Question 3: Modeling a Periodic Context
Part B: Graphing calculator not allowed — 6 points

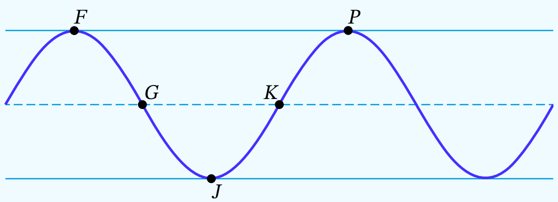
For a guitar to make a sound, the strings need to vibrate, or move up and down or back and forth, in a motion that can be modeled by a periodic function.

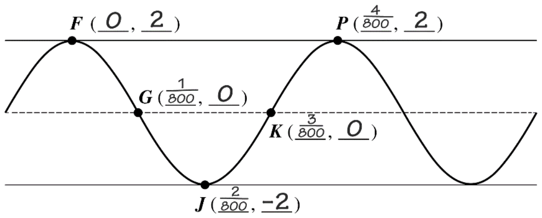
At time $t = 0$ seconds, point X on one vibrating guitar string starts at its highest position, 2 millimeters above its resting position. Then it passes through its resting position and moves to its lowest position, 2 millimeters below the resting position. Point X then passes through its resting position and returns to 2 millimeters above the resting position. This motion occurs 200 times in 1 second.

The sinusoidal function h models how far point X is from its resting position, in millimeters, as a function of time t, in seconds. A positive value of $h(t)$ indicates the point is above the resting position; a negative value of $h(t)$ indicates the point is below the resting position.

	Model Solution	Scoring

A The graph of h and its dashed midline for two full cycles is shown. Five points, F, G, J, K, and P, are labeled on the graph. No scale is indicated, and no axes are presented.
Determine possible coordinates $(t, h(t))$ for the five points: F, G, J, K, and P.





Note: t-coordinates will vary. A correct set of coordinates for one full cycle of h as pictured is acceptable.

$h(t)$-coordinates	**Point A1**
t-coordinates	**Point A2**

Scoring Notes for Part A

- No supporting work is required.
- $h(t)$-coordinates and/or t-coordinates may appear in a list.
- Negative t-coordinates are acceptable. Fractions do not need to be reduced; equivalent fractions and exact decimal values are acceptable.
- t-coordinates must be $0 + \frac{4}{800}k$, $\frac{1}{800} + \frac{4}{800}k$, $\frac{2}{800} + \frac{4}{800}k$, $\frac{3}{800} + \frac{4}{800}k$, $\frac{4}{800} + \frac{4}{800}k$ for a specific integer k.
- If the graph is used to record coordinates, that work is scored. In this case, other work is considered scratchwork and is not scored. Use of the graph is not required.

Partial Credit for Part A

A response that **does not** earn either **Point A1** or **Point A2** is eligible for **partial credit** in part A if the response meets one of the following criteria:

- All 5 points are in the form $(h(t), t)$ with correct input values and correct output values swapped.
- 3 out of the 5 points are correct.
- All 5 points $(t, h(t))$ meet these requirements:
  - t-coordinates are in arithmetic sequence with $\Delta t = \frac{1}{800}$.
  - $h(t)$-coordinates are such that
    1. F and P have **same** $h(t)$-coordinate.
    2. G and K have **same** $h(t)$-coordinate, which is **less than** $h(t)$-coordinate of F and P.
    3. Difference in $h(t)$-coordinates for F and G **equals**
       Difference in $h(t)$-coordinates for G and J.

Partial credit response is scored **0** for **Point A1** and **1** for **Point A2**.

B The function h can be written in the form $h(t) = a\sin(b(t + c)) + d$. Find values of constants a, b, c, and d.

$\frac{2\pi}{b} = \frac{1}{200}$, so $b = 400\pi$

$d = 0$

Method 1:

Using $a = 2$,

$c = \frac{1}{800}$ OR $c = \frac{1}{800} + \frac{4}{800}k$, for any integer k

$a =$ 2
$b =$ 400π
$c =$ $\frac{1}{800}$
$d =$ 0

For example, $h(t) = 2\sin\left(400\pi\left(t + \frac{1}{800}\right)\right)$. Based on horizontal shifts, there are other correct forms for $h(t)$.

Vertical transformations: Values of a and d	**Point B1**
Horizontal transformations: Values of b and c	**Point B2**

Method 2:

Using $a = -2$,

$c = -\frac{1}{800}$ OR $c = -\frac{1}{800} + \frac{4}{800}k$, for any integer k

$a =$	-2
$b =$	400π
$c =$	$-\frac{1}{800}$
$d =$	0

For example, $h(t) = -2\sin\left(400\pi\left(t - \frac{1}{800}\right)\right)$. Based on horizontal shifts, there are other correct forms for $h(t)$.

Scoring Notes for Part B

- No supporting work is required.
- Points are earned for correct values in a list OR for correct values in an expression for $h(t)$. Only one of these answer presentations is required.
- Fractions do not need to be reduced; equivalent fractions and exact decimal values are acceptable.
- If the answer box is used to record values, that work is scored. In this case, other work is considered scratchwork and is not scored. Use of the answer box is not required.
- **Point B1** and **Point B2** may be earned based on the correct use of an imported response from part A that meets these criteria:
  - $a \neq 1$, $b \neq 1$, and $c \neq 0$.
  - All 5 points $(t, h(t))$ from part A meet these requirements:
    - t-coordinates are in arithmetic sequence with $\Delta t = \frac{1}{800}$.
    - $h(t)$-coordinates are such that
      1. F and P have **same** $h(t)$-coordinate.
      2. G and K have **same** $h(t)$-coordinate, which is **less than** $h(t)$-coordinate of F and P.
      3. Difference in $h(t)$-coordinates for F and G **equals** Difference in $h(t)$-coordinates for G and J.

Partial Credit for Part B

A response that **does not** earn either **Point B1** or **Point B2** is eligible for **partial credit** in part B if the response meets one of the following criteria:

- Correct values of a and b [Values of a and b could be $\pm$]
- Correct values of b and d [Value of b could be $\pm$]
- Response uses $h(t) = a\mathbf{cos}(b(t + c)) + d$ with values as follows:
  - $a = 2$; $b = 400\pi$; $c = 0 + \frac{4}{800}k$, for a specific integer k; $d = 0$
  - $a = -2$; $b = 400\pi$; $c = -\frac{2}{800} + \frac{4}{800}k$, for a specific integer k; $d = 0$

Partial credit response is scored **1** for **Point B1** and **0** for **Point B2**.

C Refer to the graph of h in part A. The t-coordinate of G is t_1, and the t-coordinate of J is t_2.

(i) On the interval (t_1, t_2), which of the following is true about h?

a. h is positive and increasing.

b. h is positive and decreasing.

c. h is negative and increasing.

d. h is negative and decreasing.

(ii) On the interval (t_1, t_2), describe the concavity of the graph of h and determine whether the rate of change of h is increasing or decreasing.

(i) Choice d.	Function behavior	**Point C1**
(ii) The graph of h is concave up on the interval (t_1, t_2), and the rate of change of h is increasing on the interval (t_1, t_2).	Concavity of graph and behavior of rate of change	**Point C2**

Scoring Notes for Part C

- No supporting work is required.
- **Point C1** is earned only for a correct answer of "d" OR "negative and decreasing." If both the letter choice and written description are included, the written description is scored.
- To earn **Point C2**, both descriptions must be correct. **Point C2** is not earned for a response that only includes "the graph of h is concave up" OR only includes "the rate of change of h is increasing."
- To earn **Point C2**, "concave up" AND "increasing" is acceptable.
- To earn **Point C2**, "concave up" AND "function h is decreasing at an increasing rate" is acceptable.
- A response with an isolated statement "decreasing at an increasing rate" does not earn **Point C2**. The implied subject is "the rate of change of h."
- A response with a statement that "the rate of change of h is increasing at an increasing (or decreasing) rate" does not earn **Point C2**. Analysis to make such a conclusion requires calculus.
- **Point C2** cannot be earned if there are any errors in part C (ii).

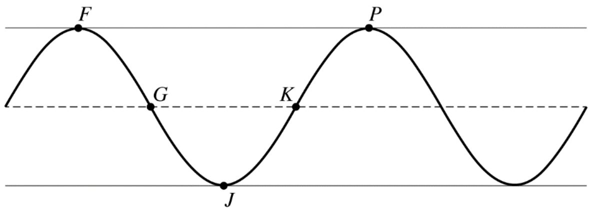
Question 3: Modeling a Periodic Context
Part B: Graphing calculator not allowed — 6 points

Note: Figure not drawn to scale.

The tire of a car has a radius of 9 inches, and a person rolls the tire forward at a constant rate on level ground, as shown in the figure. Point W on the edge of the tire touches the ground at time $t = \frac{1}{2}$ second. The tire completes a full rotation, and the next time W touches the ground is at time $t = \frac{5}{2}$ seconds. The maximum height of W above the ground is 18 inches. As the tire rolls, the height of W above the ground periodically increases and decreases.

The sinusoidal function h models the height of point W above the ground, in inches, as a function of time t, in seconds.

	Model Solution	Scoring	
(A)	The graph of h and its dashed midline for two full cycles is shown. Five points, F, G, J, K, and P, are labeled on the graph. No scale is indicated, and no axes are presented. Determine possible coordinates $(t, h(t))$ for the five points: F, G, J, K, and P.		

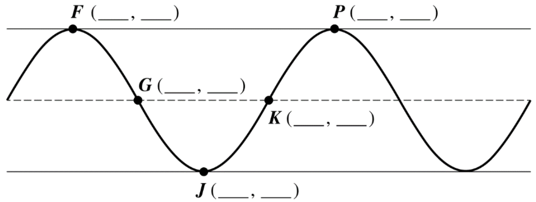


Model Solution	Scoring	
F has coordinates $\left(\frac{3}{2}, 18\right)$. G has coordinates $(2, 9)$. J has coordinates $\left(\frac{5}{2}, 0\right)$. K has coordinates $(3, 9)$. P has coordinates $\left(\frac{7}{2}, 18\right)$. OR	$h(t)$-coordinates	**1 point**
	t-coordinates	**1 point**

Model Solution	Scoring
F has coordinates $\left(-\frac{1}{2}, 18\right)$. G has coordinates $(0, 9)$. J has coordinates $\left(\frac{1}{2}, 0\right)$. K has coordinates $(1, 9)$. P has coordinates $\left(\frac{3}{2}, 18\right)$. Note: t-coordinates will vary. A correct set of coordinates for one full cycle of h as pictured is acceptable.	

Scoring notes:

- No supporting work is required.
- $h(t)$-coordinates and/or t-coordinates may appear in a list. Negative t-coordinates are acceptable.
- t-coordinates must be $-\frac{1}{2} + 2k, 0 + 2k, \frac{1}{2} + 2k, 1 + 2k, \frac{3}{2} + 2k$, for a specific integer k.
- If the graph is used to record coordinates, that work is scored. In this case, other work is considered scratchwork and is not scored. Use of the graph is not required.



- A response that does not earn either point in Part (A) is eligible for **partial credit** in Part (A) if the response meets one of the following criteria. Partial credit response is scored **0-1** in Part (A).
  - All 5 points in the form $(h(t), t)$ with correct input values and correct output values swapped
  - 3 correct points out of the 5 points
  - All 5 points $(t, h(t))$ meet these requirements:
    - t-coordinates in arithmetic sequence with $\Delta t = \frac{1}{2}$
    - $h(t)$-coordinates are such that
      (1) F and P have same $h(t)$-coordinate
      (2) G and K have same $h(t)$-coordinate, which is less than $h(t)$-coordinate of F and P
      (3) Difference in $h(t)$-coordinates for F and G equals difference in $h(t)$-coordinates for G and J

(B) The function h can be written in the form $h(t) = a\sin(b(t + c)) + d$. Find values of constants a, b, c, and d.

$h(t) = a\sin(b(t+c)) + d$ $a = 9$ $\frac{2\pi}{b} = 2$, so $b = \frac{2\pi}{2} = \pi$	Vertical transformations: Values of a and d	**1 point**
$c = -1$ $d = 9$ $h(t) = 9\sin(\pi(t-1)) + 9$ OR $a = -9$ $\frac{2\pi}{b} = 2$, so $b = \frac{2\pi}{2} = \pi$ $c = 0$ $d = 9$ $h(t) = -9\sin(\pi t) + 9$ Note: Based on horizontal shifts and reflections, there are other correct forms for $h(t)$.	Horizontal transformations: Values of b and c	**1 point**

Scoring notes:

- No supporting work is required.
- Points are earned for correct values in a list OR for correct values in an expression for $h(t)$. Only one of these answer presentations is required.
- If the answer box is used to record values, that work is scored. In this case, other work is considered scratchwork and is not scored. Use of the answer box is not required.

$a =$ ____________

$b =$ ____________

$c =$ ____________

$d =$ ____________

- Other correct values of c:
  - $h(t) = 9\sin(\pi(t+c)) + 9 \Rightarrow c = -1 + 2k$, for any integer k
  - $h(t) = -9\sin(\pi(t+c)) + 9 \Rightarrow c = 0 + 2k$, for any integer k
- Full credit for Part (B) is possible based on the correct use of an imported response from Part (A) that meets these criteria:
  - $a \neq 1$, $b \neq 1$, $d \neq 0$, and if $a > 0$, then $c \neq 0$
  - All 5 points $(t, h(t))$ meet these requirements:
    - t-coordinates in arithmetic sequence with $\Delta t = \frac{1}{2}$
    - $h(t)$-coordinates are such that

      (1) F and P have same $h(t)$-coordinate

      (2) G and K have same $h(t)$-coordinate, which is less than $h(t)$-coordinate of F and P

      (3) Difference in $h(t)$-coordinates for F and G equals difference in $h(t)$-coordinates for

- A response that does not earn either point in Part (B) is eligible for **partial credit** in Part (B) if the response meets one of the following criteria. Partial credit response is scored **1-0** in Part (B).
  - Values of a and b [Values of a and b could be $\pm$]
  - Values of b and d [Value of b could be $\pm$]
  - Response uses $h(t) = a\cos(b(t+c)) + d$ with values as follows:
    - $a = 9$; $b = \pi$; $c = -\frac{3}{2} + 2k$, for a specific integer k; $d = 9$
    - $a = -9$; $b = \pi$; $c = -\frac{1}{2} + 2k$, for a specific integer k; $d = 9$

(C) Refer to the graph of h in part (A). The t-coordinate of K is t_1, and the t-coordinate of P is t_2.

(i) On the interval (t_1, t_2), which of the following is true about h?

a. h is positive and increasing.
b. h is positive and decreasing.
c. h is negative and increasing.
d. h is negative and decreasing.

(ii) Describe how the rate of change of h is changing on the interval (t_1, t_2).

(i) Choice a.	Function behavior	**1 point**
(ii) Because the graph of h is concave down on the interval (t_1, t_2), the rate of change of h is decreasing on the interval (t_1, t_2).	Change in rate of change	**1 point**

Scoring notes:

- No supporting work is required.
- The first point is earned for a correct answer of "a" OR "positive and increasing." If both the letter choice and written description are included, the written description is scored.
- To earn the second point, "decreasing" OR "function h is increasing at a decreasing rate" is acceptable. If concavity of the graph of h is referenced, it must be correct.
- The second point is not earned for a response that only includes "the graph of h is concave down."
- A response with a statement that the rate of change of h is decreasing at an increasing (or decreasing) rate does not earn the second point. Analysis to make such a conclusion requires calculus.
- The second point is not earned for a response that states "increasing at a decreasing rate" without a subject. The implied subject is "the rate of change of h."
- The second point cannot be earned if there are any errors in Part (C) (ii).

Total for question 3 **6 points**

Question 4: Symbolic Manipulations
Part B: Graphing calculator not allowed — 6 points

Directions:

- Unless otherwise specified, the domain of a function f is assumed to be the set of all real numbers x for which $f(x)$ is a real number. Angle measures for trigonometric functions are assumed to be in radians.
- Solutions to equations must be real numbers. Determine the exact value of any expression that can be obtained without a calculator. For example, $\log_2 8$, $\cos\left(\frac{\pi}{2}\right)$, and $\sin^{-1}(1)$ can be evaluated without a calculator.
- Unless otherwise specified, combine terms using algebraic methods and rules for exponents and logarithms, where applicable. For example, $2x + 3x$, $5^2 \cdot 5^3$, $\frac{x^5}{x^2}$, and $\ln 3 + \ln 5$ should be rewritten in equivalent forms.
- For each part of the question, show the work that leads to your answers.

Model Solution	Scoring	
(A) The functions g and h are given by $g(x) = e^{(x+3)}$, $h(x) = \arcsin\left(\frac{x}{2}\right)$. (i) Solve $g(x) = 10$ for values of x in the domain of g. (ii) Solve $h(x) = \frac{\pi}{4}$ for values of x in the domain of h.		
(i) $g(x) = 10$ $e^{(x+3)} = 10$ $\ln e^{(x+3)} = \ln 10$ $x + 3 = \ln 10$ $x = -3 + \ln 10$	Solution to $g(x) = 10$	**1 point**
(ii) $h(x) = \frac{\pi}{4}$ $\arcsin\left(\frac{x}{2}\right) = \frac{\pi}{4}$ $\frac{x}{2} = \sin\left(\frac{\pi}{4}\right)$ $\frac{x}{2} = \frac{\sqrt{2}}{2}$ (OR $\frac{1}{\sqrt{2}}$) $x = \sqrt{2}$	Solution to $h(x) = \frac{\pi}{4}$	**1 point**

Scoring notes:

- Supporting work is required in (i) and (ii). “Scratchwork” can be ignored; the use of a variable other than x is acceptable. Arithmetic errors following a correct solution may be considered scratchwork.
- Supporting work in (ii) must include $\frac{x}{2} = \sin\left(\frac{\pi}{4}\right)$ OR $\frac{x}{2} = \frac{\sqrt{2}}{2}$ OR $\frac{x}{2} = \frac{1}{\sqrt{2}}$.
- An alternate solution for (i) is $x = \frac{\log_b 10}{\log_b e} - 3$, where $b > 0$, $b \neq 1$, and the result is evaluated according to bullets 2 and 3 in the directions.
- Where applicable, answers that have not been evaluated according to bullets 2 and 3 in the directions do not earn the point. Rationalizing denominators is not required.
- A response that does not earn either point in Part (A) is eligible for **partial credit** in Part (A) if the response has one criteria from the first column AND one criteria from the second column. Partial credit response is scored **1-0** in Part (A).

First Column	**Second Column**
Correct answer in (i) without supporting work	Correct answer in (ii) without supporting work
Correct answer in (i) with supporting work, but the answer has not been evaluated according to bullets 2 and 3 in the directions. No incorrect work.	Correct answer in (ii) with supporting work, but the answer has not been evaluated according to bullets 2 and 3 in the directions. No incorrect work.
Answer in (i) is reported as $x + 3 = \ln 10$. No incorrect work follows.	Answer in (ii) is reported as $\frac{x}{2} = \sin\left(\frac{\pi}{4}\right)$. No incorrect work follows.
	Answer in (ii) is reported as $\frac{x}{2} = \frac{\sqrt{2}}{2}$. No incorrect work follows.
	Answer in (ii) is reported as $\frac{x}{2} = \frac{1}{\sqrt{2}}$. No incorrect work follows.

(B) The functions j and k are given by

$$j(x) = \log_{10}\left(8x^5\right) + \log_{10}\left(2x^2\right) - 9\log_{10} x$$

$$k(x) = \left(\frac{1 - \sin^2 x}{\sin x}\right)\sec x .$$

(i) Rewrite $j(x)$ as a single logarithm base 10 without negative exponents in any part of the expression. Your result should be of the form $\log_{10}(\text{expression})$.

(ii) Rewrite $k(x)$ as a single term involving $\tan x$.

(i) $j(x) = \log_{10}\left(8x^5\right) + \log_{10}\left(2x^2\right) - 9\log_{10} x$ $j(x) = \log_{10}\left(8x^5 \cdot 2x^2\right) - \log_{10} x^9$ $j(x) = \log_{10}\left(\frac{8x^5 \cdot 2x^2}{x^9}\right)$ $j(x) = \log_{10}\left(\frac{16x^7}{x^9}\right)$ $j(x) = \log_{10}\left(\frac{16}{x^2}\right),\ x > 0$	Expression for $j(x)$	**1 point**
(ii) $k(x) = \left(\frac{1 - \sin^2 x}{\sin x}\right)\sec x$ $k(x) = \left(\frac{\cos^2 x}{\sin x}\right)\left(\frac{1}{\cos x}\right)$ $k(x) = \left(\frac{\cos x}{\sin x}\right)$ $k(x) = \frac{1}{\tan x},\ \sin x \neq 0,\ \cos x \neq 0$	Expression for $k(x)$	**1 point**

Scoring notes:

- Supporting work is required in (i) and (ii). "Scratchwork" can be ignored; the use of a variable other than x is acceptable.
- Domain restrictions are not required to be included and are not scored.
- Where applicable, answers that have not been evaluated according to bullets 2 and 3 in the directions do not earn the point.
- To earn the first point, use of "$\log$" rather than "$\log_{10}$" is acceptable.
- The expression $j(x) = \log_{10}\left(\frac{4}{x}\right)^2$ earns the point in (i) with supporting work.
- A logarithmic expression that is missing one or both parentheses around the full argument of the logarithm is still eligible to earn the point.
- If a response is presented as a complex fraction, the complex fraction must be unambiguous in structure. Parentheses must be used correctly, and/or the fraction bars must be clearly and correctly proportioned.

- A response that does not earn either point in Part (B) is eligible for **partial credit** in Part (B) if the response has one criteria from the first column AND one criteria from the second column. Partial credit response is scored **1-0** in Part (B).

First Column	**Second Column**
Correct expression in (i) without supporting work	Correct expression in (ii) without supporting work
Expression in (i) is reported as $\log_{10}\left(\frac{8x^5 \cdot 2x^2}{x^9}\right)$. No incorrect work follows.	Expression in (ii) is reported as $\frac{\cos x}{\sin x}$ OR $\cot x$. No incorrect work follows.
Expression in (i) is reported as $\log_{10}\left(\frac{16x^7}{x^9}\right)$. No incorrect work follows.	Expression in (ii) includes a correct application of a Pythagorean identity with no incorrect work.
Expression in (i) is reported as $2\log_{10}\left(\frac{4}{x}\right)$. No incorrect work follows.	
Expression in (i) is reported as $-\log_{10}\left(\frac{x^2}{16}\right)$. No incorrect work follows.	
Expression in (i) is reported as $-2\log_{10}\left(\frac{x}{4}\right)$. No incorrect work follows.	
The expression in (i) is reported using natural logarithm and has the correct argument OR any of the expressions in partial credit rows two through six above are presented with natural logarithm.	

(C) The function m is given by

$$m(x) = \cos^{-1}(\tan(2x)).$$

Find all values in the domain of m that yield an output value of 0.

$m(x) = 0 \Rightarrow \cos^{-1}(\tan(2x)) = 0$ $\tan(2x) = \cos(0)$	One value of x	**1 point**
$\tan(2x) = 1$ $2x = \frac{\pi}{4} + \pi n$ $x = \frac{\pi}{8} + \frac{\pi}{2}n$, where n is any integer	All values of x	**1 point**

Scoring notes:

- Supporting work is required. "Scratchwork" can be ignored; the use of a variable other than x is acceptable.
- A response with supporting work that gives all correct values for x, such as $x = \frac{\pi}{8} + \pi n$ and $x = \frac{5\pi}{8} + \pi n$, earns both points.
- When expressing a general solution for all values for x (e.g., $x = \frac{\pi}{8} + \frac{\pi}{2}n$), the response can use i, k, n, or any letter except x, which is the variable used in the function.
- To earn the second point, "where n is any integer" is not required to be included.
- To earn the second point, no incorrect values for x are included.

Total for question 4 **6 points**