

Probability and Random Variables

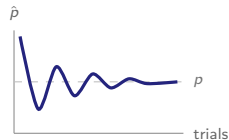
AP Statistics ■ Unit 4

AP Statistics



What we will cover

- 1 Randomness and probability
- 2 Probability rules
- 3 Conditional and independence
- 4 Random variables
- 5 Combining variables
- 6 Binomial and geometric





Randomness and probability

Probability is the long-run **relative frequency** of an outcome.

- short runs vary by **chance**
- **simulation** estimates a hard probability

Law of large numbers :
more trials $\Rightarrow \hat{p}$ settles toward the true p .



Playing cards



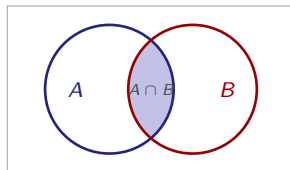
Probability rules

From the **sample space**, every **event** obeys $0 \leq P \leq 1$, summing to 1.

$$P(A^c) = 1 - P(A)$$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Mutually exclusive events share nothing, so $P(A \cap B) = 0$ and the last term drops.





Conditional probability and independence

A **conditional probability** restricts to the times B happens:

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}$$

The **multiplication rule** :

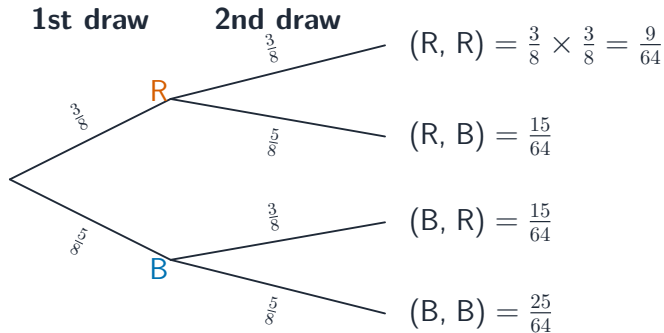
$P(A \cap B) = P(A) P(B \mid A)$. The chance of A **once you know** B happened. Two-way tables make these easy: restrict to B 's row or column, then find A 's share.

	girl	boy
sport	24	36
none	16	24

$$P(\text{girl} \mid \text{sport}) = \frac{24}{60}$$

Independent : $P(A \mid B) = P(A)$, so $P(A \cap B) = P(A)P(B)$. Not the same as mutually exclusive!

On a tree diagram



On a tree diagram, multiply the probabilities along the branches



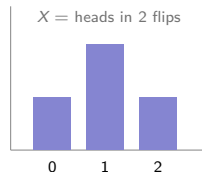
Random variables

A **random variable** X takes a number from a chance process (**discrete** or **continuous**).

- its **distribution** lists each x with $P(x)$, summing to 1

$$\mu_X = \sum x_i P(x_i) \quad \sigma_X^2 = \sum (x_i - \mu_X)^2 P(x_i)$$

μ_X is the **expected value** – the long-run average outcome.



Randomness, conditional probability, and expected value

random

individual outcomes are **uncertain**, but a regular pattern emerges over **many** repetitions

conditional probability

$$P(A \mid B) = \frac{P(A \text{ and } B)}{P(B)}$$

what it means

the chance of A **among the cases where B happened** – B becomes the new denominator

independence

A and B are independent when $P(A \mid B) = P(A)$

a probability distribution

lists every value of a random variable with its probability; they sum to 1

the mean (expected value)

$\mu_X = \sum x_i P(x_i)$ – the long-run average, not a value you expect to see

An expected value need not be attainable: the expected number of heads in three tosses is 1.5. It is an average over many repetitions.

Estimating Probabilities Using Simulation

A **simulation** imitates a chance process using random digits or technology.

Steps: state the model, assign digits to outcomes, run many trials, and record the proportion of trials meeting the condition.



Combining random variables

A **linear transformation** :

$$\mu_{aX+b} = a\mu_X + b \quad \sigma_{aX+b} = |a| \sigma_X$$

For a **sum or difference** of **independent** variables, means combine and **variances add** – for *both*:

$$\mu_{X \pm Y} = \mu_X \pm \mu_Y \quad \sigma_{X \pm Y}^2 = \sigma_X^2 + \sigma_Y^2$$

Never subtract variances for a difference
– variances always **add**.

Two packages

$\sigma_X = 0.3$, $\sigma_Y = 0.4$, independent.

$$\sigma_{X+Y} = \sqrt{0.3^2 + 0.4^2} = 0.5$$

The difference $X - Y$ has the *same* $\sigma = 0.5$.



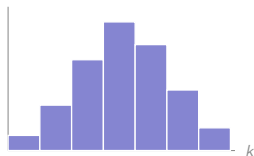
The binomial distribution

The **binomial setting** counts successes – remember **BINS**: **B**inary, **I**ndependent, fixed **N**umber, **S**ame p .

$$P(X = k) = \binom{n}{k} p^k (1 - p)^{n-k}$$

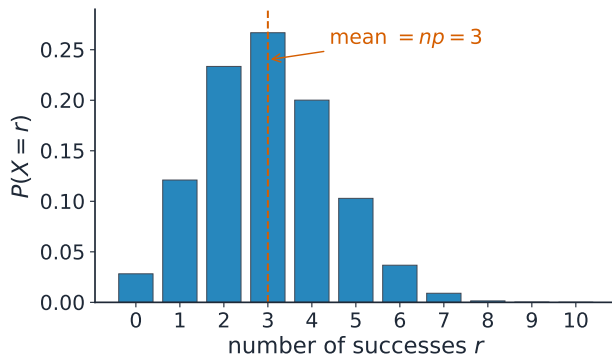
$$\mu_X = np \quad \sigma_X = \sqrt{np(1 - p)}$$

Shape: symmetric at $p = 0.5$, skewed otherwise; more bell-shaped as n grows.



Binomial: count of successes.

Introduction to the Binomial Distribution



The binomial distribution, with mean n times p

Worked example – binomial by hand

A free-throw shooter makes 80%. In $n = 10$ attempts, find $P(X = 8)$ and describe the distribution's centre and spread.

$$P(X = 8) = \binom{10}{8} (0.8)^8 (0.2)^2$$

$$= 45(0.1678)(0.04) \approx 0.302$$

$$\mu = np = 10(0.8) = 8 \text{ makes}$$

$$\sigma = \sqrt{10(0.8)(0.2)} \approx 1.26$$

$\binom{10}{8} = 45$ counts **which** 8 shots dropped – forget it and you find one single ordering (0.0067), not the event. Check **BINS** first: without independence it is not binomial at all.

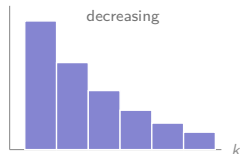


The geometric distribution

The **geometric setting** keeps going *until* the first success (no fixed n).

$$P(X = k) = (1 - p)^{k-1} p \quad \mu_X = \frac{1}{p}$$

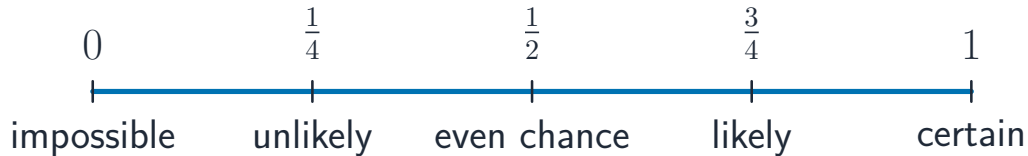
X counts **trials until** the first success; a binomial counts successes in a **fixed** number of trials.



Roll until a six:

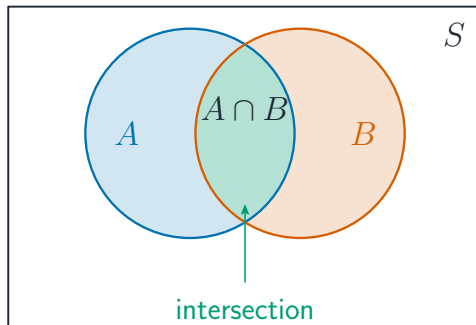
$$\mu = 6.$$

Introduction to Probability



Probability runs from 0 (impossible) to 1 (certain)

Mutually Exclusive Events



A Venn diagram: the overlap is the intersection of two events

Independent Events and Unions of Events

		die 2					
+		1	2	3	4	5	6
die 1	1	2	3	4	5	6	7
	2	3	4	5	6	7	8
	3	4	5	6	7	8	9
	4	5	6	7	8	9	10
	5	6	7	8	9	10	11
	6	7	8	9	10	11	12

the six highlighted cells all give a total of 7

A sample space diagram lists every equally likely outcome

Exam tips

- A probability lies in $[0, 1]$; use the **complement** $(1 - P)$ and add mutually exclusive events.
- For independent events multiply; for “and/or” use the general addition and conditional rules.
- **Expected value** $= \sum(\text{value} \times \text{probability})$.
- Recognise **binomial** (fixed n , two outcomes, constant p) and **geometric** settings.
- Draw a tree or table for multi-stage problems and multiply along branches.



Unit 4 – key takeaways

1

Probability

long-run relative frequency; law of large numbers

2

Rules

addition for or; multiplication for and; conditional

3

Random vars

$\mu = \sum xP(x)$; variances add for independent

4

Distributions

binomial $\mu = np$; geometric $\mu = 1/p$

Check yourself

- 1 Write the complement rule for $P(A)$.
- 2 For independent X, Y , how do you combine their variances?
- 3 What does BINS stand for?
- 4 A binomial has $n = 20$, $p = 0.5$: what is μ_X ?



Answers 1. $P(A^c) = 1 - P(A)$ 2. add them: $\sigma_X^2 + \sigma_Y^2$ 3. Binary, Independent, Number, Same p 4. $np = 10$