exists.

Applying it

- In the example h=5: changing a to 5 gives one right triangle, to 4 gives none, and to 10 gives one with A=B=30°.
- Check the angle sum, side order and cosine-rule reconstruction for every candidate.
- State both configurations when the given information does not select one, and keep exact lengths/areas where possible.

In your course

- AQA A-level Mathematics (7357), specification section E.
- Every paper allows a calculator, but method marks still need the expression or equation written before its value.

■ **Key idea**

- **Ambiguous case**: Given triangle data that can produce two different valid triangles.
- Let A be opposite a, with b another known side.
- Watch for: A second inverse-sine value is only a candidate: it may make the remaining angle zero or negative.

E · Key-point sketches and transformed tangent domains

Which input angles locate its peaks, crossings and breaks?

- A shifted and stretched trig graph can hide its key points.
- This lesson studies **asymptote** 渐近线: A line approached by a graph while its values grow without bound or tend toward a limit.

The method

- For a sine/cosine graph a f(kx+c)+d, amplitude is |a|, centre line y=d and period $2\pi/|k|$ for $k\neq 0$.
- Transform the five quarter-cycle base inputs to locate key points.
- A negative a reflects vertically.

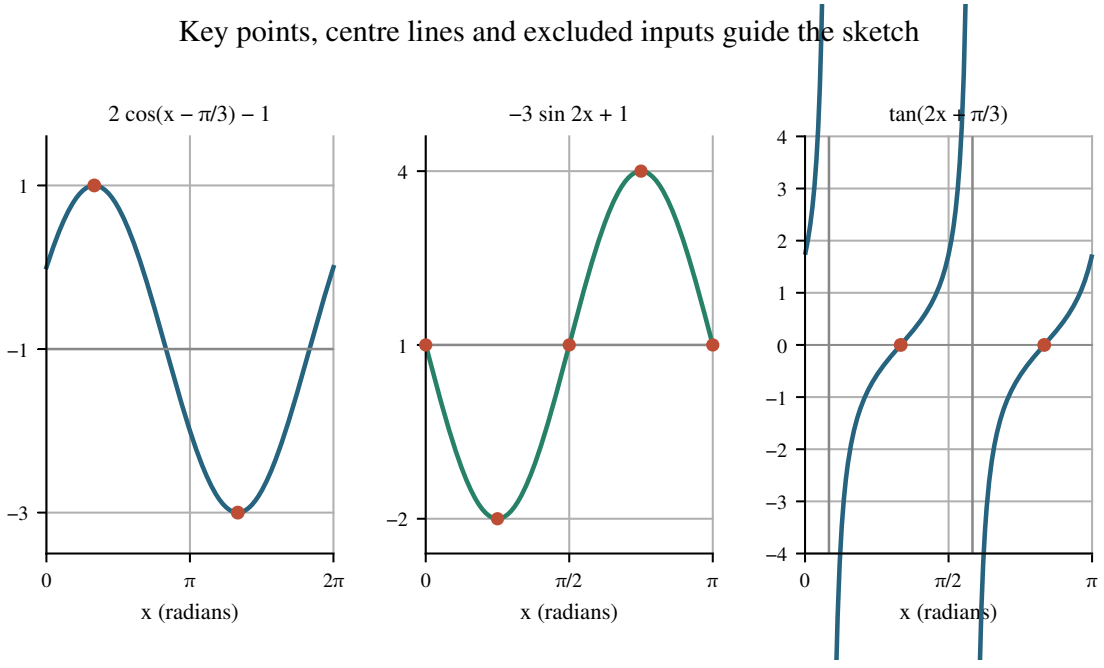
- For tangent the period is $\pi/|k|$, there is no amplitude and the range is all real numbers; exclude every input giving a base-angle $\pi/2+j\pi$.

A worked case

$$T_{\sin,\cos} = \frac{2\pi}{|k|}, \quad T_{\tan} = \frac{\pi}{|k|} \quad (k \neq 0)$$

Worked example

For $y=2\cos(x-\pi/3)-1$, one cycle has points $(\pi/3,1),(5\pi/6,-1),(4\pi/3,-3),(11\pi/6,-1),(7\pi/3,1)$. The range is $[-3,1]$ and period 2π. For $y=-3\sin 2x+1$, the points $x=0,\pi/4,\pi/2,3\pi/4,\pi$ give $y=1,-2,1,4,1$; the period is π and range $[-2,4]$. For $y=\tan(2x+\pi/3)$, zeros satisfy $2x+\pi/3=j\pi$, hence $x=j\pi/2-\pi/6$. Asymptotes satisfy $2x+\pi/3=\pi/2+j\pi$, hence $x=\pi/12+j\pi/2$. On $0\le x<\pi$, the zeros are $\pi/3$ and $5\pi/6$ and the asymptotes $\pi/12$ and $7\pi/12$.



Where it goes wrong

- The cosine maximum need not occur at $x=0$ after a horizontal shift.
- For the stated tangent function, draw separate increasing branches between asymptotes.
- A negative horizontal or vertical factor reverses the direction; never join branches across a break.

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- Vertical translations change outputs but not the period or excluded x values.
- A displayed interval may contain only part of a complete shifted cycle.

■ **Watch out**

Not true: A transformed tangent graph has a finite amplitude and can be joined across its vertical asymptotes. A tangent graph has no amplitude, and its branches must not be joined across asymptotes.

Applying it

- Find period and centre line first, then map exact base-angle key points.
- Add enough repeats to cover the requested interval and label scale, extremes and asymptotes.
- To find x-axis crossings, solve the output equation rather than assuming every centre-line crossing has y=0.
- Check a sample point in each tangent branch and preserve the original domain.

In your course

- AQA A-level Mathematics (7357), specification section E.
- Every paper allows a calculator, but method marks still need the expression or equation written before its value.

■ **Key idea**

- **Asymptote**: A line approached by a graph while its values grow without bound or tend toward a limit.
- For a sine/cosine graph a f(kx+c)+d, amplitude is |a|, centre line y=d and period $2\pi/|k|$ for $k\neq0$.
- Watch for: the cosine maximum need not occur at x=0 after a horizontal shift.

F · Exponentials, logarithms and modelling

Why does a decay model stay positive?

- A medicine concentration falls by the same percentage each hour. A constant subtraction would eventually predict a negative amount.
- This lesson studies **half-life** 半衰期: The time for a decaying quantity to fall to half its initial value.

The method

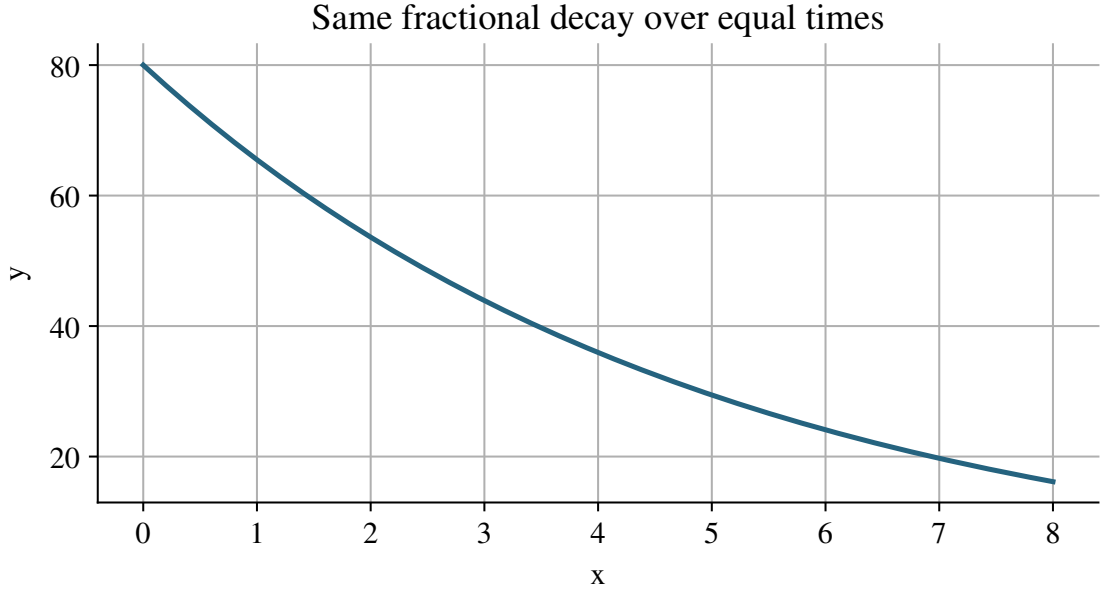
- For y=Ae^(kt), k is a proportional rate.
- Taking logs gives ln y=ln A+kt.
- Logarithms require positive arguments, and log(x+y) is not log x+log y.

A worked case

$$y = Ae^{kt}, \qquad \ln y = \ln A + kt$$

Worked example

If y=80e^(-0.2t), y=40 gives e^(-0.2t)=1/2. Hence t=ln2/0.2≈3.466. For 3^x=20, x=ln20/ln3. Plotting ln y against t linearises this exponential model.



Where it goes wrong

- A fitted exponential is a model, not a guarantee.
- Specify the time units and range of use.
- A negative k describes decay; a negative starting amount usually contradicts the context.

■ **Watch out**

Not true: For positive x and y, ln(x+y)=ln x+ln y. The log of a sum has no simple law; the product rule is ln(xy)=ln x+ln y.

Applying it

- Compare actual observations with the model.
- Systematic departures may indicate changing conditions.
- In a report, explain what the rate and initial value mean, rather than giving a bare equation.

In your course

- AQA A-level Mathematics (7357), specification section F.
- Every paper allows a calculator, but method marks still need the expression or equation written before its value.

■ **Key idea**

- **Half-life**: The time for a decaying quantity to fall to half its initial value.
- For y=Ae^(kt), k is a proportional rate.
- Watch for: A fitted exponential is a model, not a guarantee.

F · Exponential graphs and logarithmic inverses

Which graph reverses that input-output pair?

- An exponential maps an exponent to a positive amount.
- This lesson studies **logarithm** 对数: The exponent to which a stated valid base must be raised to produce a positive input.

The method

- For a>0, y=a^x has every real input and positive outputs.
- It passes through (0,1).
- For a>1 it increases; for 0<a<1 it decreases.
- For $a = 1$ it is the constant 1, with no inverse function.
- For a>0 and a≠1, y=log_a x is the inverse of a^x: domain $x > 0$, range all real numbers. ln x is log_e x and reverses e^x.

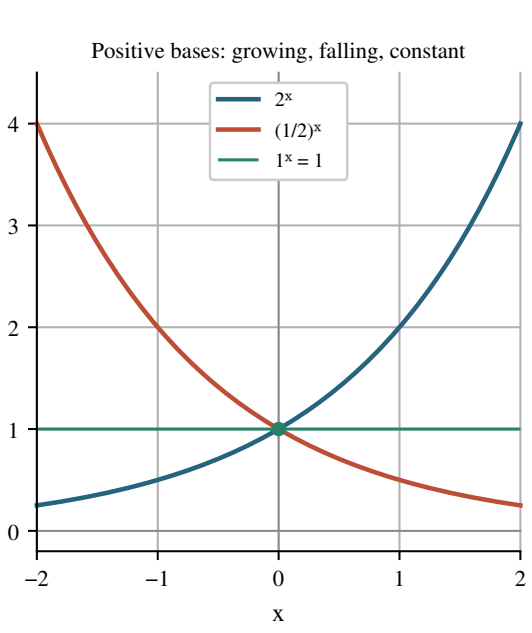
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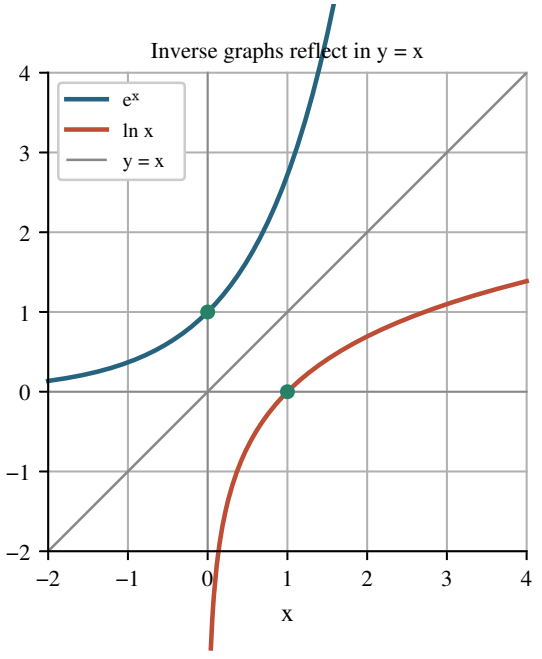
A worked case

$$y = a^x \iff x = \log_a y \quad (a > 0,\ a \neq 1,\ y > 0)$$

■ Worked example

The graph 2^x contains (−1,1/2),(0,1),(1,2),(2,4). Its inverse $\log_2$x contains (1/2,−1),(1,0),(2,1),(4,2), reflected in y=x. The graph (1/2)^x gives 2,1,1/2 at x=−1,0,1, while its inverse log_(1/2)x is decreasing. For e^x and ln x, the points (0,1) and (1,0) are reflected partners; ln e=1 and ln(e^3)=3. For any real u, ln(e^u)=u; for $x > 0$, e^(ln x)=x. Logarithms approach but do not reach their vertical boundary x=0; e^x approaches y=0 as x tends to negative infinity.





Where it goes wrong

- Neither ln0 nor the logarithm of a negative real number exists in the real-number course.
- A negative logarithm output is allowed: ln(1/e)=−1.
- Base 1 is permitted for an exponential constant but not for a logarithm.
- For a base below 1 the exponential approaches zero as x tends to positive infinity, and the inverse has the opposite increasing/decreasing direction from the a>1 case.

■ **Watch out**

Not true: The constant exponential 1^x has a logarithmic inverse for all positive inputs. 1^x is constant (always 1), so it is not one-to-one and has no inverse.

Applying it

- State the base, domain and range before taking an inverse.
- Swap the coordinates of exact exponential points to construct the log graph.
- Preserve open boundaries and asymptotes.
- The source F3 prints x≥0; the valid logarithm input condition used here is $x > 0$, since a^u is positive and never zero.

In your course

- AQA A-level Mathematics (7357), specification section F.
- Every paper allows a calculator, but method marks still need the expression or equation written before its value.

■ **Key idea**

- **Logarithm**: The exponent to which a stated valid base must be raised to produce a positive input.
- For a>0, y=a^x has every real input and positive outputs.
- Watch for: neither ln0 nor the logarithm of a negative real number exists in the real-number course.

F · Logarithm laws, negative powers and equation domains

Which original input condition rejects it?

- Combining two logs can produce a polynomial with an extra root.
- This lesson studies **argument** 真数: The input inside a logarithm, which must be positive for a real logarithm.

The method

- For a>0, a≠1 and positive u,v: log_a(uv)=log_a u+log_a v, log_a(u/v)=log_a u−log_a v, and log_a(u^k)=k log_a u for real k.
- These follow from exponent multiplication/division/powers.
- In particular k=−1 gives a reciprocal and k=−1/2 gives an inverse square root.

- Solve a^x=b, with b>0, by x=ln b/ln a.

A worked case

$$\log_a(uv) = \log_a u + \log_a v, \qquad x = \frac{\ln b}{\ln a}\ (a^x = b)$$

Worked example

For $x > 0$, ln(x^(-1))=−ln x and ln(x^(-1/2))=−(1/2)ln x. Solve ln(x−1)+ln(x+1)=ln8: the original domain is x>1. Combining gives ln(x²−1)=ln8, hence x²=9 and candidates ±3; only x=3 satisfies the original arguments. For 3^x=20, x=ln20/ln3. If 2^(2x)−2^x−2=0, let u=2^x > 0: (u−2)(u+1)=0, so u=2 gives x=1, while u=−1 is impossible. Also ln(2+3)=ln5, whereas ln2+ln3=ln6; a sum inside a log cannot be split this way.

A log-law step preserves the original positive arguments

$$\ln(x-1) + \ln(x+1) = \ln 8$$

$$x > 1 \quad \text{before combining}$$

$$x^2 - 1 = 8 \implies x = \pm 3$$

$$\text{Accept } x = 3; \quad \text{reject } x = -3$$

Where it goes wrong

- Combining or expanding logs does not erase their original domains. ln(u²)=2ln|u| for u≠0; writing 2lnu additionally needs u>0.
- Do not take a real logarithm of a zero or negative right-hand side in a^x=b.
- For base 1, 1^x=b has all real x when b=1 and no solutions otherwise; the log quotient is not available.

■ **Watch out**

Not true: Every polynomial root obtained after combining logarithms satisfies the original log equation. Logarithms need positive arguments, so check each root in the original equation.

Applying it

- List positivity restrictions first, use one common base, then apply a valid law.
- After algebra, substitute every candidate into the original logarithmic/exponential equation.
- Keep log quotients exact unless a decimal is requested.
- Changing the log base does not change an exponential equation's solution when both quotient logs use the same valid base.

In your course

- AQA A-level Mathematics (7357), specification section F.
- Every paper allows a calculator, but method marks still need the expression or equation written before its value.

■ **Key idea**

- **Argument**: The input inside a logarithm, which must be positive for a real logarithm.
- For a>0, a≠1 and positive u,v: log_a(uv)=log_a u+log_a v, log_a(u/v)=log_a u−log_a v, and log_a(u^k)=k log_a u for real k.
- Watch for: combining or expanding logs does not erase their original domains. $\ln(u^2)=2\ln|u|$ for u≠0; writing 2lnu additionally needs u>0.

F · Logarithmic graphs for power and exponential fits

Which horizontal axis distinguishes a power model from an exponential one?

- A curve can become a straight line after taking logs.
- This lesson studies **intercept** 截距: The transformed vertical value when the chosen horizontal variable is zero.

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The method

- For positive observations and parameters, y=ax^p with $x > 0$ gives ln y=ln a+p ln x.
- Plot ln y against ln x: gradient p, intercept ln a.
- For y=kb^x with k,b>0, ln y=ln k+x ln b.
- Plot ln y against x: gradient ln b, intercept ln k.
- Use the same log base for both logarithmic axes and recover parameters with that base's inverse.

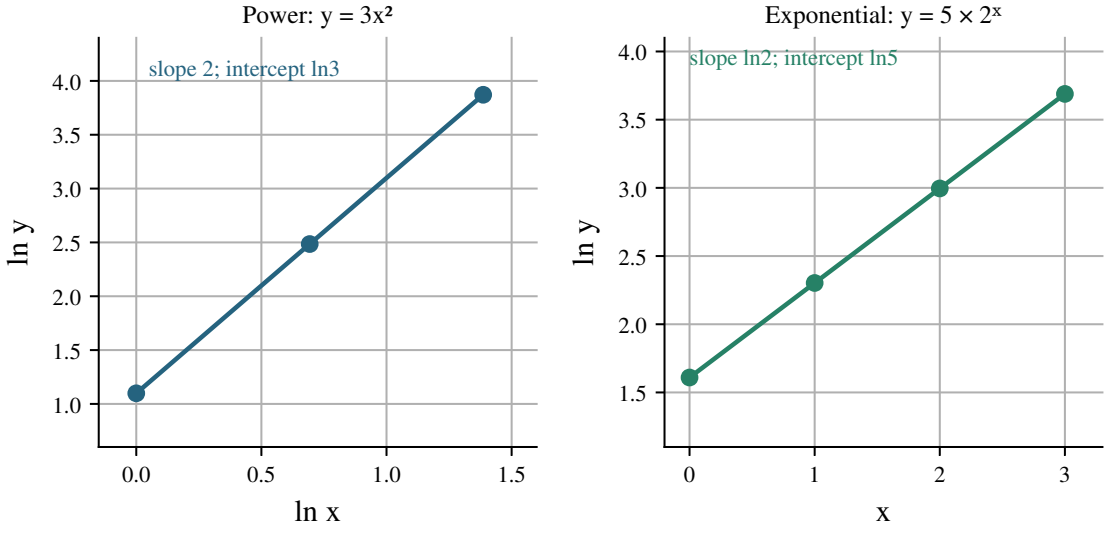
A worked case

$$\ln y = \ln a + p \ln x, \qquad \ln y = \ln k + x \ln b$$

■ Worked example

Power data (x,y)=(1,3),(2,12),(4,48) give gradient [ln48−ln3]/[ln4−ln1]=2, intercept ln3, hence y=3x^2. Exponential data (0,5),(1,10),(2,20),(3,40) give a semilog gradient ln2 and intercept ln5, hence y=5×2^x. With base-10 logs the same exponential gradient is $\log_{10}2 \approx 0.30103$, so b=10^0.30103≈2 rather than e^0.30103. A new power-data observation (3,30) exceeds the fitted prediction 27 by residual 3; a perfectly straight calibration set does not prove the model beyond its observed range.

Different horizontal axes recover different parameters



Where it goes wrong

- For the power fit the gradient is the exponent, not its logarithm.
- For the exponential fit the gradient is the logarithm of the base, not the base itself.
- The power graph's intercept corresponds to x=1 because ln x=0 there; the exponential intercept corresponds to x=0.
- Nonpositive observations cannot be logged directly.
- An exponential log-y plot may include x=0 or negative x; the log-log power plot cannot.

■ Watch out

Not true: Plotting ln y against x linearises every power model y=ax^p. A power model is linear in ln y against ln x; ln y against x linearises exponential models.

Applying it

- Label transformed axes and estimate gradient from well-separated points on the fitted line, not from a single data point.
- Recover each intercept parameter before returning to original units.
- Substitute predictions into the original model and examine residuals there.
- State the fitted range, measurement uncertainty and whether extrapolation is justified; systematic departures can suggest a different model.

In your course

- AQA A-level Mathematics (7357), specification section F.
- Every paper allows a calculator, but method marks still need the expression or equation written before its value.

■ Key idea

- **Intercept**: The transformed vertical value when the chosen horizontal variable is zero.
- For positive observations and parameters, y=ax^p with $x > 0$ gives ln y=ln a+p ln x.
- Watch for: for the power fit the gradient is the exponent, not its logarithm.

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F · Proportional rates, half-life and model limits

Why does one fixed half-life not mean a fixed subtraction?

- A decaying amount loses more per hour when more remains.
- This lesson studies **proportional rate** 比例变化率: A rate of change equal to a constant multiplied by the current amount.

The method

- For y=Ae^(kt), dy/dt=ke^(kt)A=ky.
- A>0 is the initial amount; k is a constant proportional rate in inverse time units.
- Positive k gives growth, negative k decay, zero k a constant.
- Over elapsed time Δt the multiplier is e^(kΔt).
- Doubling time is ln2/k for k>0; half-life is ln2/|k| for k<0.

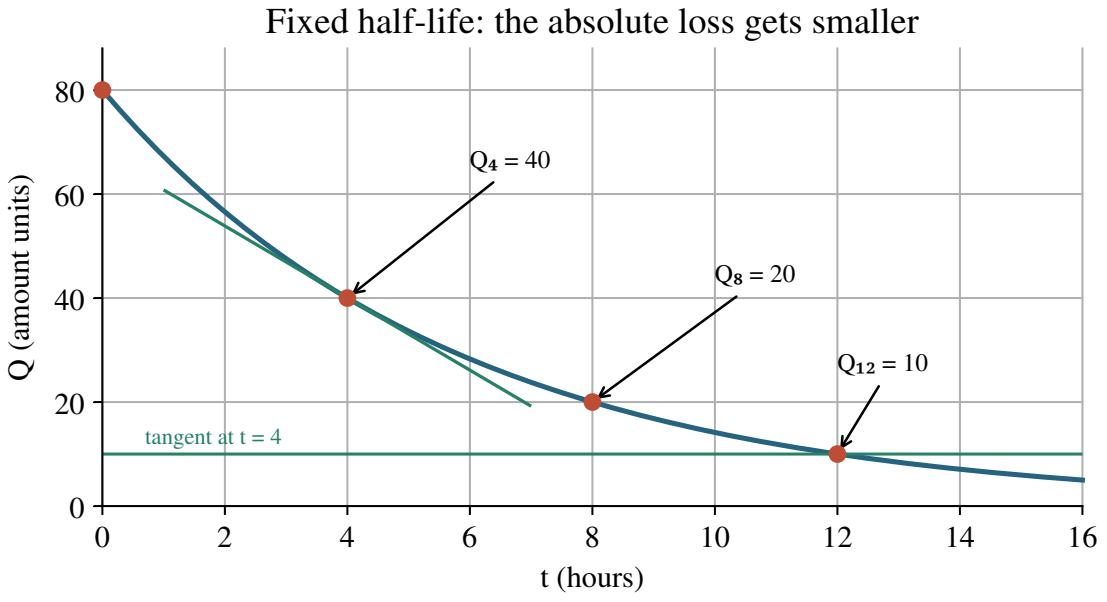
A worked case

$$y = Ae^{kt}, \qquad \frac{dy}{dt} = ky, \qquad T_{1/2} = \frac{\ln 2}{|k|} \ (k < 0)$$

■ Worked example

Use an illustrative decay Q=80e^(−λt), t in hours and λ=ln2/4. Then Q_4=40, Q_8=20 and Q_{12}=10: each four-hour step halves the amount. At t=4 the rate is Q'=−40λ=−10ln2 amount units per hour. Solving Q≤10 gives t≥12; the strict condition Q<10 instead needs t>12. For an illustrative mathematical account P=100e^(0.1t), t in years, P_1≈110.517 and the one-year effective growth is about 10.517%, not 10%. If the given effective yearly growth is 10%, use k=ln1.1 instead. Continuous compounding and a stated annual multiplier must not be confused.

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Where it goes wrong

- The exponent must be dimensionless: changing hours to minutes changes k by a factor 1/60.
- A half-life is not the time to reach zero; positive A times an exponential stays positive for finite time.
- Absolute losses vary with the amount although the proportional rate is fixed.
- A first whole-period threshold needs adjacent integer checks rather than only a continuous crossing time.

■ Watch out

Not true: A constant exponential proportional rate means the same absolute amount is lost in each equal time interval. A constant proportional rate loses the same percentage each interval, so the absolute amount lost falls.

Applying it

- State time/amount units, initial conditions and the constant-rate assumption.
- Compare observations with predictions before extrapolating.
- Limited resources, changing environmental conditions, fees or withdrawals can invalidate a simple growth model; changing decay conditions can require a varying-rate or multi-stage model.

- These are illustrative mathematical models, not product terms or treatment instructions.

In your course

- AQA A-level Mathematics (7357), specification section F.
- Every paper allows a calculator, but method marks still need the expression or equation written before its value.

■ **Key idea**

- **Proportional rate**: A rate of change equal to a constant multiplied by the current amount.
- For y=Ae^(kt), dy/dt=ke^(kt)A=ky.
- Watch for: the exponent must be dimensionless: changing hours to minutes changes k by a factor 1/60.

G · Derivatives and stationary points

What is the slope at one point?

- A curved road has different slopes at different positions. An average gradient cannot describe every point.
- This lesson studies **derivative** 导数: The instantaneous rate of change, also the gradient of a tangent.

The method

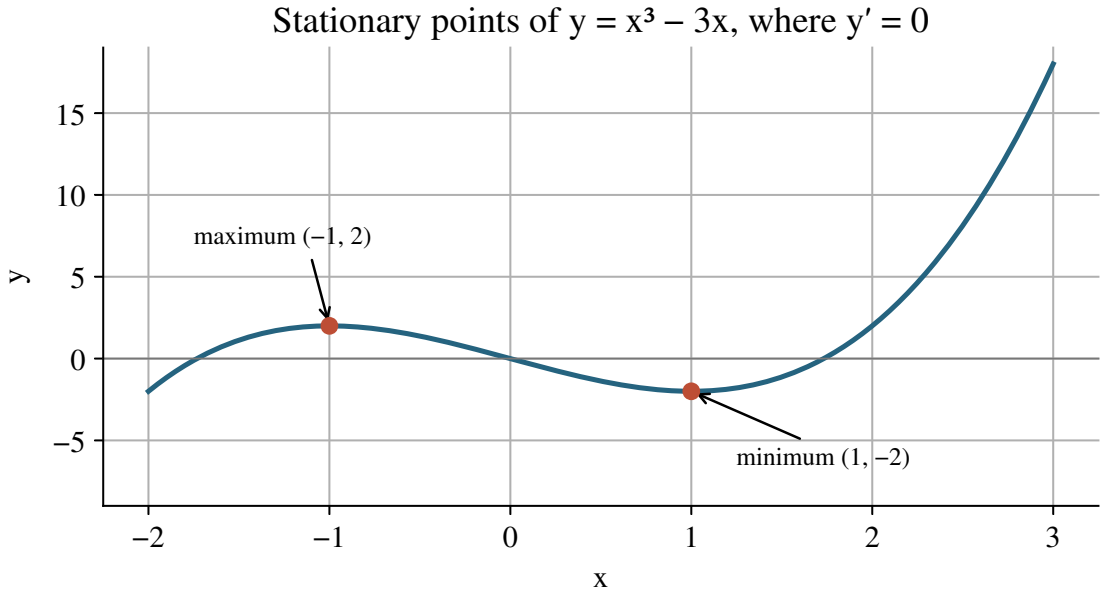
- For y=ax^n, dy/dx=anx^(n-1).
- A stationary point satisfies dy/dx=0.
- Check the sign change of the derivative, or the second derivative when it is nonzero, to classify it.

A worked case

$$\frac{d}{dx}(ax^n) = anx^{n-1}, \qquad f'(x) = 0$$

Worked example

For $y=x^3-3x$, $dy/dx=3x^2-3$. At $x=1$, the gradient is 0 and $y=-2$. The second derivative is $6x$, positive at $x=1$, so this is a local minimum. At $x=-1$, $y=2$ and the second derivative is negative, giving a local maximum.



Where it goes wrong

- A zero derivative does not always mean a maximum or minimum: $y=x^3$ is stationary at 0 but continues increasing.
- An endpoint can also produce an extreme value on a restricted domain.

Watch out

Not true: Every point with zero derivative is a local maximum or minimum. A zero derivative can also be a point of inflection: $y=x^3$ at $x=0$ is neither a maximum nor a minimum.

Applying it

- At GCSE/IGCSE use only the polynomial scope allowed by the tier; do not add chain, product or quotient rules there.
- At advanced level, connect the derivative to rates and optimization with a valid domain.

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In your course

- AQA A-level Mathematics (7357), specification section G.
- Every paper allows a calculator, but method marks still need the expression or equation written before its value.

Key idea

- **Derivative**: The instantaneous rate of change, also the gradient of a tangent.
- For y=ax^n, dy/dx=anx^(n-1).
- Watch for: A zero derivative does not always mean a maximum or minimum: $y=x^3$ is stationary at 0 but continues increasing.

G · Chain, product, quotient and implicit differentiation

Does the inside expression change too?

- A cost curve is a power of a changing expression. Differentiating the outer power alone misses the rate of its input.
- This lesson studies **chain rule** 链式法则: The rule that multiplies the outer derivative by the inner derivative for a composite function.

The method

- For y=f(g(x)), y prime=f prime(g(x))g prime(x).
- For uv, differentiate to u prime v+uv prime.
- For u/v, use (u prime v-uv prime)/v^2, where $v\neq0$.

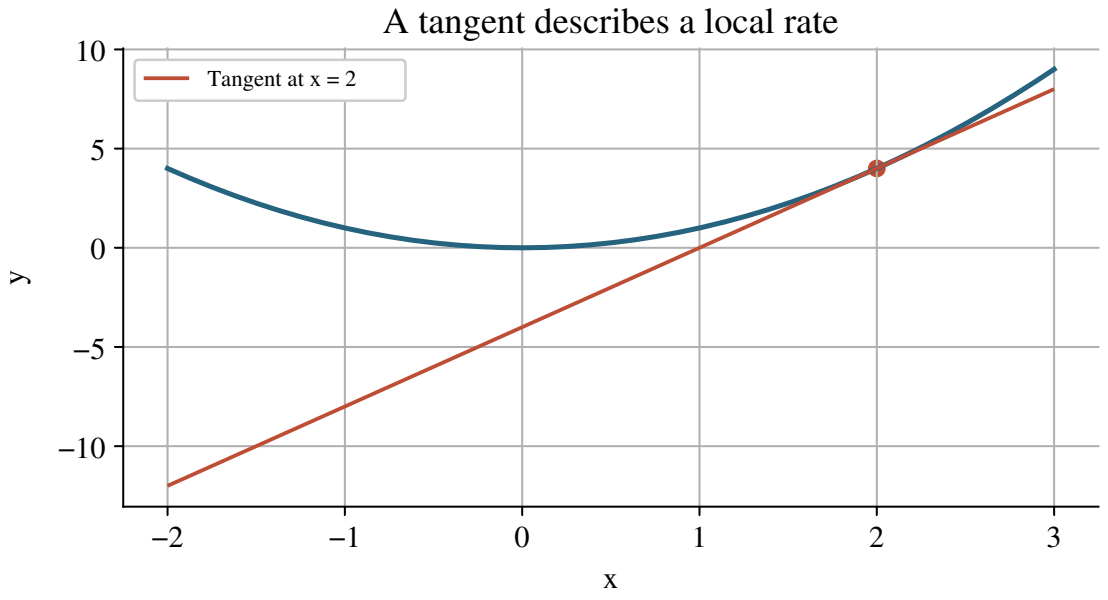
A worked case

$$\frac{d}{dx}f(g(x)) = f'(g(x))g'(x)$$

Worked example

For y=(3x+1)^4, y prime=4(3x+1)^3×3=12(3x+1)^3. At x=0, the gradient is 12. For $x^2+y^2=25$, differentiate implicitly: 2x+2y y prime=0, so y prime=-x/y when $y\neq0$.

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Where it goes wrong

- A derivative of a product is not the product of the derivatives.
- In implicit differentiation, every differentiated function of y brings a dy/dx factor.
- A quotient's denominator is squared.

> ■ **Watch out**
>
> **Not true:** The derivative of u(x)v(x) is u prime times v prime. The product rule is $(uv)'=u'v+uv'$: for $x \cdot x$ the derivative is 2x, not $1\times1=1$.

Applying it

- Choose a useful form before differentiating: expanding a short polynomial may be simpler.
- For related rates, write the relation in symbols, differentiate with respect to time, then substitute measured values.

In your course

- AQA A-level Mathematics (7357), specification section G.
- Every paper allows a calculator, but method marks still need the expression or equation written before its value.

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■ **Key idea**

- **Chain rule**: The rule that multiplies the outer derivative by the inner derivative for a composite function.
- For y=f(g(x)), y prime=f prime(g(x))g prime(x).
- Watch for: A derivative of a product is not the product of the derivatives.

G · Secants, limits and first-principles power derivatives

What happens to their average rate as the interval shrinks?

- A distance sensor gives two positions a short time apart.
- This lesson studies **difference quotient** 差商: The change in function output divided by a nonzero change in input.

The method

- For h≠0, the secant gradient is [f(x+h)−f(x)]/h.
- The derivative f′(x) is its limit as h tends to zero, if the two-sided finite limit exists.
- It gives the tangent gradient and instantaneous rate.
- The variable h may approach zero from either sign; simplify before taking the limit.

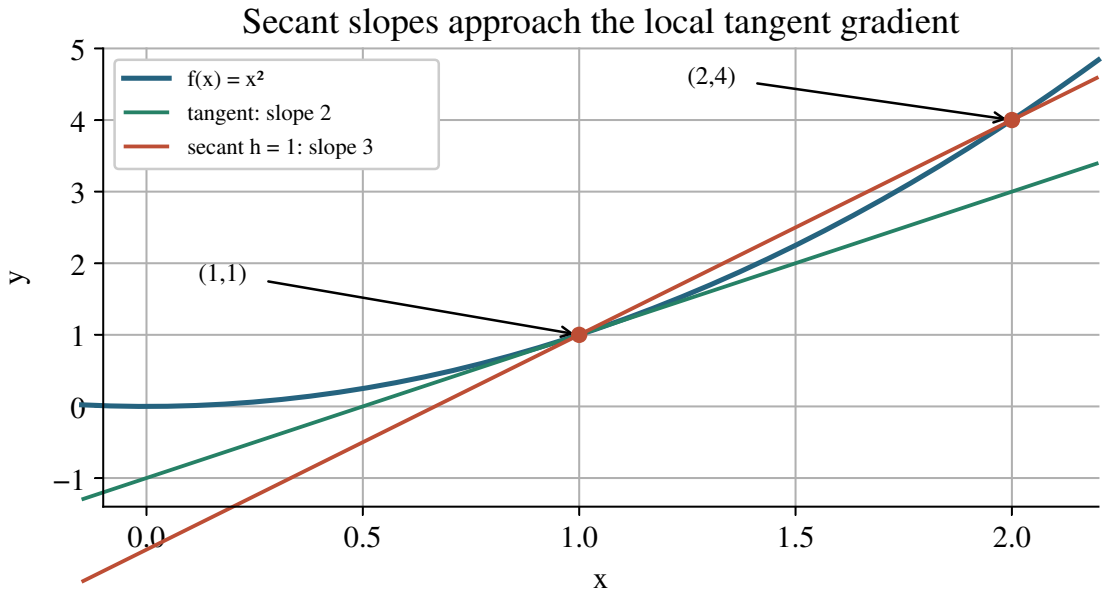
A worked case

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

■ **Worked example**

For f(x)=x², [(x+h)²−x²]/h=2x+h, so f′(x)=2x. At x=1, h=1,0.5,−0.5 give secant gradients 3,2.5,1.5, tending to 2 from both sides. The tangent through (1,1) is y−1=2(x−1). For f(x)=x³, expanding gives 3x²+3xh+h², hence f′(x)=3x². A constant gives zero and f(x)=x gives one. For any positive integer n, the binomial expansion leaves nx^(n−1) plus terms containing h, so the limit is nx^(n−1). This argument alone does not prove the rule for rational powers.

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Where it goes wrong

- Substituting h=0 into the unsimplified quotient gives 0/0, not a gradient.
- Cancelling h is valid for h≠0 before the limit.
- Continuity alone is insufficient: at x=0, |x| has right quotient 1 and left quotient −1, so no derivative there.
- The tangent is a local line, not an exact replacement for the entire curve.

■ Watch out

Not true: A difference quotient can be evaluated by setting h=0 before simplification. Setting h=0 first gives 0/0; simplify and cancel h, then let h tend to 0.

Applying it

- Write the difference quotient, expand, cancel only with h≠0 stated, and take the limit.
- Interpret derivative units as output units divided by input units.
- When a question says first principles, merely quoting nx^(n−1) does not establish the limit.
- At a corner check both approach directions.

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In your course

- AQA A-level Mathematics (7357), specification section G.
- Every paper allows a calculator, but method marks still need the expression or equation written before its value.

■ Key idea

- **Difference quotient**: The change in function output divided by a nonzero change in input.
- For h≠0, the secant gradient is [f(x+h)−f(x)]/h.
- Watch for: substituting h=0 into the unsimplified quotient gives 0/0, not a gradient.

G · First-principles sine and cosine derivatives

How do its tiny coordinate changes lead to exact derivative formulas?

- A rotating point has sine and cosine coordinates.
- This lesson studies **radian limit** 弧度极限: A small-angle limit used with angles measured in radians.

The method

- In radians, sin h/h tends to 1 and (cos h−1)/h tends to 0 as h tends to zero.
- Using angle-addition formulas in the difference quotient gives d(sin x)/dx=cos x and d(cos x)/dx=−sin x.
- These limits determine the tangent gradients; they require radian input.

A worked case

$$\lim_{h\to 0}\frac{\sin h}{h}=1, \qquad (\sin x)'=\cos x, \quad (\cos x)'=-\sin x$$

■ Worked example

For sin x, [sin(x+h)−sin x]/h=sin x[(cos h−1)/h]+cos x[sin h/h], tending to cos x. For cos x the quotient is cos x[(cos h−1)/h]−sin x[sin h/h], tending to −sin x. Near zero, the unit-circle inequality sin h<h<tan h for 0<h<π/2 bounds sin h/h between cos h and 1; negative h has the same ratio. Also (cos h−1)/h=−2sin^2(h/2)/h tends to zero, using sin(h/2)/(h/2)→1. At x=π/6, the sine gradient is $\sqrt{3}$/2 and the cosine

gradient is $-1/2$. For sine at x=0, h=0.1 gives $\sin(0.1)/0.1 \approx 0.998334$; at h=−0.1 the ratio is the same.

Separate the fixed x factors from the shrinking h factors

$$\sin(x+h) - \sin x = \sin x(\cos h - 1) + \cos x \sin h$$

$$\cos(x+h) - \cos x = \cos x(\cos h - 1) - \sin x \sin h$$

Divide by $h \neq 0$, then take the radian limits:

$$\frac{\sin h}{h} \to 1; \quad \frac{\cos h - 1}{h} \to 0$$

$$(\sin x)' = \cos x; \quad (\cos x)' = -\sin x$$

Where it goes wrong

- A small-angle approximation is not an exact finite-h equality.
- The cosine derivative has a minus sign.
- If a variable θ is measured in degrees, $\mathrm{d}\sin(\pi\theta/180)/\mathrm{d}\theta = (\pi/180)\cos(\pi\theta/180)$; the radian formula cannot be copied without that factor.
- Do not use the derivative formula itself to prove the starting limit.

> ■ **Watch out**
>
> **Not true:** The derivative of cos x is positive sin x for radian x. The derivative of cos x is −sin x.

Applying it

- Start with the original difference quotient and use angle addition before applying the two radian limits.
- Retain the fixed sin x/cos x factors and their signs.
- Check exact-point gradients against whether the graph is rising or falling; the proof applies at any real radian x, not only near x=0.

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In your course

- AQA A-level Mathematics (7357), specification section G.
- Every paper allows a calculator, but method marks still need the expression or equation written before its value.

■ Key idea

- **Radian limit**: A small-angle limit used with angles measured in radians.
- In radians, sin h/h tends to 1 and (cos h−1)/h tends to 0 as h tends to zero.
- Watch for: A small-angle approximation is not an exact finite-h equality.

G · Gradient graphs, concavity and inflection tests

Which gradient graph tells the difference?

- Two curves can both have zero slope, yet one turns and the other keeps rising.
- This lesson studies **point of inflection** 拐点: A point where a continuous curve changes its direction of concavity.

The method

- The sign of f′ identifies increasing or decreasing intervals.
- Zeros of f′ are stationary candidates.
- A change + to − gives a local maximum; − to + gives a local minimum.
- The second derivative f″ measures how the gradient changes: positive means an increasing gradient and a convex (concave-up) curve; negative means a decreasing gradient and a concave (concave-down) curve.
- An inflection needs a change of concavity, not only f″=0.

A worked case

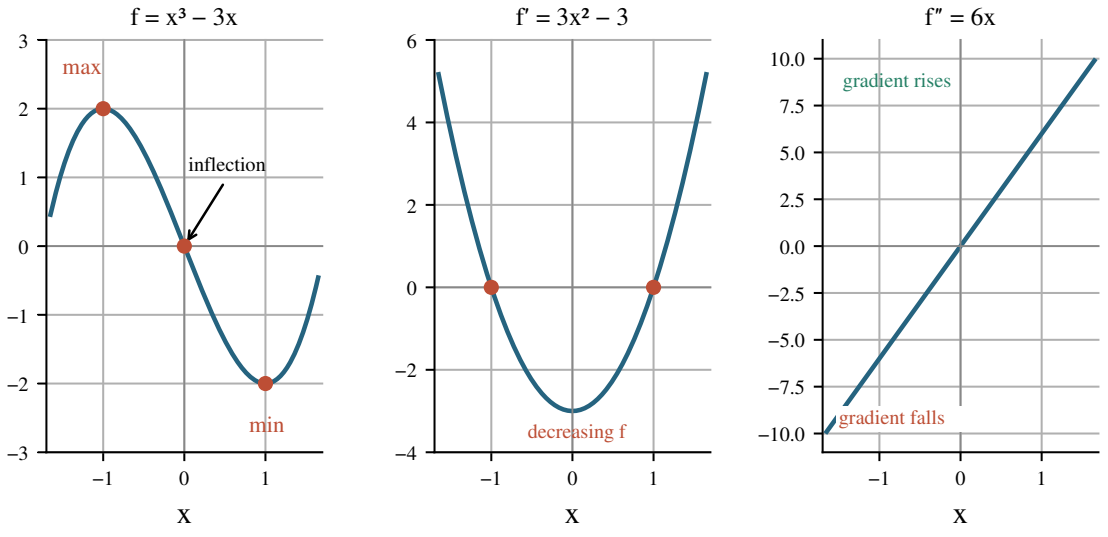
$$f = x^3 - 3x, \qquad f' = 3x^2 - 3, \qquad f'' = 6x$$

■ Worked example

For f=x^3−3x, f′=3x^2−3 and f″=6x. The gradient graph is above zero for x<−1 or x>1 and below for −1<x<1. Thus (−1,2) is a local maximum and (1,−2) a local minimum. The second derivative changes from negative to positive at x=0, so (0,0) is an inflection with nonzero gradient −3. For g=x^3, g′=3x^2 stays positive on both sides of zero: (0,0) is a stationary inflection, not a turning point. For q=x^4, q′=4x^3 changes − to + and q″=12x^2 is nonnegative: (0,0) is a minimum, but not an inflection even

though $q''(0)=0$.

Sign changes distinguish extrema from inflections



Where it goes wrong

- Do not confuse the height of f with its gradient f′.
- A decreasing curve can have an increasing gradient while staying below zero in the gradient graph.
- The test $f''(a)=0$ is inconclusive; inspect signs on each side.
- A point of inflection need not be stationary, and a stationary point need not be an extremum.
- Endpoint extrema on a restricted interval also require checking the endpoints.

■ Watch out

Not true: Every point at which the second derivative is zero is a point of inflection. At an inflection the concavity must change; $y=x^4$ has a zero second derivative at 0 but no inflection.

Applying it

- Align the x-axes of the original curve and gradient graph.
- Mark zeros of f′ and build sign intervals before naming extrema.
- Read concavity from whether the gradient graph rises or falls.

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- For a twice-differentiable polynomial, use an actual sign change of f″ to establish an inflection and evaluate the original function for its coordinates.

In your course

- AQA A-level Mathematics (7357), specification section G.
- Every paper allows a calculator, but method marks still need the expression or equation written before its value.

Key idea

- **Point of inflection**: A point where a continuous curve changes its direction of concavity.
- The sign of f′ identifies increasing or decreasing intervals.
- Watch for: do not confuse the height of f with its gradient f′.

G · Rational-power derivatives and valid domains

Why must a derivative have its own domain?

- A square-root curve has a finite height at zero but an unbounded slope nearby.
- This lesson studies **rational power** 有理数指数: An exponent that can be written as a fraction of integers, interpreted with a valid real root.

The method

- For a rational exponent p, d(x^p)/dx=p x^(p−1) wherever the real function is defined and differentiable.
- Multiply by constant coefficients; differentiate sums and differences term by term.
- A constant has derivative zero.
- Rewrite radicals and reciprocals as powers before subtracting one from the exponent.
- Positive x is valid for all rational powers; negative and zero inputs need separate checks.

A worked case

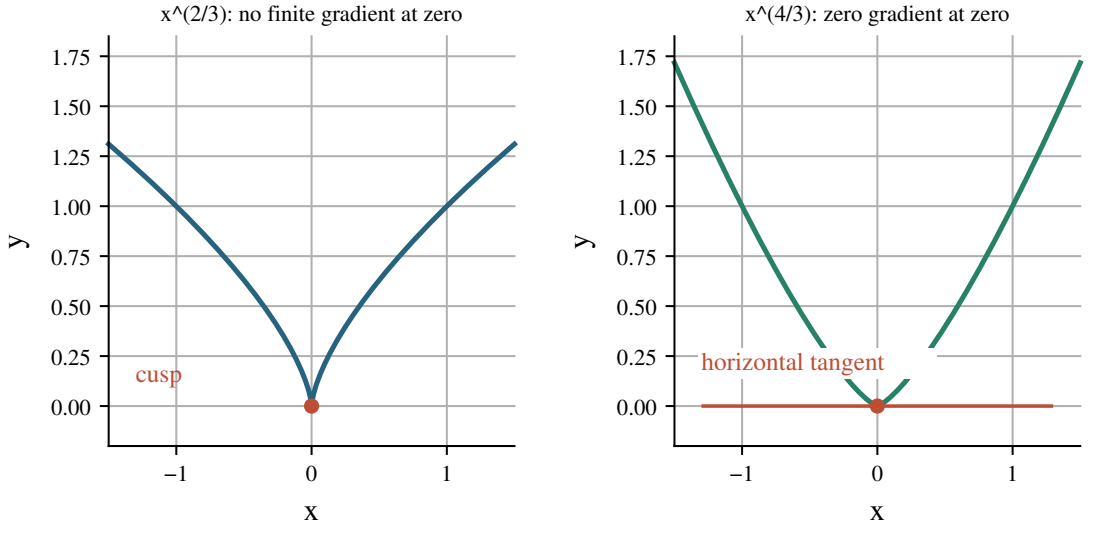
$$\frac{d}{dx}x^p = px^{p-1} \quad \text{on its differentiable real domain}$$

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■ Worked example

For $x > 0$, f=3√x−4/x+1/√x gives f′=(3/2)x^(−1/2)+4x^(−2)−(1/2)x^(−3/2). At x=4 this is 3/4+1/4−1/16=15/16. For q=x^(2/3)=(cube root x)^2, negative x is allowed; q′=(2/3)x^(−1/3) for x≠0, so q′(−8)=−1/3. At zero q has a cusp: the difference quotient |h|^(2/3)/h diverges with opposite signs. For r=x^(4/3), [r(h)−r(0)]/h=cube root h tends to zero, so r′(0)=0. Thus a fractional power does not automatically exclude zero from the derivative.

A finite function value does not decide differentiability



Where it goes wrong

- The original root domain and derivative domain can differ.
- √x is defined at x=0, but 1/(2√x) has no finite value there; 1/√x is not defined there at all.
- A real negative base needs a reduced rational exponent with an odd denominator.
- For x<0, x^(2/3) means the square of its real cube root, not an invalid real logarithm calculation.
- Do not differentiate a reciprocal by keeping the same exponent or dropping the minus sign.

■ Watch out

Not true: Every rational-power function defined at zero has a finite derivative there. x^(2/3) is defined at 0, but its derivative (2/3)x^(−1/3) is undefined there.

Applying it

- State the original domain first, rewrite each term, multiply its coefficient by the exponent and subtract one from that exponent.
- Return to radicals if requested.
- Check boundary inputs with the original difference quotient when the formal derivative is undefined or inconclusive.
- Keep exact fractional coefficients until the final evaluation.

In your course

- AQA A-level Mathematics (7357), specification section G.
- Every paper allows a calculator, but method marks still need the expression or equation written before its value.

■ Key idea

- **Rational power**: An exponent that can be written as a fraction of integers, interpreted with a valid real root.
- For a rational exponent p, d(x^p)/dx=p x^(p−1) wherever the real function is defined and differentiable.
- Watch for: the original root domain and derivative domain can differ.

G · Exponential, logarithmic and trigonometric derivatives

Which factor must appear in its derivative?

- A wave with twice the input frequency has a steeper initial slope.
- This lesson studies **inner rate** 内函数变化率: The derivative of the changing input inside a composite function.

The method

- For constant k and a>0: (e^(kx))′=k e^(kx), (a^(kx))′=k ln(a)a^(kx), (sin kx)′=k cos kx, (cos kx)′=−k sin kx and (tan kx)′=k $\sec^2$(kx), where sec u=1/cos u.
- Trigonometric inputs are radians.
- (ln x)′=1/x for $x > 0$.
- Add differentiated terms and retain their coefficients.

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A worked case

$$(a^{kx})' = k\ln(a)a^{kx}, \quad (\tan kx)' = k\sec^2(kx), \quad (\ln x)' = \frac{1}{x}$$

■ Worked example

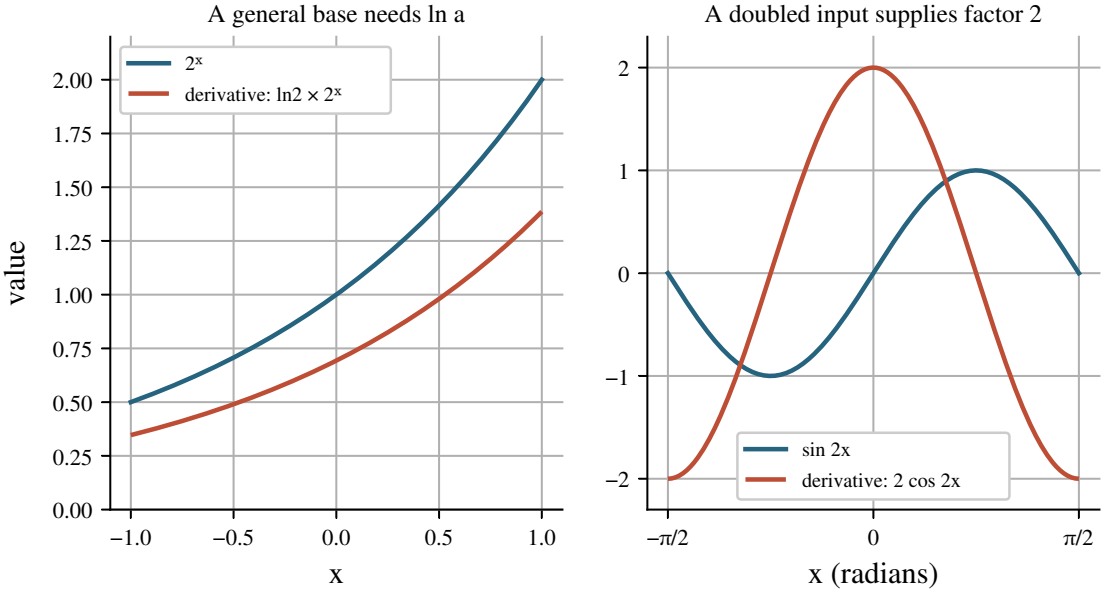
For $x > 0$ and $\cos(x/2) \neq 0$, let

$$f = 2e^{3x} + 3^{2x} - 4\ln x \\ + \sin 2x - 2\cos 3x + \tan(x/2).$$

Then

$$f' = 6e^{3x} + 2\ln 3 \cdot 3^{2x} - \frac{4}{x} \\ + 2\cos 2x + 6\sin 3x + \frac{1}{2}\sec^2(x/2).$$

The term $-2\cos 3x$ gives a positive $6\sin 3x$ because its coefficient and derivative sign are both negative. For $g = 5^{-2x}$, $g' = -2\ln 5 \cdot 5^{-2x}$, so $g'(0) = -2\ln 5$. For $h = \sin 2x - 2\cos 3x + \tan(x/2)$, $h'(0) = 2 + 0 + \frac{1}{2} = \frac{5}{2}$. For $p = 2e^{3x}$, $p'(0) = 6$. Base $a = 1$ gives the constant 1 and derivative zero; $k = 0$ also produces constant functions.



Where it goes wrong

- The exponential base is constant but its exponent changes: a^(kx) needs ln a as well as k.
- The derivative of ln x is 1/x, not ln x/x.
- Tangent is undefined where cos(kx)=0 and its derivative retains those exclusions.
- Derivative formulas in radians need $\pi/180$ factors for degree variables.
- A negative k changes the inner-rate sign; it must not disappear.

■ Watch out

Not true: The derivative of a^(kx) is k a^(kx) for every positive base a. The derivative of a^(kx) is k · ln a · a^(kx); the factor ln a is 1 only for base e.

Applying it

- Identify the function family, preserve its coefficient, differentiate the outer function and multiply by k.
- List log and tangent domain exclusions before evaluating at a point.
- Check the sign against an increasing/declining exponential or the local wave direction.
- Derivatives of linear combinations are obtained by addition, not by multiplying terms.

In your course

- AQA A-level Mathematics (7357), specification section G.
- Every paper allows a calculator, but method marks still need the expression or equation written before its value.

■ Key idea

- **Inner rate**: The derivative of the changing input inside a composite function.
- For constant k and a>0: $(e^{kx})'=k\,e^{kx}$, $(a^{kx})'=k\ln(a)a^{kx}$, $(\sin kx)'=k\cos kx$, $(\cos kx)'=-k\sin kx$ and $(\tan kx)'=k\sec^2(kx)$, where $\sec u=1/\cos u$.
- Watch for: the exponential base is constant but its exponent changes: a^(kx) needs ln a as well as k.

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G · Product and quotient rules with retained exclusions

Why does multiplying their separate derivatives miss most of the rate?

- Both factors in a changing area can vary.
- This lesson studies **quotient rule** 商法则: The rule for differentiating a ratio of two changing functions when the denominator is nonzero.

The method

- For differentiable u,v, (uv)′=u′v+uv′.
- For v≠0, (u/v)′=(u′v−uv′)/v^2.
- The product rule has two contributions: change in either factor while the other provides its current value.
- The quotient rule uses the derivative of the numerator times the denominator, minus the numerator times the derivative of the denominator, all over v^2.

A worked case

$$(uv)' = u'v + uv', \qquad \left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$$

■ Worked example

For y=x^{2}e^x, y′=2xe^x+x^{2}e^x=e^x(x^2+2x), so y′(1)=3e. For z=(x^2+1)/(x−1), x≠1, z′=[2x(x−1)−(x^2+1)]/(x−1)^2=(x^2−2x−1)/(x−1)^2; at x=2 it is −1. Polynomial division gives z=x+1+2/(x−1), hence z′=1−2/(x−1)^2, an independent check. For w=(x^2−1)/(x−1), x≠1, cancellation gives w=x+1 and w′=1 only on that original domain. The missing point at x=1 is not restored; its continuous extension would be a different function.

Two product contributions; quotient order and domain matter

Product: $(x^2 e^x)' = 2x e^x + x^2 e^x$

Quotient: $z = (x^2 + 1)/(x - 1)$, $x \neq 1$

$z' = [2x(x - 1) - (x^2 + 1)]/(x - 1)^2$

Check: $z = x + 1 + 2/(x - 1)$

Hence $z' = 1 - 2/(x - 1)^2$, still $x \neq 1$

Where it goes wrong

- Multiplying $u'v'$ is not the derivative of uv.
- Reversing the quotient numerator flips its sign.
- The squared denominator does not eliminate the original pole or hole.
- Simplifying before differentiating can help, but it must preserve exclusions.
- A stationary quotient point must also lie in the original domain.

Watch out

Not true: Cancelling a factor from a quotient restores every excluded input. Cancelling does not change the original domain; the excluded inputs stay excluded.

Applying it

- Name u,v and write u',v' before substitution.
- Keep brackets around both quotient numerator terms until expansion is complete.
- Factor a product derivative only after collecting both contributions.
- Compare an expanded or divided form when available and substitute into the original domain before evaluating or solving $y'=0$.

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In your course

- AQA A-level Mathematics (7357), specification section G.
- Every paper allows a calculator, but method marks still need the expression or equation written before its value.

■ Key idea

- **Quotient rule**: The rule for differentiating a ratio of two changing functions when the denominator is nonzero.
- For differentiable u,v, (uv)′=u′v+uv′.
- Watch for: multiplying u′v′ is not the derivative of uv.

G · Chain rules for nested functions and restricted inputs

How do we keep every link in its rate of change?

- A sensor converts a changing input through several formulas.
- This lesson studies **composite derivative** 复合函数导数: The outer derivative evaluated at the inner output, multiplied by the derivative of that inner input.

The method

- For y=f(u) and u=g(x), dy/dx=f′(g(x))g′(x).
- Evaluate the outer derivative at the actual inner expression.
- For several nested functions, multiply every inner-rate factor.
- Domain restrictions come from each stage; the differentiated expression cannot restore a forbidden original input.

A worked case

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}, \qquad (e^{x^2})' = 2xe^{x^2}$$

■ Worked example

For y=(3x−1)^4, set u=3x−1: y′=4u³×3=12(3x−1)³, giving y′(0)=−12. For z=e^(x²), z′=e^(x²)×2x; z′(0)=0 although the outer exponential derivative is never zero. For w=ln(2x+1), x>−1/2, w′=2/(2x+1); w′(1)=2/3. For r=sin((2x+1)²), first set u=(2x+1)²: r′=cos((2x+1)²)×2(2x+1)×2=4(2x+1)cos((2x+1)²). For s=√(5−x²), the original domain is −√5≤x≤√5, but s′=−x/√(5−x²) is finite only in the open interval.

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Every nested input contributes its changing rate

$y = \sin((2x + 1)^2)$

Outer derivative: $\cos((2x + 1)^2)$

Middle rate: $2(2x + 1)$

Inner rate: 2

Multiply: $y' = 4(2x + 1)\cos((2x + 1)^2)$

Where it goes wrong

- Do not stop after differentiating the outer power or exponential.
- For $\sin(u^2)$, the cosine input stays u^2; it is not replaced by 2u.
- Missing one link in a nested derivative loses a factor.
- A zero inner derivative can give zero total derivative, so dividing by it without checking can lose valid cases.
- A square root can exist at an endpoint without a finite derivative there.

> ■ **Watch out**
>
> **Not true:** The derivative of $e^{(x^2)}$ is $e^{(x^2)}$ without an inner-rate factor. The chain rule multiplies by the inner derivative: the derivative of $e^{(x^2)}$ is $2x \cdot e^{(x^2)}$.

Applying it

- Work from the outside inward and record each input change.
- Multiply the factors, then simplify.
- For a short polynomial, expansion gives an independent alternative check.
- Retain every log/root restriction and interpret whether a zero rate comes from the outer function, the inner input or both.

In your course

- AQA A-level Mathematics (7357), specification section G.
- Every paper allows a calculator, but method marks still need the expression or equation written before its value.

■ Key idea

- **Composite derivative**: The outer derivative evaluated at the inner output, multiplied by the derivative of that inner input.
- For y=f(u) and u=g(x), dy/dx=f′(g(x))g′(x).
- Watch for: do not stop after differentiating the outer power or exponential.

G · Inverse derivatives at corresponding points

How do we reverse its local rate without taking the reciprocal of its output?

- A calibration converts length to a reading.
- This lesson studies **inverse derivative** 反函数导数: The rate of the inverse output with respect to its input, evaluated at the corresponding original point.

The method

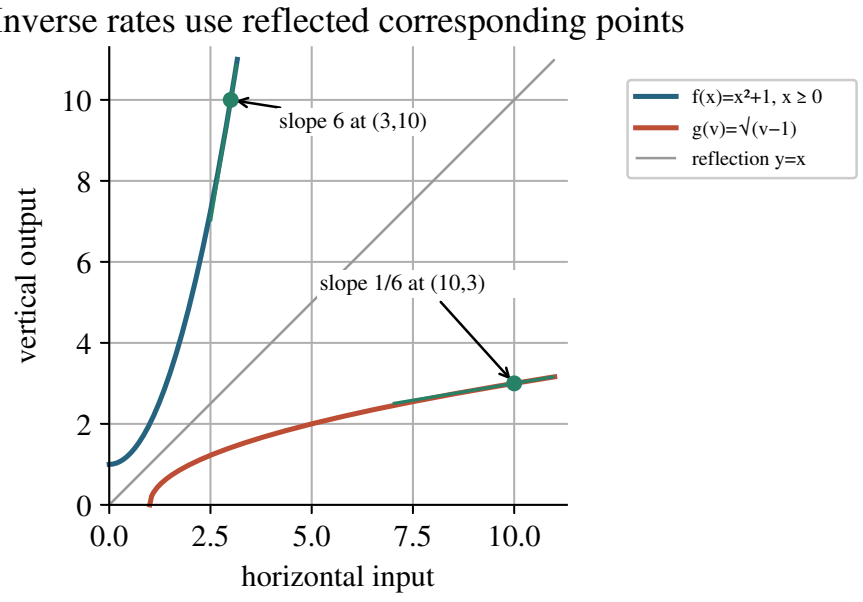
- Let f be one-to-one on a stated interval and g=f^{-1}.
- The identity f(g(v))=v gives f′(g(v))g′(v)=1.
- When the derivatives exist and f′(g(v))≠0, g′(v)=1/f′(g(v)).
- The inverse is evaluated at the original output v, not automatically at the same original input x.
- The reciprocal 1/f(x) is a different function.

A worked case

$$(f^{-1})'(v) = \frac{1}{f'(f^{-1}(v))}$$

■ Worked example

For $f(x)=x^2+1$ on $x\geq 0$, $g(v)=\sqrt{(v-1)}$, $v\geq 1$. At $v=10$, $g(10)=3$ and $f'(3)=6$, so $g'(10)=1/6$. On $x\leq 0$ the inverse branch is $-\sqrt{(v-1)}$, with $g(10)=-3$ and $g'(10)=-1/6$. For $f=e^{\wedge}x$, $g=\ln v$: $f'(g(v))=e^{\wedge}(\ln v)=v$, hence $g'=1/v$ for $v>0$. For $f=\sin x$ on $-\pi/2\leq x\leq \pi/2$, $g=\arcsin v$. Within $-1<v<1$, $\cos(g(v))=\sqrt{(1-v^2)}>0$, so $g'=1/\sqrt{(1-v^2)}$. The nonnegative cosine follows from the restricted sine branch; it is not an arbitrary sign choice.



Where it goes wrong

- Do not replace the corresponding-point denominator $f'(g(v))$ by $f'(v)$.
- One-to-one behaviour alone does not ensure a finite inverse gradient: x^3 is invertible, but its derivative is zero at zero and cube root has no finite derivative there.
- Root/arcsine inverse endpoints can also lack finite gradients.
- Different inverse branches can give different signs.

■ Watch out

Not true: The derivative of an inverse is $1/f'(v)$, evaluated at the inverse input v. The derivative of the inverse at y is $1/f'(x)$, where x is the input with $f(x)=y$.

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Applying it

- Restrict the original domain, find the inverse input/output pair, differentiate f and evaluate its derivative at the recovered original input.
- Then take the reciprocal only if that finite gradient is nonzero.
- Swap horizontal and vertical units for the inverse rate.
- Check the result by differentiating an explicit inverse when one is available.

In your course

- AQA A-level Mathematics (7357), specification section G.
- Every paper allows a calculator, but method marks still need the expression or equation written before its value.

■ Key idea

- **Inverse derivative**: The rate of the inverse output with respect to its input, evaluated at the corresponding original point.
- Let f be one-to-one on a stated interval and $g=f^{-1}$.
- Watch for: do not replace the corresponding-point denominator $f'(g(v))$ by $f'(v)$.

G · Connected rates, geometry and time units

Does its radius increase at a steady rate too?

- A spherical object gains volume at a steady rate.
- This lesson studies **connected rates** 关联变化率: Rates linked by a relationship between quantities changing with the same time variable.

The method

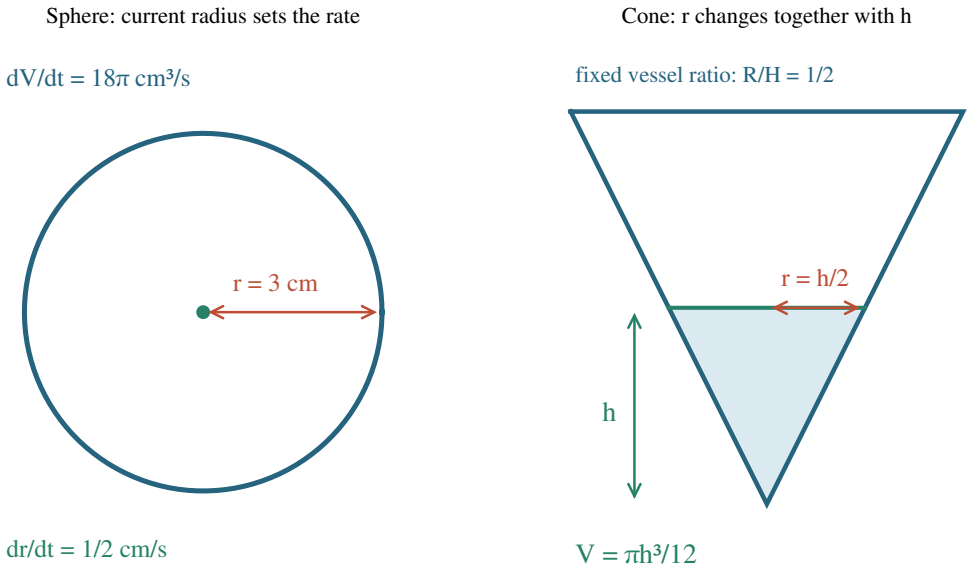
- If $y=f(x)$ and x varies with time t, $dy/dt=(dy/dx)(dx/dt)$.
- Derive a relation between changing quantities, differentiate with respect to time, then substitute the measured values at that instant.
- Keep the time variable and each rate's sign and units.
- For a sphere $V=(4/3)\pi r^3$ and $A=4\pi r^2$, so $dV/dt=4\pi r^2\ dr/dt$ and $dA/dt=8\pi r\ dr/dt$.

A worked case

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt}, \qquad \frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$$

■ Worked example

A sphere has radius 3 cm and gains volume at 18π cm^3/s. Its instantaneous radius rate is $18\pi/(4\pi\times9)=1/2$ cm/s, and surface-area rate is $8\pi\times3\times(1/2)=12\pi$ cm^2/s. This is a local rate; as r grows, the same volume rate gives a smaller radius rate. In a conical vessel whose full radius is half its full height, similarity gives r=h/2 at every water level. Thus $V=\pi h^3/12$ and $dV/dt=(\pi h^2/4)dh/dt$. At h=4 cm and inflow 8π cm^3/s, dh/dt=2 cm/s. For a fixed 5 m ladder, $x^2+y^2=25$; at (x,y)=(3,4) m with dx/dt=0.2 m/s, dy/dt=−(3/4)×0.2=−0.15 m/s: the top descends.



Where it goes wrong

- Do not substitute the current radius or height before differentiating: that would falsely turn the relation into a constant. dV/dt is not dV/dr.
- A changing cone radius must follow the similarity relation, not stay fixed at its current value.
- Convert a per-minute inflow to a per-second rate before combining it with second-based data.
- Negative rate means decrease, not a negative length.

■ **Watch out**

Not true: Substitute a measured radius into the volume formula before differentiating it with respect to time. Differentiate the formula with respect to time first, then substitute the measured radius; substituting first turns the radius into a constant.

Applying it

- Draw the geometry, identify constants and state the valid time/measurement units.
- Eliminate an extra changing variable using similarity or a constraint, then differentiate.
- Use the values for the same instant and describe the result's direction.
- Physical assumptions, such as a fixed ladder length or a true sphere, limit the model; rates need not remain constant after that instant.

In your course

- AQA A-level Mathematics (7357), specification section G.
- Every paper allows a calculator, but method marks still need the expression or equation written before its value.

■ **Key idea**

- **Connected rates**: Rates linked by a relationship between quantities changing with the same time variable.
- If y=f(x) and x varies with time t, dy/dt=(dy/dx)(dx/dt).
- Watch for: do not substitute the current radius or height before differentiating: that would falsely turn the relation into a constant. dV/dt is not dV/dr.

G · Implicit first derivatives and tangent domains

How can we find its tangent gradient without solving for a square root first?

- A circle has an upper and lower branch.
- This lesson studies **implicit relation** 隐式关系: An equation relating x and y without isolating one as an explicit function of the other.

The method

- Treat y as a differentiable local function of x.
- Differentiate the whole relation with respect to x: a function of y brings a dy/dx factor by the chain rule.
- Collect terms containing dy/dx and solve when its coefficient is nonzero.
- If the differentiated relation becomes A(x,y)+B(x,y)y′=0, then y′=−A/B where B≠0; the gradient belongs to a chosen local branch.

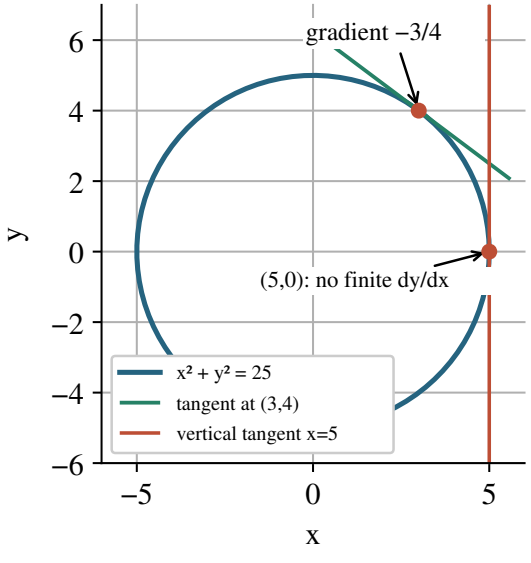
A worked case

$$2x + 2y\frac{dy}{dx} = 0, \qquad \frac{dy}{dx} = -\frac{x}{y} \quad (y \neq 0)$$

■ Worked example

For $x^2+y^2=25$, differentiate to $2x+2y\,y'=0$, hence $y'=-x/y$ for $y\neq0$. At (3,4), the tangent gradient is $-3/4$ and $y-4=-(3/4)(x-3)$; the normal gradient is $4/3$. At (5,0) the tangent is vertical, $x=5$, and no finite dy/dx exists. For $x^2+xy+y^2=7$, differentiate the product xy to $y+x\,y'$. Then $2x+y+(x+2y)y'=0$, so $y'=-(2x+y)/(x+2y)$. At (1,2), which satisfies the relation, the gradient is $-4/5$. For e^y+x=3, e^y y′+1=0, giving $y'=-e^{(-y)}$; at (2,0) the gradient is -1.

Implicit slopes describe a chosen local curve branch



Where it goes wrong

- Differentiating y^2 gives $2y\,y'$, not $2y$ or $2(y')^2$.
- The product xy needs both y and $x\,y'$.
- Verify a supplied point lies on the original relation before assigning its tangent.
- A zero denominator may indicate a vertical tangent or a singular point; the fraction alone cannot classify every case.
- For example $x^2+y^2=0$ has only one isolated real point, not a smooth circle tangent.

■ Watch out

Not true: Differentiating y^2 with respect to x gives $2y$ without a dy/dx factor. By the chain rule, the derivative of y^2 with respect to x is $2y \cdot dy/dx$.

Applying it

- Differentiate every term, retain all chain/product factors, collect the y' terms and state the resulting denominator condition.
- Evaluate at the original point and use point-slope form for the tangent.
- A horizontal tangent has zero finite gradient and a vertical normal; a vertical tangent needs its own line equation.
- This lesson uses first derivatives only, matching G5.

In your course

- AQA A-level Mathematics (7357), specification section G.
- Every paper allows a calculator, but method marks still need the expression or equation written before its value.

■ Key idea

- **Implicit relation**: An equation relating x and y without isolating one as an explicit function of the other.
- Treat y as a differentiable local function of x.
- Watch for: differentiating y^2 gives $2y\,y'$, not $2y$ or $2(y')^2$.

G · Parametric first derivatives and vertical tangents

How do their separate rates combine into the curve's slope?

- Two coordinates depend on the same slider.

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- This lesson studies **parametric gradient** 参数曲线斜率: The ratio of the y-rate to the x-rate along a parametrised curve, when the x-rate is nonzero.

The method

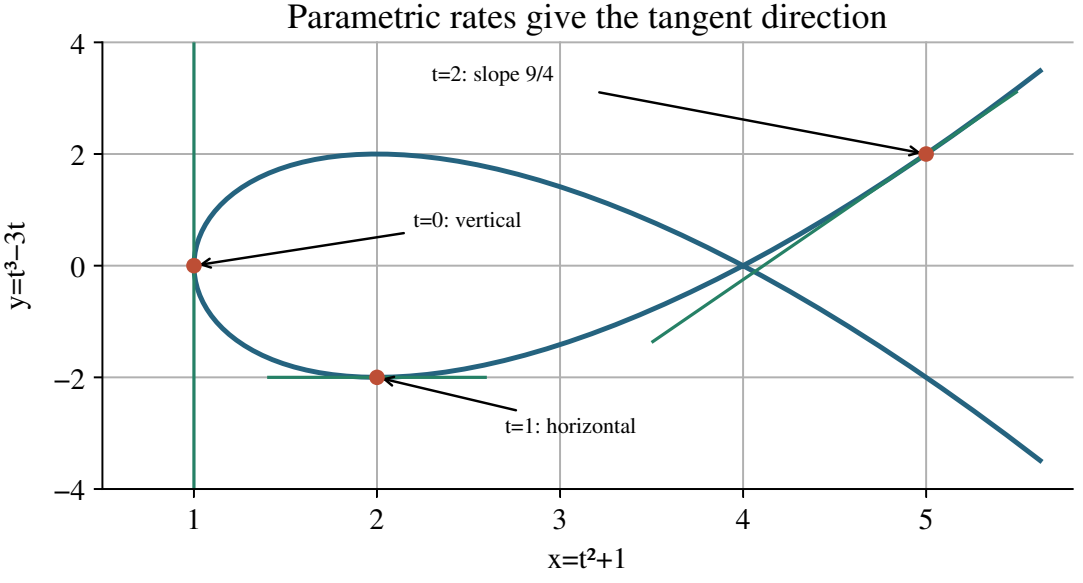
- For x=x(t), y=y(t), the chain rule gives dy/dt=(dy/dx)(dx/dt), so dy/dx=(dy/dt)/(dx/dt) when dx/dt≠0.
- Differentiate the two coordinates separately, then divide in y-over-x order.
- The point is obtained from the same parameter value.
- A parameter need not represent physical time.

A worked case

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} \quad (dx/dt \neq 0)$$

Worked example

For $x=t^2+1$, $y=t^3-3t$, $dx/dt=2t$ and $dy/dt=3t^2-3$. At $t=2$ the point is $(5,2)$ and gradient is $9/4$, so the tangent is $y-2=(9/4)(x-5)$. At $t=1$, the point $(2,-2)$ has horizontal tangent $y=-2$ because $dy/dt=0$ while $dx/dt=2$. At $t=0$, $(1,0)$ has vertical tangent $x=1$: $dx/dt=0$ but $dy/dt=-3\neq0$. A regular local parametrisation then changes y while x has zero instantaneous change. For $x=2\cos t, y=3\sin t$, at $t=\pi/4$ the point is $(\sqrt{2},3\sqrt{2}/2)$ and slope is $-3/2$. At $t=0$ the ellipse tangent is vertical, $x=2$.



Where it goes wrong

- The ratio is dy/dt divided by dx/dt, not its reciprocal.
- If both rates vanish, 0/0 is inconclusive: $x=t^3$,$y=t^3$ still traces y=x with slope 1 through zero.
- If dx/dt=0 alone, inspect the local curve rather than report a finite numerical gradient.
- Repeated x values can correspond to different y values or branches; keep the stated parameter value.

■ Watch out

Not true: When both parametric coordinate rates are zero, the curve must have no tangent. When both rates are zero, dy/dx=0/0 is undetermined; examine the curve more closely, because a tangent may still exist.

Applying it

- Find the coordinate pair, calculate both first rates, then test whether the ratio is valid.
- State horizontal/vertical tangent lines separately when appropriate and preserve the parameter range.
- Check by eliminating the parameter when that gives a simple local relation.
- G5 requires first derivatives only; a second parametric derivative is outside this lesson's stated scope.

In your course

- AQA A-level Mathematics (7357), specification section G.
- Every paper allows a calculator, but method marks still need the expression or equation written before its value.

■ Key idea

- **Parametric gradient**: The ratio of the y-rate to the x-rate along a parametrised curve, when the x-rate is nonzero.
- For x=x(t), y=y(t), the chain rule gives dy/dt=(dy/dx)(dx/dt), so dy/dx=(dy/dt)/(dx/dt) when dx/dt≠0.
- Watch for: the ratio is dy/dt divided by dx/dt, not its reciprocal.

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G · Tangent and normal equations and gradient conditions

Why must both lines use the same curve point?

- A road's tangent gives its local direction; a cross-road follows the normal.
- This lesson studies **normal line** 法线: A line through the curve point perpendicular to its local tangent.

The method

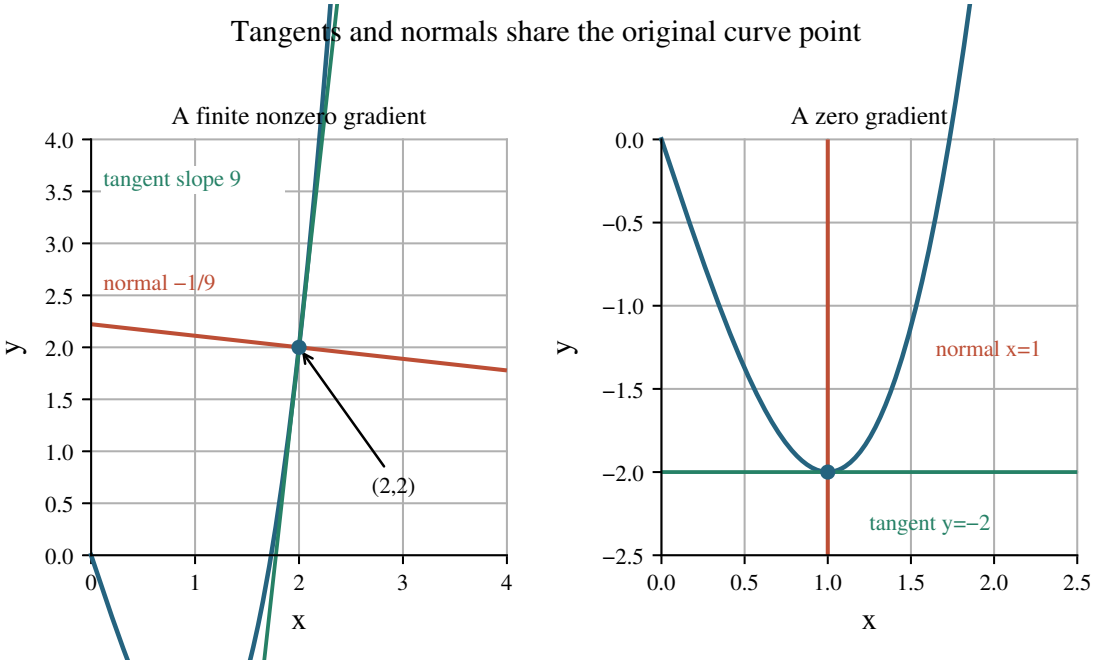
- At a differentiable point (a,f(a)), the tangent gradient is m=f′(a) and its equation is y−f(a)=m(x−a).
- For a finite nonzero m, the normal gradient is $-1/m$.
- Find points satisfying a given gradient condition by solving for f′(x), then evaluate the original f.
- Parallel lines have equal gradients; finite nonzero perpendicular gradients multiply to -1.

A worked case

$$y - f(a) = f'(a)(x - a), \qquad m_{\text{normal}} = -\frac{1}{f'(a)} \quad (f'(a) \neq 0)$$

■ Worked example

For f=x^3−3x at x=2, f(2)=2 and f′(2)=9. The tangent is y−2=9(x−2), or y=9x−16. The normal is y−2=−(1/9)(x−2). Tangents parallel to y=9x+4 satisfy $3x^2-3=9$, giving x=±2: points (2,2) and (−2,−2). The second tangent is y+2=9(x+2). At x=1 the tangent is horizontal y=−2 and the normal is vertical x=1; no finite negative reciprocal of zero exists. For the circle at (5,0), the tangent is vertical x=5 and the normal horizontal y=0. The secant through (−2,−2) and (2,2) has gradient 1, unlike either local tangent gradient 9.



Where it goes wrong

- The derivative supplies a gradient, not the y-coordinate.
- Use the original curve for the point.
- The normal's reciprocal includes a minus sign and is unavailable for a horizontal tangent; give a vertical line instead.
- A tangent can meet the curve again elsewhere.
- A given line's intercept does not affect a parallel-gradient condition, but both tangent and normal must pass through the stated curve point.

■ Watch out

Not true: A normal to a horizontal tangent has finite gradient zero. The normal to a horizontal tangent is vertical, so its gradient is undefined.

Applying it

- Find the point and gradient separately, then use point-slope form.
- If a gradient condition gives several inputs, retain every allowed point and line.
- State horizontal/vertical cases without inventing an infinite numerical gradient.
- Verify the line passes through the point and that a finite nonzero tangent/normal gradient pair multiplies to -1.

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In your course

- AQA A-level Mathematics (7357), specification section G.
- Every paper allows a calculator, but method marks still need the expression or equation written before its value.

■ Key idea

- **Normal line**: A line through the curve point perpendicular to its local tangent.
- At a differentiable point (a,f(a)), the tangent gradient is m=f′(a) and its equation is y−f(a)=m(x−a).
- Watch for: the derivative supplies a gradient, not the y-coordinate.

G · Optimisation with constraints, boundaries and rejected roots

Which cut actually gives the largest valid volume?

- Cutting larger corner squares makes a box taller but its base smaller.
- This lesson studies **feasible domain** 可行域: The allowed input values after the original physical or mathematical constraints are applied.

The method

- Express the quantity to optimise in one variable, with its feasible domain.
- Solve a derivative-zero condition for interior candidates, reject invalid roots, and classify using derivative signs or a nonzero second derivative.
- A global maximum/minimum also needs boundary values or limiting behaviour; local classification alone does not settle a restricted-domain question.

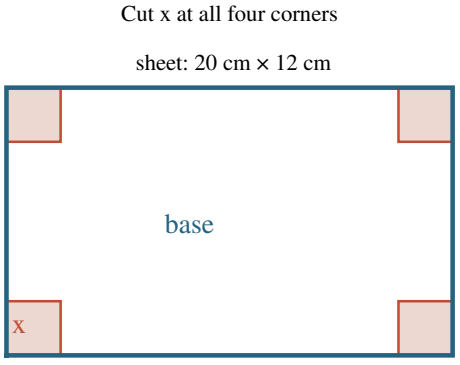
A worked case

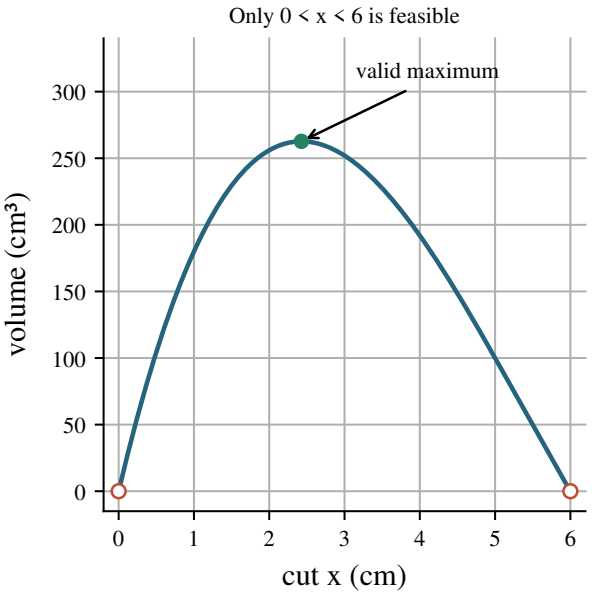
$$V = x(20 - 2x)(12 - 2x), \quad 0 < x < 6, \quad V' = 240 - 128x + 12x^2$$

■ Worked example

Cut squares of side x cm from a 20 cm by 12 cm sheet and fold an open box. Its dimensions are x,20−2x,12−2x, so V=240x−64x^2+4x^3 and 0<x<6. V′=240−128x+12x^2 gives roots x=(16±2√19)/3. The smaller root α≈2.4274 lies in the domain; the larger ≈8.2393 makes the short base side negative and is rejected. V′ is positive on (0,α) and negative on (α,6), while V tends to zero at both ends. Thus α gives the global maximum. A separate rectangle with perimeter 24 cm has sides x,12−x, area A=12x−x^2

and $0<x<12$; its maximum is 36 cm^2 at $x=6$. For $f=(x-1)^2$ on $[0,3]$, the interior stationary value is the minimum 0, but the global maximum is the endpoint value $f(3)=4$.





Where it goes wrong

- A stationary root outside the feasible domain is not an alternative design.
- A second-derivative test gives a local classification, not every global comparison.
- Open endpoints are limiting values, not permitted designs.
- For a closed interval check actual endpoint values.
- If dimensions must use a specified discrete step, compare the neighbouring allowed choices rather than reporting an unattainable continuous optimum.

■ Watch out

Not true: Every solution of $V'=0$ is a feasible global maximum of the box volume. A stationary point must lie inside the feasible domain, and the endpoints must be checked for the global maximum.

Applying it

- Write units and constraints before calculus, keep exact roots through classification, and give the requested final dimensions or quantity rather than only the input.
- State idealisations such as negligible material thickness and exact folds.

- If the model has discrete cuts or measurement tolerance, explain how the final feasible choice and its comparison would change.

In your course

- AQA A-level Mathematics (7357), specification section G.
- Every paper allows a calculator, but method marks still need the expression or equation written before its value.

■ **Key idea**

- **Feasible domain**: The allowed input values after the original physical or mathematical constraints are applied.
- Express the quantity to optimise in one variable, with its feasible domain.
- Watch for: A stationary root outside the feasible domain is not an alternative design.

G · Constructing differential equations from rate laws

Which parts belong in the rate equation?

- A population grows in proportion to its current size while a fixed number leaves each day.
- This lesson studies **differential equation** 微分方程: A relationship involving an unknown function and one or more of its derivatives.

The method

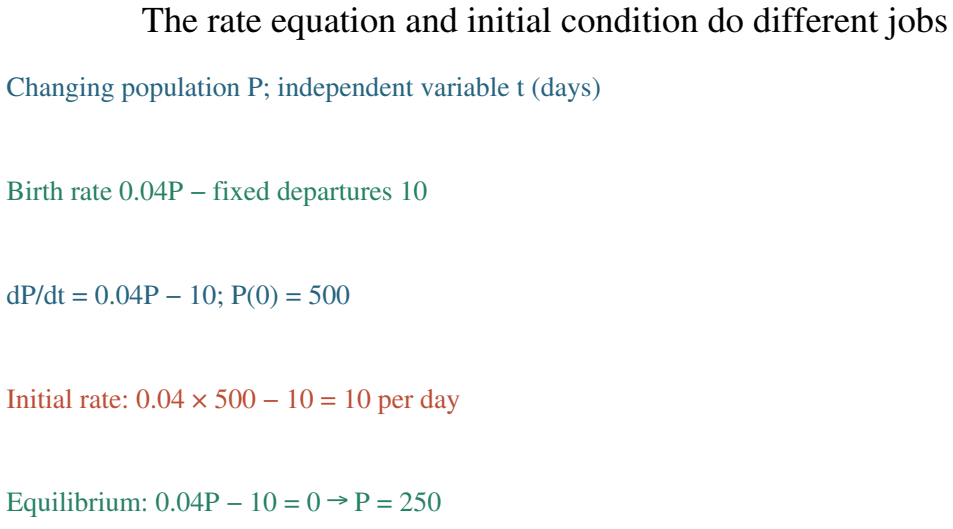
- Choose independent and dependent variables and convert the stated rate law into a derivative.
- Proportional to y means ky; proportional to y^2 means ky^2.
- A fixed inflow minus proportional loss gives dy/dt=a−ky; a rate proportional to the gap from a target A gives dy/dt=k(A−y).
- An initial condition fixes a function value, not a derivative formula.
- Constants must have units making both sides the same kind of rate.

A worked case

$$\frac{dP}{dt} = 0.04P - 10, \qquad P(0) = 500$$

Worked example

For an illustrative population P, t in days, births occur at 0.04P per day and 10 individuals leave per day. The model is dP/dt=0.04P−10, P(0)=500. Initially the rate is 10 individuals/day; P=250 is an equilibrium, and below it the same equation predicts decline. The model must stop or be refined if it predicts a negative population. For a velocity v in m/s with t in seconds, acceleration 6−0.2v gives dv/dt=6−0.2v, v(0)=0: initial acceleration is 6 m/s^2 and equilibrium speed 30 m/s. In a hypothetical demand model, quantity Q falls at a price-rate proportional to Q. At price p=12, Q=200 and dQ/dp=−10, so dQ/dp=−0.05Q with Q(12)=200; the independent variable is price, not time. In pure maths, slope 2xy with y(1)=3 gives dy/dx=2xy and initial slope 6.



Where it goes wrong

- Proportional rate is not constant absolute change.
- Fixed departures need a subtraction term, not a second multiplier of P.
- Do not replace a changing P with its initial value throughout the equation.
- An equilibrium solves the rate-equals-zero condition; it need not equal the initial value or be approached from every start.
- A price derivative and a time derivative are different quantities with different units.

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■ **Watch out**

Not true: A statement that the rate is proportional to the current amount means the same absolute change at all times. Proportional to the current amount means dN/dt=kN, so the change is larger when N is larger.

Applying it

- Define each variable, translate every inflow/outflow or proportional term with its sign, state constant assumptions and attach the initial condition separately.
- Evaluate the initial rate as a check and find any equilibrium before interpreting the model.
- These are illustrative constructions, not compulsory context lists: G6 examples are optional.
- Solving the resulting equation is a separate integration task.

In your course

- AQA A-level Mathematics (7357), specification section G.
- Every paper allows a calculator, but method marks still need the expression or equation written before its value.

■ **Key idea**

- **Differential equation**: A relationship involving an unknown function and one or more of its derivatives.
- Choose independent and dependent variables and convert the stated rate law into a derivative.
- Watch for: proportional rate is not constant absolute change.

H · Integrals, area and accumulation

How much change has accumulated?

- A velocity graph tells us motion at an instant. How can we recover the displacement accumulated over time?
- This lesson studies **antiderivative** 原函数: A function whose derivative equals the given integrand.

The method

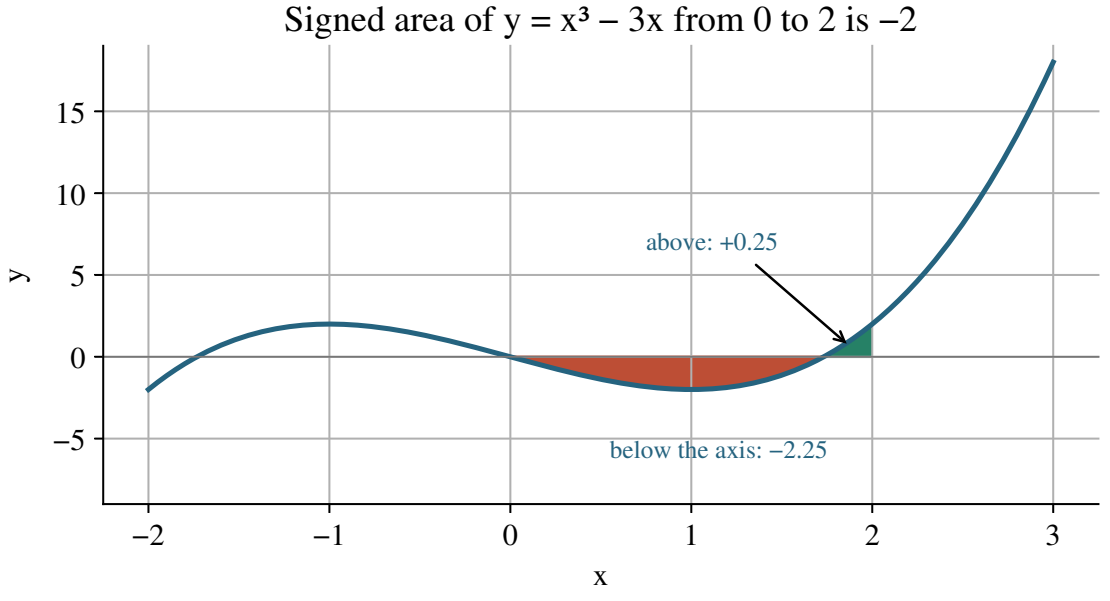
- For n≠-1, the integral of ax^n is ax^(n+1)/(n+1)+C.
- A definite integral is signed accumulation.
- Split at sign changes when total area or distance is required.

A worked case

$$\int ax^n \, dx = \frac{ax^{n+1}}{n+1} + C \quad (n \neq -1)$$

Worked example

For v(t)=3t²-3 over 0≤t≤2, displacement=[t³-3t]_0^2=2. Since v changes sign at t=1, distance=-[t³-3t]_0^1+[t³-3t]_1^2=2+4=6. The constants cancel only for a definite integral.



Where it goes wrong

- The integral of 1/x is ln|x|+C, not the power rule with n=-1.
- Signed area can be zero even when the total enclosed area is positive.

■ **Watch out**

Not true: A definite integral equals the total positive area, even when the graph crosses the axis. Area below the axis counts as negative, so a definite integral can be smaller than the total area, or even zero.

Applying it

- Identify what the integral means and include the correct units.
- A rate measured per second integrates to the underlying quantity, not to another rate.
- Differentiate an antiderivative to check it.

In your course

- AQA A-level Mathematics (7357), specification section H.
- Every paper allows a calculator, but method marks still need the expression or equation written before its value.

■ **Key idea**

- **Antiderivative**: A function whose derivative equals the given integrand.
- For n≠-1, the integral of ax^n is ax^(n+1)/(n+1)+C.
- Watch for: the integral of 1/x is ln|x|+C, not the power rule with n=-1.

H · Substitution, parts and partial fractions

Can we integrate the two factors separately?

- A simple-looking product such as xe^x cannot be integrated by separately integrating each factor.
- This lesson studies **integration by parts** 分部积分法: An integration method based on the derivative of a product.

The method

- Use substitution when an inner derivative appears as a factor.
- By parts, integral u v prime =uv-integral u prime v.
- For a rational function, divide first if needed, then decompose into partial fractions.

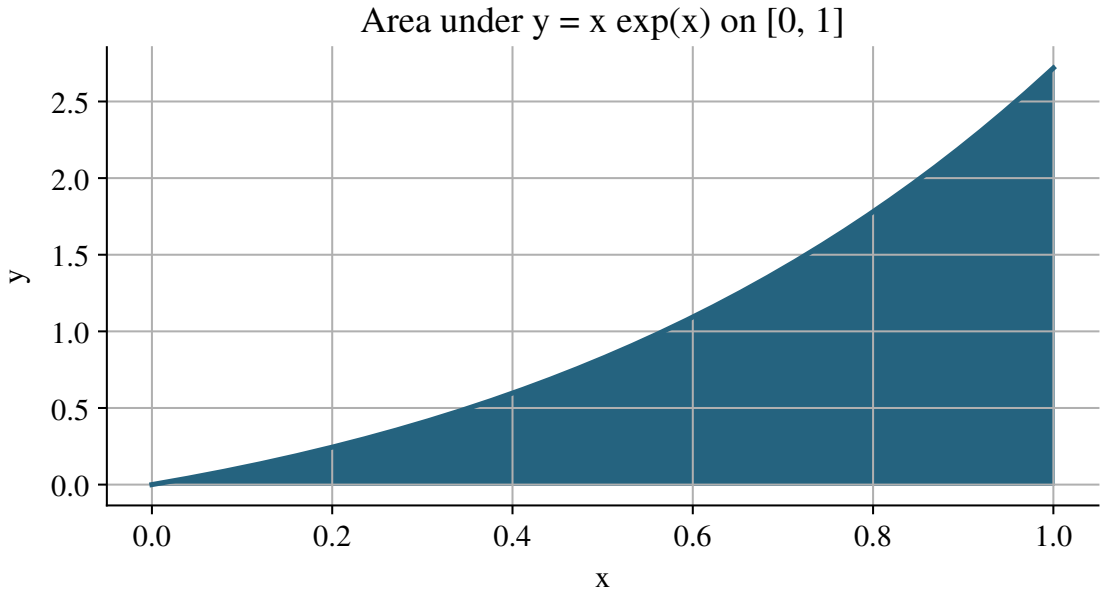
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A worked case

$$\int uv'\,dx = uv - \int u'v\,dx$$

Worked example

For integral xe^x dx, take u=x and v prime=e^x. Then the integral is xe^x-e^x+C. For integral 2x/(x²+1) dx, substitute w=x²+1, dw=2x dx; the answer is ln(x²+1)+C.



Where it goes wrong

- Choosing u and v prime well matters: the remaining integral should become simpler.
- In a definite substitution, either change the limits or return to x before applying the original limits.

Watch out

Not true: The integral of a product equals the product of its separate integrals. A product needs a method such as integration by parts or substitution: ∫x · x dx=x³/3, not (x²/2)(x²/2).

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Applying it

- AQA H5 introduces substitution and parts; H6 integrates partial fractions with linear denominators.
- The single worked parts/substitution examples are introductory support, not complete repeated-parts or partial-fraction coverage.

In your course

- AQA A-level Mathematics (7357), specification section H.
- Every paper allows a calculator, but method marks still need the expression or equation written before its value.

■ **Key idea**

- **Integration by parts**: An integration method based on the derivative of a product.
- Use substitution when an inner derivative appears as a factor.
- Watch for: choosing u and v prime well matters: the remaining integral should become simpler.

H · Differential equations and numerical solutions

What does the starting value decide?

- A growing population's rate is proportional to its current size. The rate equation describes a whole family until an initial population is supplied.
- This lesson studies **initial condition** 初始条件: A specified value that selects a particular solution from a family.

The method

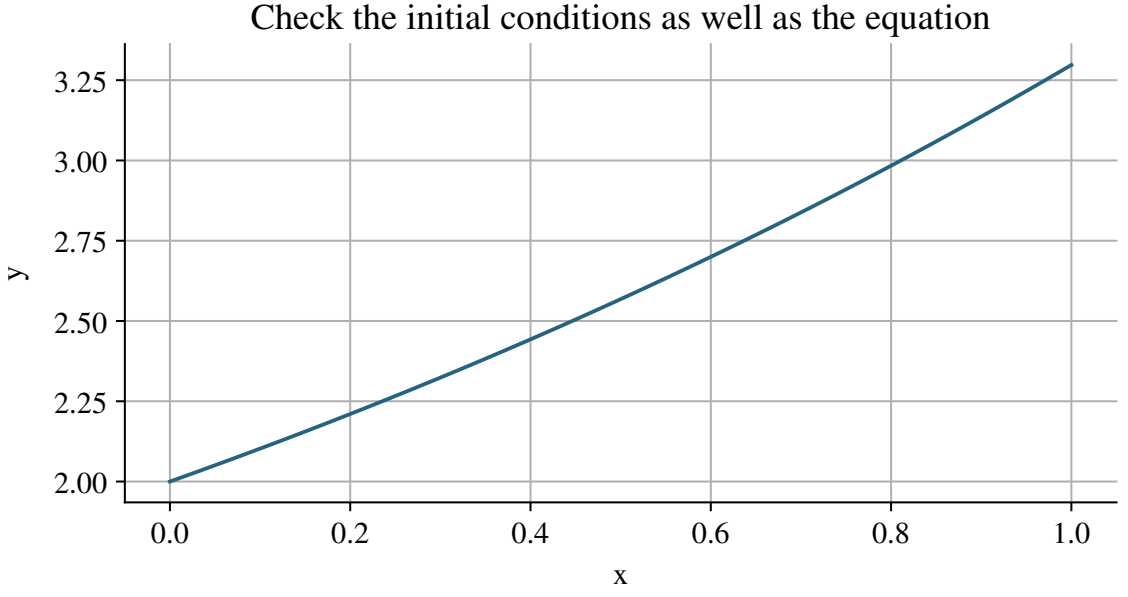
- Separate the variables in dy/dx=ky, integrate both sides and apply an initial condition.
- Include any equilibrium solution excluded when dividing by y.

A worked case

$$\frac{dy}{dx} = ky, \qquad y = Ae^{kx}$$

■ Worked example

For dy/dx=0.5y and y(0)=2, separation gives ln|y|=0.5x+C, hence y=Ae^(0.5x). The initial value gives A=2. Differentiate the result to check the equation and substitute x=0 to check the initial condition.



Where it goes wrong

- An initial condition determines the integration constant; it is not a replacement for integrating.
- Dividing by y can omit an equilibrium solution y=0.
- Euler accuracy depends on step size and the equation.

■ Watch out

Not true: An initial condition never changes the constant in a differential-equation solution. Euler's method depends on the step size; smaller steps usually give a more accurate value.

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Applying it

- This unit uses first-order separation and initial conditions.
- Euler numerical integration and second-order complementary functions are excluded from this lesson.

In your course

- AQA A-level Mathematics (7357), specification section H.
- Every paper allows a calculator, but method marks still need the expression or equation written before its value.

■ Key idea

- **Initial condition**: A specified value that selects a particular solution from a family.
- Separate the variables in dy/dx=ky, integrate both sides and apply an initial condition.
- Watch for: an initial condition determines the integration constant; it is not a replacement for integrating.

H · The Fundamental Theorem and accumulated change

How does the accumulated volume recover that same rate when differentiated?

- A flow meter reports a changing rate.
- This lesson studies **accumulation function** 累积函数: A function whose value is the signed integral from a fixed starting point to a variable endpoint.

The method

- For continuous f on an interval, A(x)=integral from a to x of f(t) dt has $A'(x)=f(x)$.
- If $F'=f$, the definite integral from a to b is $F(b)-F(a)$.
- These two directions connect accumulation and differentiation.
- Dummy variable t is integrated; x is the moving endpoint.
- For A(g(x)), apply the chain rule: its derivative is $f(g(x))g'(x)$.

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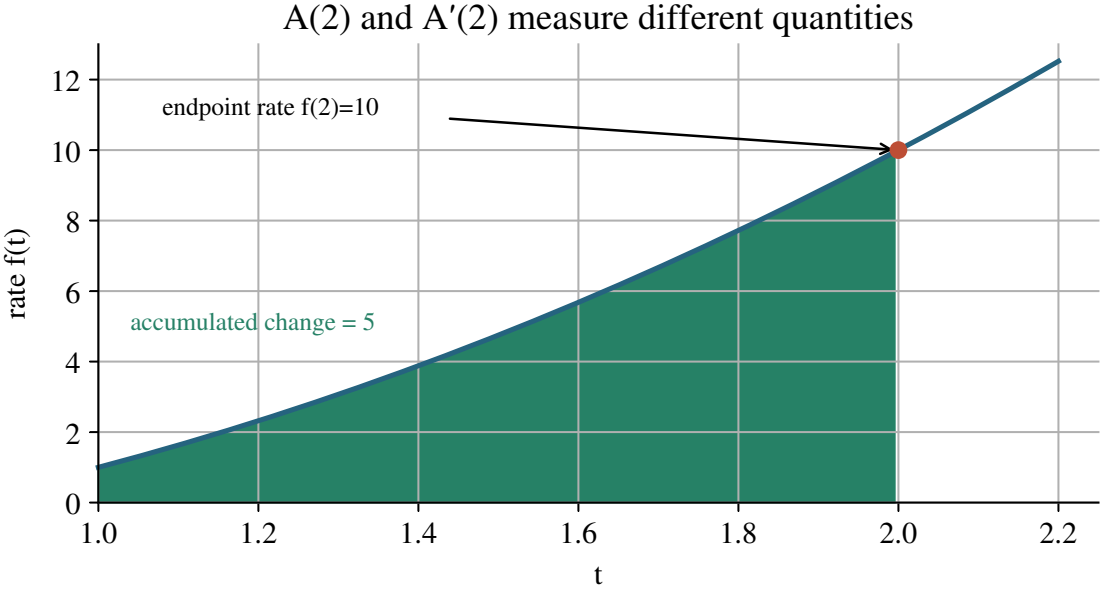
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A worked case

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x), \qquad \int_a^b f(x)\,dx = F(b) - F(a)$$

■ Worked example

Let A(x)=integral from 1 to x of $(3t^2-2)$ dt. An antiderivative is t^3-2t, so $A(x)=x^3-2x+1$, A(1)=0, A(2)=5 and $A'(2)=10$. Thus accumulated change 5 differs from endpoint rate 10. For B(x)=integral from 1 to x^2 of $(3t^2-2)$ dt, $B'(x)=(3x^4-2)2x$; at x=1 this is 2. Independently, if $F'=3x^2-2$ and F(1)=4, then $F=x^3-2x+5$. Its change F(2)−F(1)=5 agrees with A(2), although its initial value is not zero.



Where it goes wrong

- Subtract the lower antiderivative value even when the lower limit is not zero.
- A definite integral is signed change, not automatically total positive area.
- A′(x) is the integrand at the endpoint, not its antiderivative.
- Continuity on the relevant interval is a sufficient hypothesis; do not integrate across a pole such as 1/t at zero.
- A moving endpoint x^2 also contributes its chain-rule factor.

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■ **Watch out**

Not true: The derivative of an accumulation function is its accumulated total. The derivative of $\int_a^x$ f(t) dt is f(x), the integrand's value at x, not the total.

Applying it

- Use rate units multiplied by input units for accumulated change.
- Check A(a)=0 and differentiate the final expression to recover f.
- Initial values determine constants in a state function; definite changes do not depend on an arbitrary antiderivative constant.
- This lesson gives the theorem and elementary variable-bound examples; improper integrals are outside this treatment.

In your course

- AQA A-level Mathematics (7357), specification section H.
- Every paper allows a calculator, but method marks still need the expression or equation written before its value.

■ **Key idea**

- **Accumulation function**: A function whose value is the signed integral from a fixed starting point to a variable endpoint.
- For continuous f on an interval, A(x)=integral from a to x of f(t) dt has A′(x)=f(x).
- Watch for: subtract the lower antiderivative value even when the lower limit is not zero.

H · Integration as the limit of rectangle sums

What changes as the strip width tends to zero?

- A finite collection of rectangles misses part of a curved region.
- This lesson studies **Riemann sum** 黎曼和: A sum of signed rectangle contributions that approaches an integral as the partition becomes finer.

The method

- For n equal strips on [a,b], width h=(b−a)/n.
- A right-end sum is h times the sum of f(a+kh) for k=1,…,n; a left-end sum uses $k = 0$,…,n−1.

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- For continuous f, both limits as n tends to infinity equal the integral.
- A finite sum is an approximation; the limit is the exact signed accumulation.

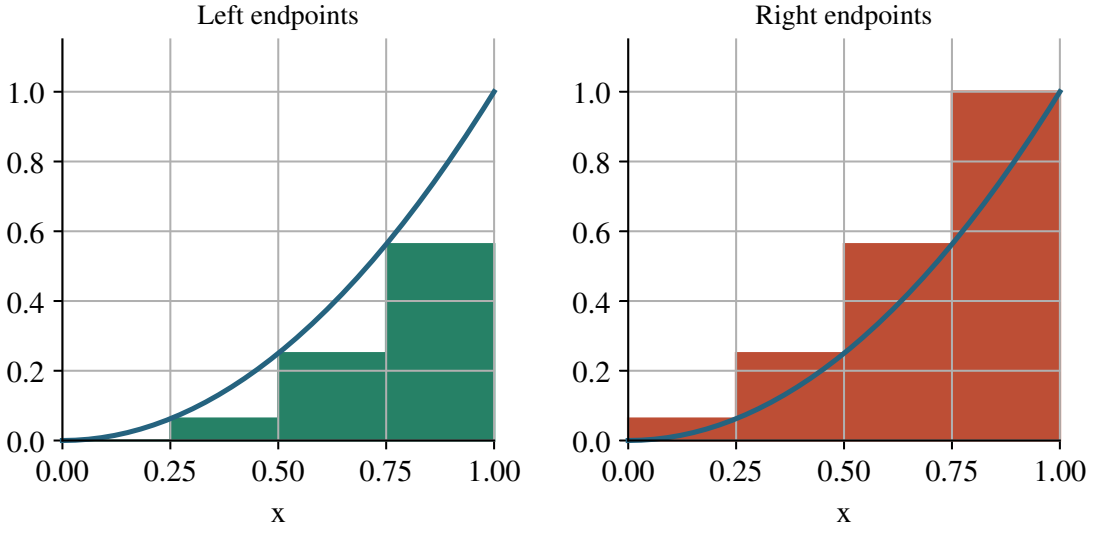
A worked case

$$\lim_{n\to\infty} \frac{1}{n^3} \sum_{k=1}^{n} k^2 = \frac{1}{3} = \int_0^1 x^2 \, dx$$

Worked example

For f(x)=x^2 on [0,1], h=1/n. The right sum R_n=(1/n^3) sum from k=1 to n of k^2=(n+1)(2n+1)/(6n^2)=1/3+1/(2n)+1/(6n^2). The left sum L_n=(n−1)n(2n−1)/(6n^3)=1/3−1/(2n)+1/(6n^2). With n=2, L_2=1/8 and R_2=5/8; both bound the exact integral 1/3 because x^2 increases on this interval. Their difference is 1/n, tending to zero. This same definition gives signed contributions: for f=−2 on [0,1], every such sum is −2, while positive geometric area is 2.

Finite rectangles approach the same signed integral



Where it goes wrong

- Multiply by strip width: adding heights alone gives the wrong units and limit.
- The left sum includes $k = 0$ and excludes k=n; the right sum does the reverse.
- Increasing functions give left/right bounds, but the rule must be checked on the actual interval.

- Rectangle sums and the trapezium rule are different finite approximations.
- Negative heights contribute negative signed area.

■ **Watch out**

Not true: A finite right-end rectangle sum is already the exact integral of every continuous curve. A finite rectangle sum is an approximation; it becomes the integral only in the limit.

Applying it

- Write the partition, evaluation points and sigma limits before simplifying.
- Use a finite-sum identity, then evaluate the limit.
- Compare finite bounds with the antiderivative result as an independent check.
- This is H4 exact limit reasoning; numerical trapezia belong to I3 and are not a substitute for the limiting argument.

In your course

- AQA A-level Mathematics (7357), specification section H.
- Every paper allows a calculator, but method marks still need the expression or equation written before its value.

■ **Key idea**

- **Riemann sum**: A sum of signed rectangle contributions that approaches an integral as the partition becomes finer.
- For n equal strips on [a,b], width h=(b−a)/n.
- Watch for: multiply by strip width: adding heights alone gives the wrong units and limit.

H · Elementary antiderivatives, coefficients and domains

How must integration undo that factor?

- Differentiation multiplies an exponential or trigonometric argument by its coefficient.
- This lesson studies **antiderivative** 原函数: A function whose derivative is the specified integrand on the stated interval.

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The method

- For real powers on an allowed differentiable domain, integrate x^n as x^(n+1)/(n+1)+C when n≠−1.
- Integrate 1/x as ln|x|+C on an interval excluding zero.
- For k≠0, integrals of e^(kx), sin(kx), cos(kx) are e^(kx)/k, −cos(kx)/k, sin(kx)/k respectively, plus constants.
- Apply linearity to sums and constant multiples; verify by differentiation.

A worked case

$$\int \cos(kx)\,dx = \frac{\sin(kx)}{k} + C \quad (k \neq 0), \qquad \int \frac{1}{x}\,dx = \ln|x| + C$$

Worked example

For f(x)=6x²−4e^(2x)+3cos(3x)−2sin(2x)+5/x with $x > 0$, F=2x³−2e^(2x)+sin(3x)+cos(2x)+5ln x+C. Each derivative recovers its matching term. On x<0, integrate 5/x as 5ln(−x), or 5ln|x|; constants can differ on disconnected intervals. The integral of 3cos(3x) from 0 to $\pi/6$ is sin(π/2)−sin 0=1. For F′=6x² and F(1)=7, F=2x³+5. If $k = 0$, e^(kx)=1 and cos(kx)=1 integrate to x+C, while sin(kx)=0 integrates to C; never divide by $k = 0$.

Differentiate each result to recover its integrand

Integrand	Antiderivative (plus C)
6x²	2x³
–4 exp(2x)	–2 exp(2x)
3 cos(3x)	sin(3x)
–2 sin(2x)	cos(2x)
5/x	5 ln|x| (x ≠ 0)

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Where it goes wrong

- The n=−1 power is a separate logarithmic case, not division by zero in the power formula.
- Keep the sign in the sine antiderivative and divide by the argument coefficient. ln x alone does not cover negative x.
- An indefinite integral needs a constant; a definite integral has a numerical value when valid limits are supplied.
- A real fractional power may restrict the domain.

■ Watch out

Not true: Integrating cos(kx) gives k sin(kx), including $k = 0$. $\int$cos(kx) dx=(1/k)sin(kx)+c for k≠0; when $k = 0$ the integrand is 1 and the integral is x+c.

Applying it

- Differentiate term by term to check coefficients and signs, and state the interval before using logarithms or fractional powers.
- The H2 named elementary forms are powers, e^(kx), 1/x, sin(kx) and cos(kx).
- Other forms require separate justification and are not asserted as additional compulsory H2 forms.

In your course

- AQA A-level Mathematics (7357), specification section H.
- Every paper allows a calculator, but method marks still need the expression or equation written before its value.

■ Key idea

- **Antiderivative**: A function whose derivative is the specified integrand on the stated interval.
- For real powers on an allowed differentiable domain, integrate x^n as x^(n+1)/(n+1)+C when n≠−1.
- Watch for: the n=−1 power is a separate logarithmic case, not division by zero in the power formula.

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How reliable is an approximation?

- A nonlinear equation has no convenient exact solution. An approximation still needs evidence that it is reliable.
- This lesson studies **iteration** 迭代: A repeated update in which each new approximation is calculated from the previous one.

The method

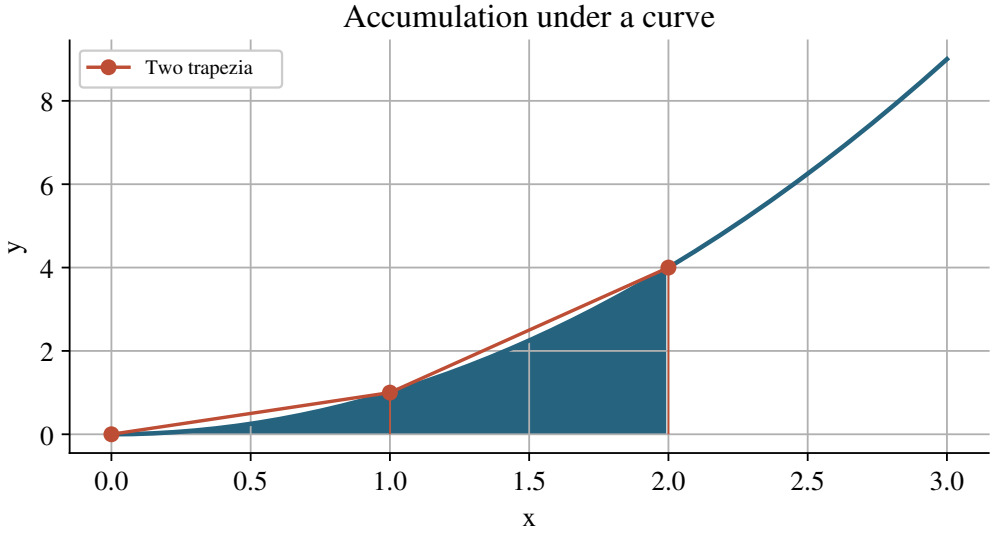
- A continuous function with opposite signs at two endpoints has a root between them.
- Newton's method uses x next=x-f(x)/f prime(x), with a nonzero derivative.
- The trapezium rule approximates a definite integral using endpoint heights.

A worked case

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Worked example

For f(x)=x^2-2 and x_0=1.5, Newton gives x_1=1.5-(2.25-2)/3=1.4166667. With y=x^2 on [0,2] and two equal strips, h=1 and trapezium area=(1/2)(0+2×1+4)=3; the exact area is 8/3.



Where it goes wrong

- A sign change across a discontinuity does not prove a root.
- Iteration can diverge or cycle.
- The trapezium rule's overestimate or underestimate depends on curvature, not just whether the function increases.

Watch out

Not true: Every sign change proves a root, including one across a discontinuity. A sign change shows a root only if the function is continuous there: 1/x changes sign across x=0 but has no root.

Applying it

- Give a stopping criterion and report sensible accuracy.
- Confirm the approximated root by a sign bracket around the stated rounded answer; explain any failure of the chosen iteration.

In your course

- AQA A-level Mathematics (7357), specification section I.
- Every paper allows a calculator, but method marks still need the expression or equation written before its value.

Key idea

- **Iteration**: A repeated update in which each new approximation is calculated from the previous one.
- A continuous function with opposite signs at two endpoints has a root between them.
- Watch for: A sign change across a discontinuity does not prove a root.

J · Vectors and transformation geometry

Why is displacement shorter than the walk?

- Walking 4 m east and 3 m north gives a displacement of 5 m, even though the travelled distance is 7 m.
- This lesson studies **resultant** 合向量: The vector sum representing the combined displacement or force.

The method

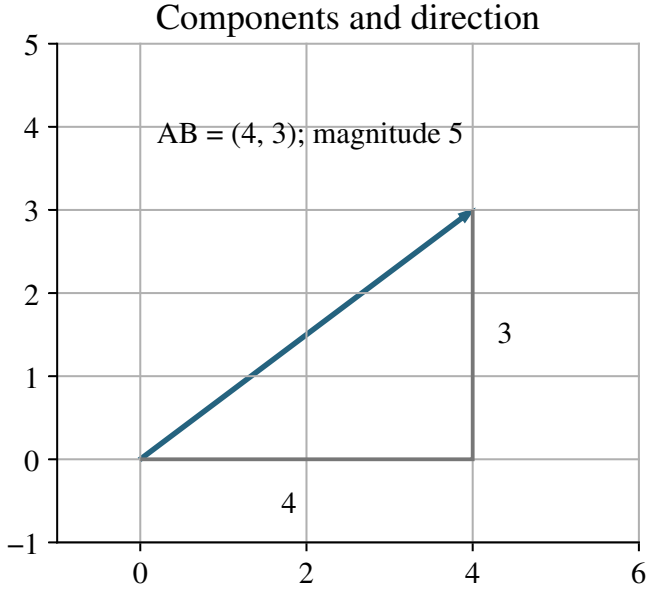
- Add corresponding vector components and subtract position vectors to find a displacement.
- A translation moves every point by the same vector; a scalar multiple changes length and possibly direction.

A worked case

$$\overrightarrow{AB} = \mathbf{b} - \mathbf{a}, \qquad |\mathbf{v}| = \sqrt{v_x^2 + v_y^2}$$

Worked example

With a=(4,1) and b=(1,3), a+b=(5,4). From A=(1,2) to B=(5,5), displacement AB=(4,3). Its magnitude is $\sqrt{(4^2+3^2)}=5$.



Where it goes wrong

- The order of subtraction matters: BA=-AB.
- Proving parallelism needs a scalar-multiple relation; a sketch alone is insufficient.
- Negative enlargement reverses position about its centre.

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■ **Watch out**

Not true: AB and BA have the same components. BA goes the opposite way to AB, so BA=−AB: every component changes sign.

Applying it

- Use vector components, scalar multiples and magnitudes for displacements and forces.
- This unit does not require the scalar product.

In your course

- AQA A-level Mathematics (7357), specification section J.
- Every paper allows a calculator, but method marks still need the expression or equation written before its value.

■ **Key idea**

- **Resultant**: The vector sum representing the combined displacement or force.
- Add corresponding vector components and subtract position vectors to find a displacement.
- Watch for: the order of subtraction matters: BA=-AB.

J · Reflections, rotations and enlargements

Where is the enlargement centre?

- A logo is enlarged around a point away from the origin. Multiplying its coordinates alone puts it in the wrong place.
- This lesson studies **centre of enlargement** 位似中心: The point from which each point's displacement is multiplied by a scale factor.

The method

- A translation adds a vector.
- A reflection reverses signed perpendicular distance from a mirror line.
- A rotation needs a centre, angle and direction.
- For enlargement from C, use new P=C+k(P-C).

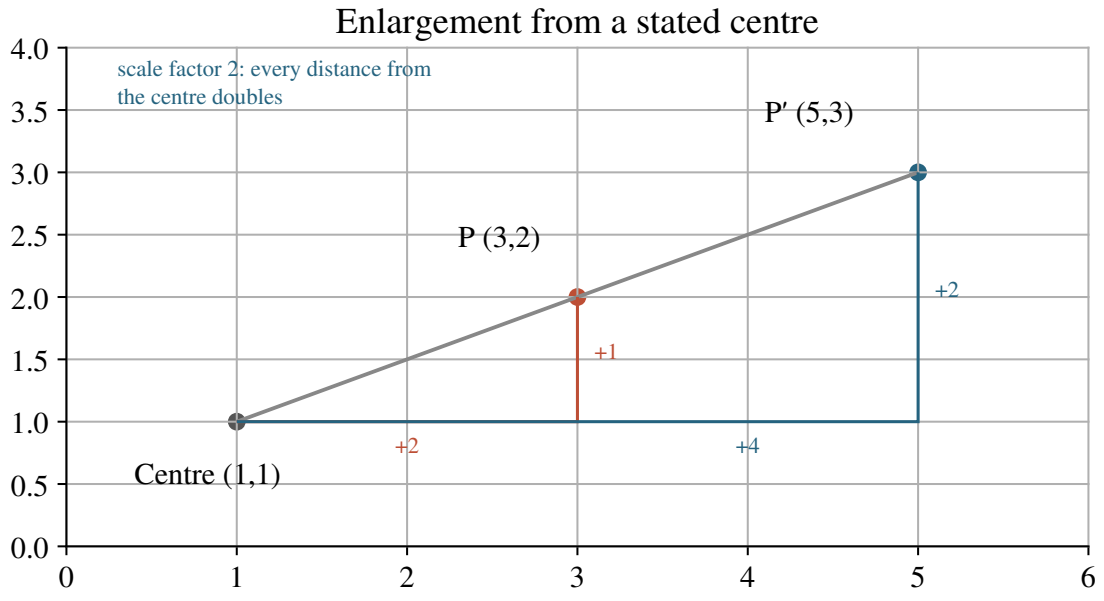
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A worked case

$$\mathbf{p}_{\text{image}} = \mathbf{c} + k(\mathbf{p} - \mathbf{c})$$

■ Worked example

For C=(1,1),P=(3,2),k=2, P-C=(2,1), so new P=(1,1)+2(2,1)=(5,3). Reflection of (3,2) in the y-axis gives (-3,2). A 90° anticlockwise rotation about the origin gives (-2,3).



Where it goes wrong

- A rotation needs its centre and direction, not just an angle.
- A negative enlargement factor places the image on the opposite side of the centre.
- A translation does not change orientation or size.

■ Watch out

Not true: Every enlargement is centred at the origin unless its scale factor is negative. An enlargement can have any centre; a negative scale factor puts the image on the other side of the centre.

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Applying it

- Describe a transformation completely before constructing the image.
- Check corresponding distances and angles.
- For combined transformations, apply them in the stated order; they usually do not commute.

In your course

- AQA A-level Mathematics (7357), specification section J.
- Every paper allows a calculator, but method marks still need the expression or equation written before its value.

■ Key idea

- **Centre of enlargement**: The point from which each point's displacement is multiplied by a scale factor.
- A translation adds a vector.
- Watch for: A rotation needs its centre and direction, not just an angle.

K · Sampling and a large data set

Who is missing from the data?

- A weather database has a missing entry and several stations. Treating each row as identical can distort a comparison.
- This lesson studies **sampling frame** 抽样框: The list or population definition from which a sample is selected.

The method

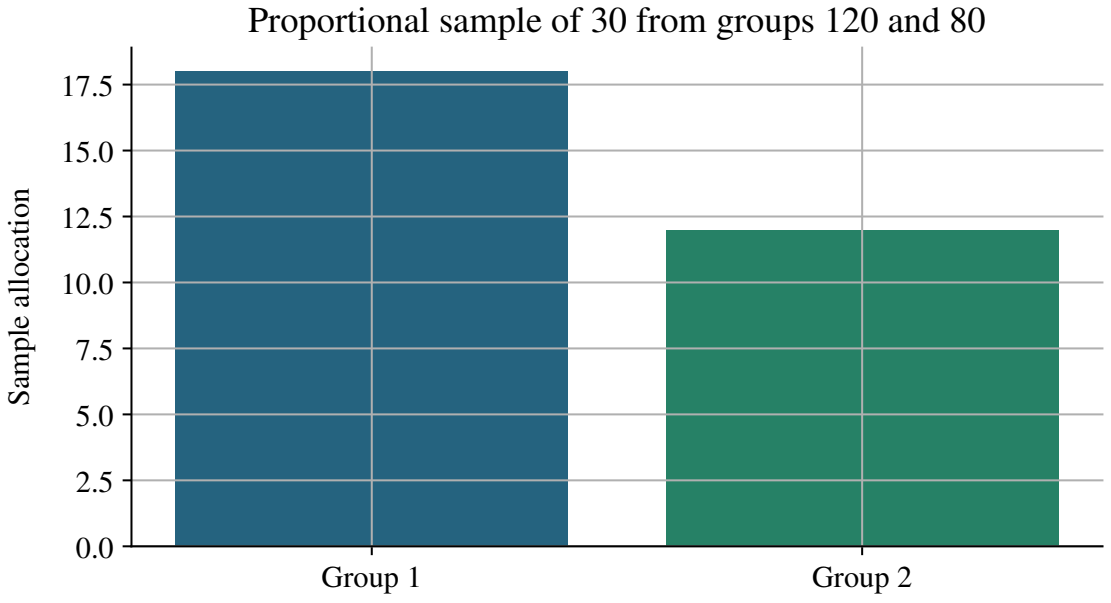
- Identify the population, sampling unit and frame.
- Distinguish random, systematic, stratified, quota and opportunity sampling.
- Missing data is not zero; verify units, dates and variable definitions before comparing samples.

A worked case

$$n_i = n\frac{N_i}{N}$$

Worked example

A population has 120 students in one group and 80 in another. A proportional stratified sample of 30 needs 30×120/200=18 from the first and 12 from the second. Random selection is then needed within each group.



Where it goes wrong

- A large biased sample remains biased.
- Stratification is not the same as selecting whoever is available from each group.
- AQA large-data-set familiarity requires the actual supplied data and metadata, not invented weather values.

Watch out

Not true: Doubling a biased sample removes its selection bias. A bigger sample collected the same biased way keeps the same bias; only better sampling removes it.

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Applying it

- For AQA, use the official large data set in a supervised spreadsheet task: identify a variable, justify a comparison, inspect missing values, create a display and explain a limitation.
- Save the decisions with the analysis.

In your course

- AQA A-level Mathematics (7357), specification section K.
- Every paper allows a calculator, but method marks still need the expression or equation written before its value.

Key idea

- **Sampling frame**: The list or population definition from which a sample is selected.
- Identify the population, sampling unit and frame.
- Watch for: A large biased sample remains biased.

L · Data summaries, histograms and interpretation

Can one average tell the whole story?

- Two groups have the same median but different spread. One summary cannot describe both location and consistency.
- This lesson studies **frequency density** 频率密度: Frequency divided by class width, used as histogram height.

The method

- Compare an appropriate average and spread in context.
- A histogram uses area for frequency, so height=frequency/class width.
- Grouped estimates assume representative values within intervals.

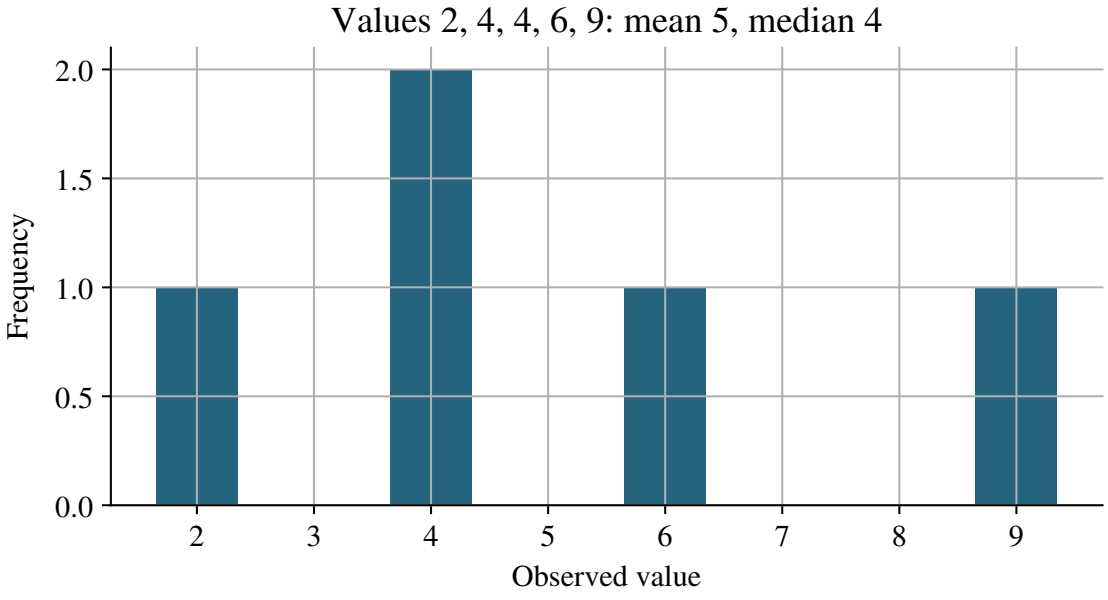
A worked case

$$\overline{x} = \frac{\sum x_i}{n}, \qquad \text{density} = \frac{\text{frequency}}{\text{class width}}$$

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■ Worked example

A class from 10 to 20 with frequency 30 has density 30/10=3. A class from 20 to 40 with frequency 20 has density 20/20=1. Its wider bar must not be mistaken for a larger density. For values 2,4,4,6,9, the median is 4 and mean is 5.



Where it goes wrong

- The tallest histogram bar need not contain the most observations.
- A grouped mean is an estimate.
- Correlation does not prove causation, and extrapolation extends beyond the observed range.

■ Watch out

Not true: A histogram bar's height equals its frequency, even with unequal class widths. With unequal class widths the height is frequency density; frequency is the area of the bar.

Applying it

- Choose a display that fits the data type.
- Give both a numerical comparison and what it means for the population; do not infer more precision than the sample supports.

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In your course

- AQA A-level Mathematics (7357), specification section L.
- Every paper allows a calculator, but method marks still need the expression or equation written before its value.

■ Key idea

- **Frequency density**: Frequency divided by class width, used as histogram height.
- Compare an appropriate average and spread in context.
- Watch for: the tallest histogram bar need not contain the most observations.

L · Regression, financial models and residuals

Is the prediction supported by the data?

- A line predicts bus travel time from distance. A small calculation error matters less than using a model outside its evidence.
- This lesson studies **residual** 残差: The observed value minus the value predicted by a fitted model.

The method

- A linear model y=a+bx has intercept a and slope b with contextual units.
- Inspect residuals and the data range.
- For repeated percentage change use a geometric model; for a loan distinguish principal, rate, repayment and period.

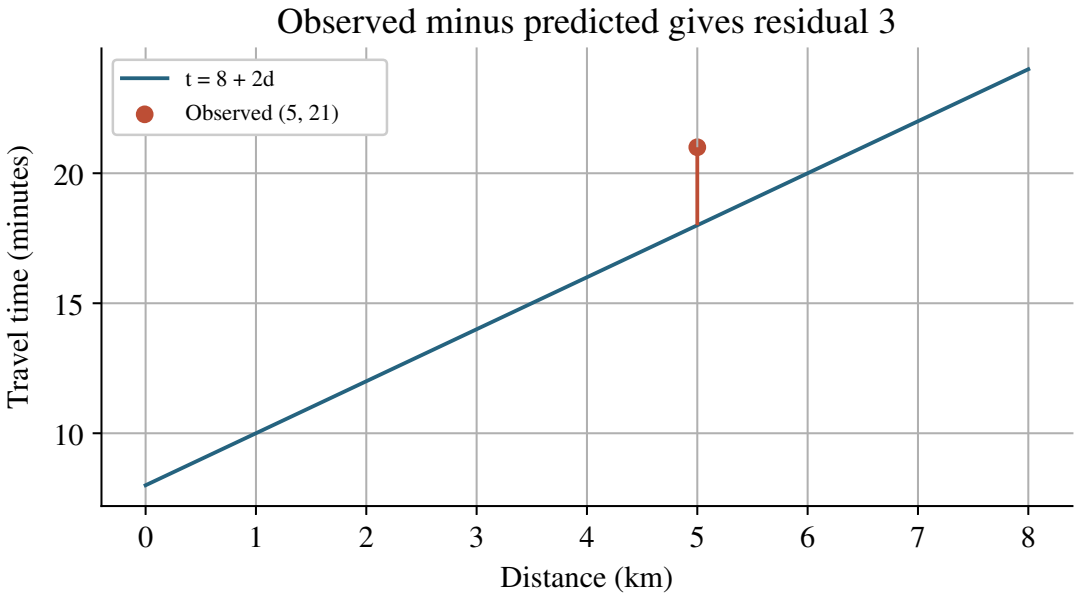
A worked case

$$\widehat{t} = 8 + 2d, \qquad e = t - \widehat{t}$$

■ Worked example

The fitted model t=8+2d predicts 18 minutes at d=5 km. If the observed time is 21, the residual is 3 minutes.

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Where it goes wrong

- A good fit does not prove a causal mechanism.
- A correlation coefficient measures linear association, not the gradient.
- Calculator output must be translated into a model, checked and interpreted.

> ■ **Watch out**
>
> **Not true:** A high correlation permits reliable extrapolation to any distance. A model fitted to data is reliable only near that data; extrapolating far outside it is unreliable.

Applying it

- Use the AQA large data set and suitable statistical displays.
- Interpret residuals, correlation and model limitations; association alone does not establish causation.

In your course

- AQA A-level Mathematics (7357), specification section L.
- Every paper allows a calculator, but method marks still need the expression or equation written before its value.

■ **Key idea**

- **Residual**: The observed value minus the value predicted by a fitted model.
- A linear model y=a+bx has intercept a and slope b with contextual units.
- Watch for: A good fit does not prove a causal mechanism.

M · Probability, trees and conditional reasoning

What changes after the first draw?

- A bag has 3 red and 2 blue counters. Taking two without replacement changes the chance of the second colour.
- This lesson studies **conditional probability** 条件概率: The probability of an event after restricting the sample space to a stated condition.

The method

- Multiply along a tree branch and add disjoint branches.
- With replacement, the composition stays fixed.
- Conditional probability is P(A given B)=P(A∩B)/P(B), for P(B)>0.

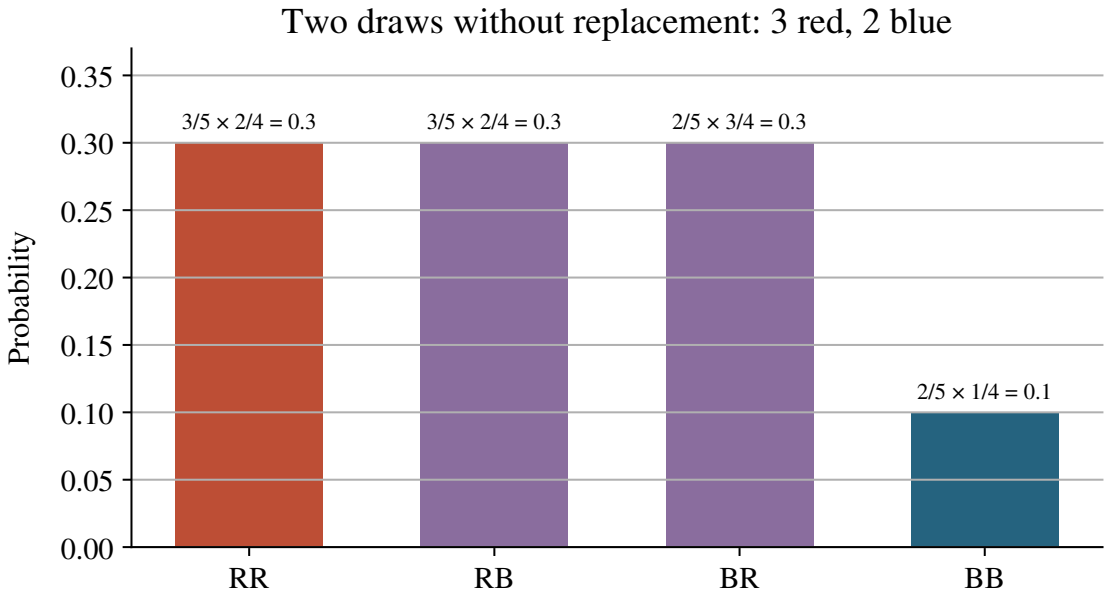
A worked case

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}, \qquad P(B) > 0$$

■ **Worked example**

Without replacement, P(two red)=3/5×2/4=3/10. P(one of each)=3/5×2/4+2/5×3/4=3/5. If P(A∩B)=0.12 and P(B)=0.3, P(A given B)=0.4.

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Where it goes wrong

- Mutually exclusive means no overlap; independent means that knowing one event does not change the other's probability.
- Two disjoint events with positive probability are not independent.

> ■ **Watch out**
>
> **Not true:** Mutually exclusive events with positive probabilities must be independent. Mutually exclusive events with positive probabilities are dependent: if one happens, the other cannot, so $P(A \text{ and } B)=0 \neq P(A)P(B)$.

Applying it

- A two-way table makes the restricted denominator visible.
- Before using a product $P(A)P(B)$, justify independence from the context or the supplied information.

In your course

- AQA A-level Mathematics (7357), specification section M.
- Every paper allows a calculator, but method marks still need the expression or equation written before its value.

■ **Key idea**

- **Conditional probability**: The probability of an event after restricting the sample space to a stated condition.
- Multiply along a tree branch and add disjoint branches.
- Watch for: mutually exclusive means no overlap; independent means that knowing one event does not change the other's probability.

N · Binomial, normal and Poisson models

What makes a count predictable?

- A quality inspector counts defective items. The number is random, but a model can describe its likely range.
- This lesson studies **expected value** 期望值: The probability-weighted mean of a random variable.

The method

- A binomial model needs fixed n, independent trials, two outcomes and constant p.
- E(X)=np and Var(X)=np(1-p).
- For a normal model use z=(x-μ)/σ.
- State the event before calculating a tail probability.

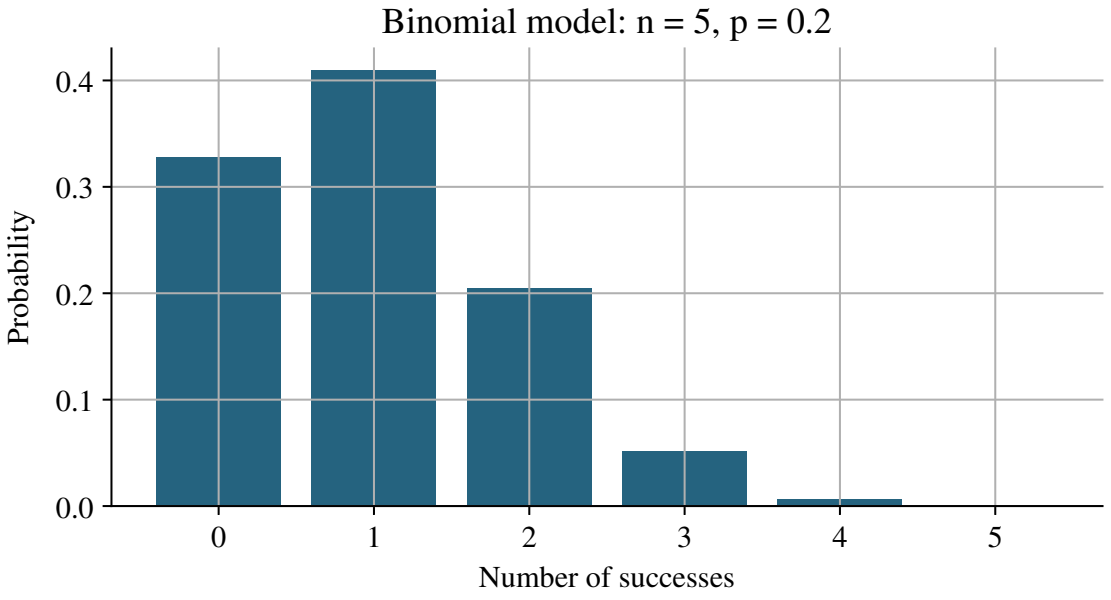
A worked case

$$P(X = k) = \binom{n}{k} p^k (1-p)^{n-k}, \qquad X \sim B(n, p)$$

■ **Worked example**

For X binomial(5,0.2), P(X=0)=0.8^5=0.32768, E(X)=1 and Var(X)=0.8. For a normal quantity with μ=100,σ=15, the value 130 has z=2. Poisson is not part of this AQA 7357 lesson.

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Where it goes wrong

- Not every count is binomial: changing p or dependence can invalidate it.
- For a continuous variable, the probability of one exact value is zero.
- Continuity correction matters when approximating a discrete distribution by a normal one.

■ Watch out

Not true: Every count of successes has a binomial distribution regardless of dependence. The binomial model needs independent trials with a fixed probability; dependent trials break it.

Applying it

- Write the event as an inequality before using calculator distribution functions.
- Distinguish $P(X<k)$, $P(X\leq k)$ and a tail complement.
- State assumptions in context.

In your course

- AQA A-level Mathematics (7357), specification section N.

■ Key idea

- **Expected value**: The probability-weighted mean of a random variable.
- A binomial model needs fixed n, independent trials, two outcomes and constant p.
- Watch for: not every count is binomial: changing p or dependence can invalidate it.

O · Hypothesis testing and contextual conclusions

Could chance explain the result?

- A factory claims that only 10% of items are defective. A sample contains more defects, but chance alone may explain some difference.
- This lesson studies **significance level** 显著性水平: The chosen probability threshold for rejecting a null hypothesis.

The method

- State H_0 and H_1 in population parameters before inspecting the outcome.
- Calculate the appropriate tail probability under H_0.
- Reject H_0 when the evidence meets the specified significance rule; otherwise say there is insufficient evidence.

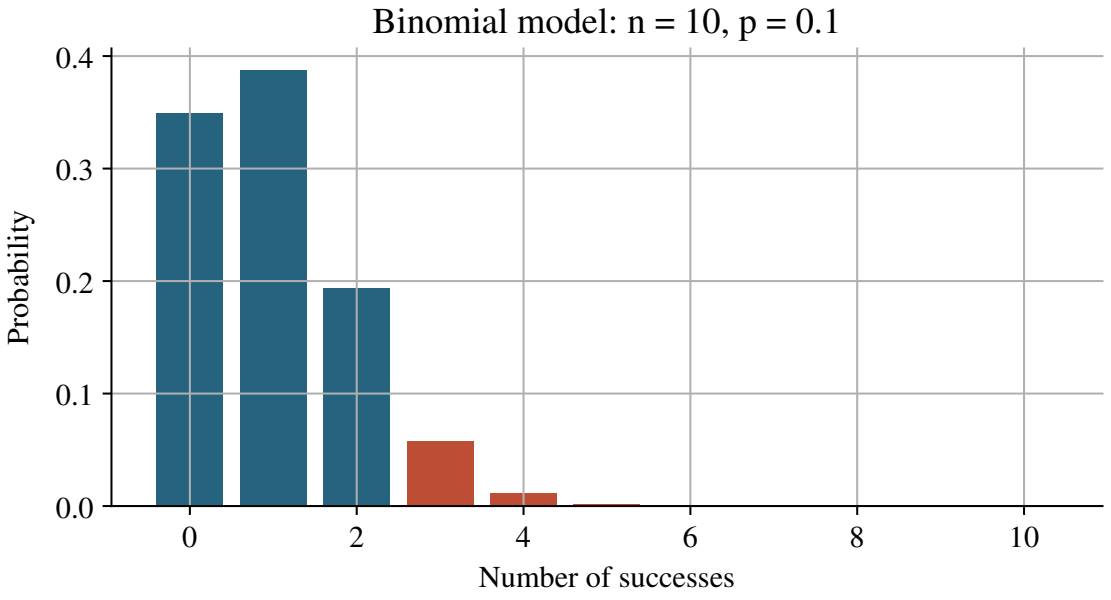
A worked case

$$H_0 : p = p_0, \qquad H_1 : p > p_0$$

■ Worked example

For H_0:p=0.1 and H_1:p>0.1 with n=10, observing at least 3 defects has probability 1-(0.9^10+10×0.1×0.9^9+45×0.1^2×0.9^8)≈0.070191. At 5%, this is insufficient evidence that the defect rate exceeds 10%.

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Where it goes wrong

- Failing to reject H_0 is not proof that H_0 is true.
- Choose the tail from H_1, not from whichever tail gives a small result.
- Statistical significance does not measure the practical size of an effect.

> ■ **Watch out**
>
> **Not true:** Failing to reject a null hypothesis proves it is true. Not rejecting the null hypothesis only means there is not enough evidence against it.

Applying it

- Finish with a sentence about the population and the original claim.
- State the model's assumptions and consider whether the sampling procedure supports them.

In your course

- AQA A-level Mathematics (7357), specification section O.
- Every paper allows a calculator, but method marks still need the expression or equation written before its value.

■ **Key idea**

- **Significance level**: The chosen probability threshold for rejecting a null hypothesis.
- State H_0 and H_1 in population parameters before inspecting the outcome.
- Watch for: failing to reject H_0 is not proof that H_0 is true.

P · Accuracy, bounds and compound measures

Is the printed measurement exact?

- A rectangular panel is labelled 8.0 cm by 5.0 cm, each to the nearest 0.1 cm. Its true area is not fixed at 40 cm^2.
- This lesson studies **lower bound** 下界: The smallest possible value consistent with a stated rounding rule.

The method

- A value rounded to the nearest unit u lies from stated value-u/2 up to, but usually not including, stated value+u/2.
- For positive quantities, combine extremes according to the operation.

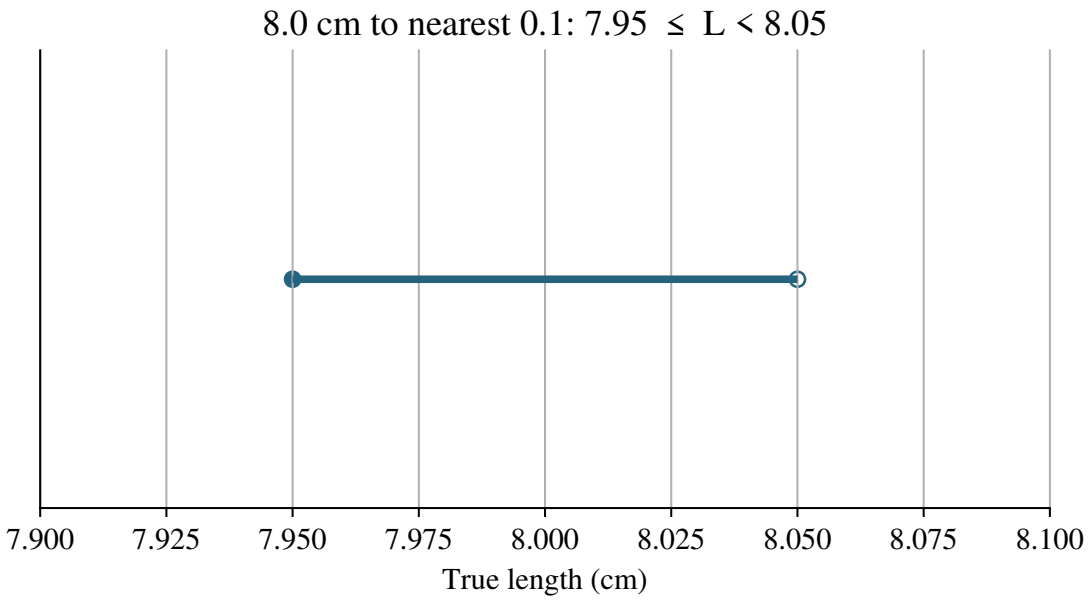
A worked case

$$A = LW, \qquad v = \frac{d}{t}$$

■ **Worked example**

The lengths satisfy $7.95 \leq L < 8.05$ and $4.95 \leq W < 5.05$. Since $A = LW$, $39.3525 \leq A < 40.6525$. For speed d/t, the largest speed uses the largest distance and smallest positive time.

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Where it goes wrong

- An upper bound is not automatically achieved.
- Dividing upper distance by upper time does not give the largest speed.
- Keep enough digits in intermediate calculations.

■ Watch out

Not true: The greatest value of positive d/t uses the greatest d and greatest t. To make d/t as large as possible, use the greatest d and the least t; a larger t makes the quotient smaller.

Applying it

- Distinguish measurement uncertainty from arithmetic rounding.
- A sensible reported precision cannot be finer than the measurements justify.

In your course

- AQA A-level Mathematics (7357), specification section P.
- Every paper allows a calculator, but method marks still need the expression or equation written before its value.

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■ **Key idea**

- **Lower bound**: The smallest possible value consistent with a stated rounding rule.
- A value rounded to the nearest unit u lies from stated value-u/2 up to, but usually not including, stated value+u/2.
- Watch for: an upper bound is not automatically achieved.

P · Kinematics and Newton's laws

Which force accelerates the trolley?

- A trolley accelerates while friction opposes its motion. The engine's force alone does not equal mass times acceleration.
- This lesson studies **resultant force** 合力: The vector sum of all external forces acting on the modelled object.

The method

- Choose a positive direction and draw a force diagram.
- Use resultant F=ma.
- For constant acceleration, v=u+at and s=ut+at^2/2.
- A particle model ignores size; a smooth surface ignores friction.

A worked case

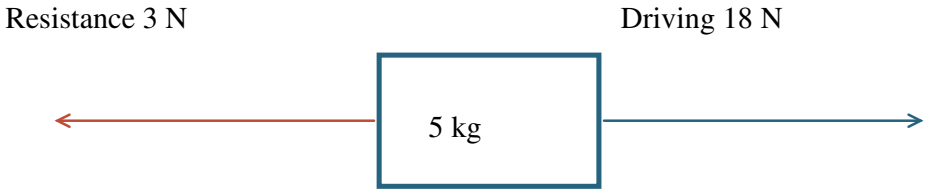
$$F_{\text{resultant}} = ma, \qquad s = ut + \frac{1}{2}at^2$$

■ **Worked example**

A 5 kg trolley has driving force 18 N and resistance 3 N. The resultant is 15 N, so a=15/5=3 m/s^2. Starting from rest for 4 s gives v=0+3×4=12 m/s and s=0×4+3×4^2/2=24 m.

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Identify the resultant before using F = ma



Where it goes wrong

- The normal reaction need not equal weight on a slope.
- Connected-particle models need one equation per particle and a consistent acceleration relation.
- The constant-acceleration formulae cannot be used for arbitrary variable acceleration.

■ Watch out

Not true: The constant-acceleration formulae apply unchanged when acceleration varies with time. The suvat formulae need constant acceleration; when acceleration varies, use calculus.

Applying it

- Explain what each idealisation permits and what it leaves out.
- For a velocity-time graph, gradient represents acceleration and signed area represents displacement.

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In your course

- AQA A-level Mathematics (7357), specification section P.
- Every paper allows a calculator, but method marks still need the expression or equation written before its value.

Key idea

- **Resultant force**: The vector sum of all external forces acting on the modelled object.
- Choose a positive direction and draw a force diagram.
- Watch for: the normal reaction need not equal weight on a slope.

Q · Kinematics and Newton's laws

Which force accelerates the trolley?

- A trolley accelerates while friction opposes its motion. The engine's force alone does not equal mass times acceleration.
- This lesson studies **resultant force** 合力: The vector sum of all external forces acting on the modelled object.

The method

- Choose a positive direction and draw a force diagram.
- Use resultant F=ma.
- For constant acceleration, v=u+at and s=ut+at^2/2.
- A particle model ignores size; a smooth surface ignores friction.

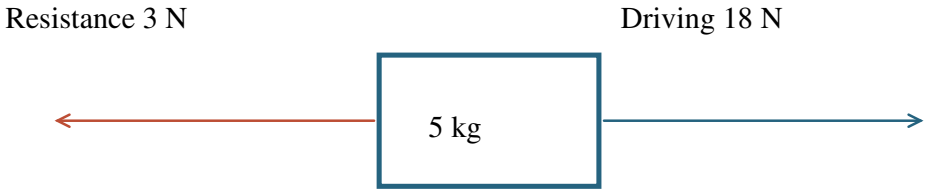
A worked case

$$F_{\text{resultant}} = ma, \qquad s = ut + \frac{1}{2}at^2$$

Worked example

A 5 kg trolley has driving force 18 N and resistance 3 N. The resultant is 15 N, so a=15/5=3 m/s^2. Starting from rest for 4 s gives v=0+3×4=12 m/s and s=0×4+3×4^2/2=24 m.

Identify the resultant before using F = ma



Where it goes wrong

- The normal reaction need not equal weight on a slope.
- Connected-particle models need one equation per particle and a consistent acceleration relation.
- The constant-acceleration formulae cannot be used for arbitrary variable acceleration.

■ Watch out

Not true: The constant-acceleration formulae apply unchanged when acceleration varies with time. The suvat formulae need constant acceleration; when acceleration varies, use calculus.

Applying it

- Explain what each idealisation permits and what it leaves out.
- For a velocity-time graph, gradient represents acceleration and signed area represents displacement.

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In your course

- AQA A-level Mathematics (7357), specification section Q.
- Every paper allows a calculator, but method marks still need the expression or equation written before its value.

■ Key idea

- **Resultant force**: The vector sum of all external forces acting on the modelled object.
- Choose a positive direction and draw a force diagram.
- Watch for: the normal reaction need not equal weight on a slope.

Q · Integrals, area and accumulation

How much change has accumulated?

- A velocity graph tells us motion at an instant. How can we recover the displacement accumulated over time?
- This lesson studies **antiderivative** 原函数: A function whose derivative equals the given integrand.

The method

- For n≠-1, the integral of ax^n is ax^(n+1)/(n+1)+C.
- A definite integral is signed accumulation.
- Split at sign changes when total area or distance is required.

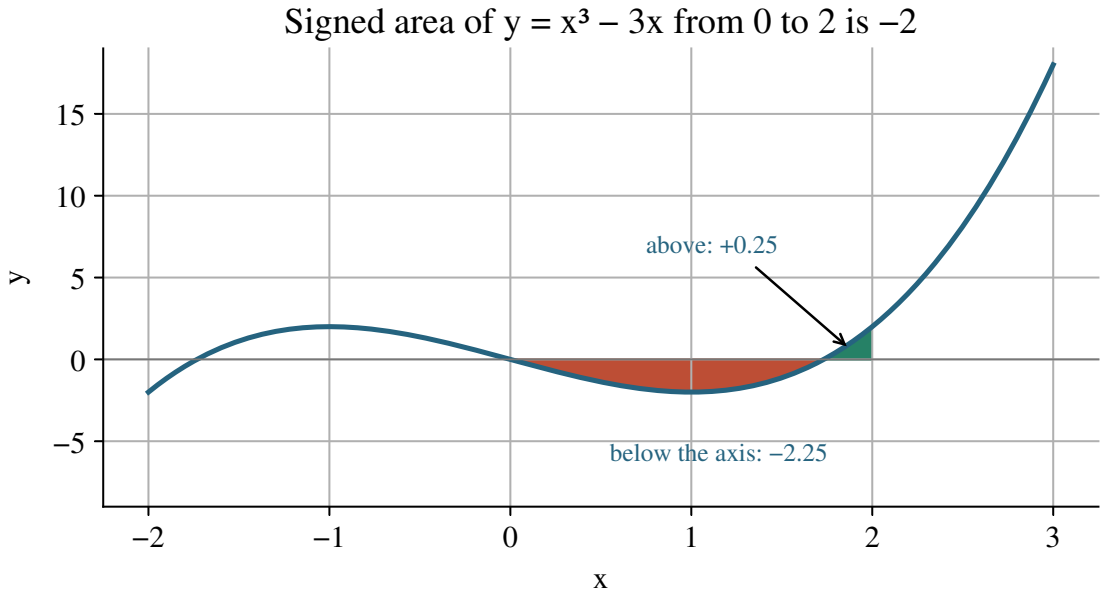
A worked case

$$\int ax^n \, dx = \frac{ax^{n+1}}{n+1} + C \quad (n \neq -1)$$

■ Worked example

For v(t)=3t^2-3 over 0≤t≤2, displacement=[t^3-3t]_0^2=2. Since v changes sign at t=1, distance=-[t^3-3t]_0^1+[t^3-3t]_1^2=2+4=6. The constants cancel only for a definite integral.

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Where it goes wrong

- The integral of 1/x is ln|x|+C, not the power rule with n=-1.
- Signed area can be zero even when the total enclosed area is positive.

> ■ **Watch out**
>
> **Not true:** A definite integral equals the total positive area, even when the graph crosses the axis. Area below the axis counts as negative, so a definite integral can be smaller than the total area, or even zero.

Applying it

- Identify what the integral means and include the correct units.
- A rate measured per second integrates to the underlying quantity, not to another rate.
- Differentiate an antiderivative to check it.

In your course

- AQA A-level Mathematics (7357), specification section Q.
- Every paper allows a calculator, but method marks still need the expression or equation written before its value.

■ **Key idea**

- **Antiderivative**: A function whose derivative equals the given integrand.
- For n≠-1, the integral of ax^n is ax^(n+1)/(n+1)+C.
- Watch for: the integral of 1/x is ln|x|+C, not the power rule with n=-1.

R · Kinematics and Newton's laws

Which force accelerates the trolley?

- A trolley accelerates while friction opposes its motion. The engine's force alone does not equal mass times acceleration.
- This lesson studies **resultant force** 合力: The vector sum of all external forces acting on the modelled object.

The method

- Choose a positive direction and draw a force diagram.
- Use resultant F=ma.
- For constant acceleration, v=u+at and s=ut+at^2/2.
- A particle model ignores size; a smooth surface ignores friction.

A worked case

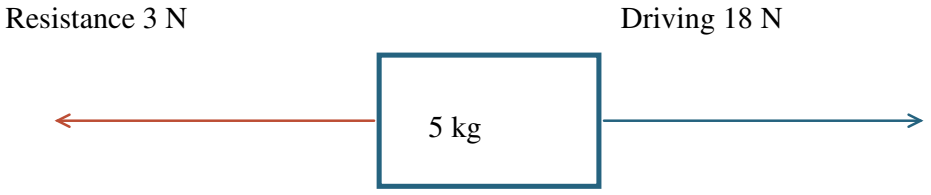
$$F_{\text{resultant}} = ma, \qquad s = ut + \frac{1}{2}at^2$$

■ **Worked example**

A 5 kg trolley has driving force 18 N and resistance 3 N. The resultant is 15 N, so a=15/5=3 m/s^2. Starting from rest for 4 s gives v=0+3×4=12 m/s and s=0×4+3×4^2/2=24 m.

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Identify the resultant before using F = ma



Where it goes wrong

- The normal reaction need not equal weight on a slope.
- Connected-particle models need one equation per particle and a consistent acceleration relation.
- The constant-acceleration formulae cannot be used for arbitrary variable acceleration.

■ Watch out

Not true: The constant-acceleration formulae apply unchanged when acceleration varies with time. The suvat formulae need constant acceleration; when acceleration varies, use calculus.

Applying it

- Explain what each idealisation permits and what it leaves out.
- For a velocity-time graph, gradient represents acceleration and signed area represents displacement.

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In your course

- AQA A-level Mathematics (7357), specification section R.
- Every paper allows a calculator, but method marks still need the expression or equation written before its value.

■ Key idea

- **Resultant force**: The vector sum of all external forces acting on the modelled object.
- Choose a positive direction and draw a force diagram.
- Watch for: the normal reaction need not equal weight on a slope.

S · Moments, equilibrium and centres of mass

Why does a longer spanner help?

- A long spanner turns a nut more easily. The turning effect depends on perpendicular distance as well as force.
- This lesson studies **moment** 力矩: Force multiplied by the perpendicular distance from its line of action to a pivot.

The method

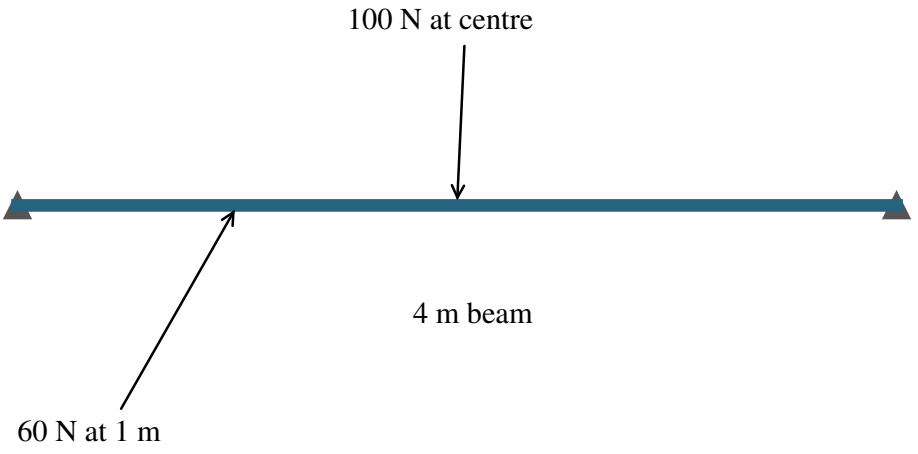
- For equilibrium, both the resultant force and resultant moment are zero.
- Take moments about a useful pivot to remove unknown forces through that point.
- A combined centre of mass is a mass-weighted mean position.

A worked case

$$M = Fd_{\perp}, \qquad \sum M = 0, \qquad \sum \mathbf{F} = \mathbf{0}$$

■ Worked example

A uniform 4 m beam weighs 100 N and is supported at both ends. An extra 60 N acts 1 m from the left. Moments about the left give 4R_right=100×2+60×1=260, so R_right=65 N. Vertical equilibrium gives R_left=95 N.



Where it goes wrong

- Use the perpendicular lever arm, not a diagonal distance to the point of application.
- A zero resultant force alone does not prevent rotation.
- A support that would need a negative normal reaction may lose contact.

> ■ **Watch out**
>
> **Not true:** Zero resultant force by itself is enough for a rigid body's equilibrium. Equilibrium also needs the resultant moment to be zero; two equal opposite forces on different lines still turn the body.

Applying it

- For a lamina made from parts, choose a common origin and tabulate each signed mass-area contribution.
- A removed region has negative mass in that calculation, not a new physical negative material.

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In your course

- AQA A-level Mathematics (7357), specification section S.
- Every paper allows a calculator, but method marks still need the expression or equation written before its value.

■ Key idea

- **Moment**: Force multiplied by the perpendicular distance from its line of action to a pivot.
- For equilibrium, both the resultant force and resultant moment are zero.
- Watch for: use the perpendicular lever arm, not a diagonal distance to the point of application.

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