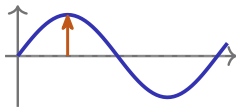


# Trigonometric & Polar Functions

AP Precalculus ■ Unit 3

AP Precalculus



## What we will cover

- 1 Periodic phenomena
- 2 Sine cosine & tangent
- 3 Sinusoidal functions
- 4 Transformations & modelling
- 5 Inverse trig & equations
- 6 Identities
- 7 Polar functions

1

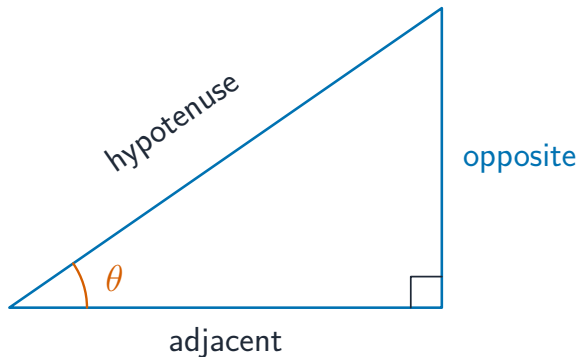
# Periodic phenomena

---

## Periodic phenomena

A **periodic** 周期 function repeats its output values over equal input intervals – tides, daylight hours, a rotating wheel.

**Period** and **frequency** 频率 are **reciprocals**.  $\sin \theta$  has period  $2\pi$ .



# Periodic functions, and angles in standard position

|                                 |                                                                                                         |
|---------------------------------|---------------------------------------------------------------------------------------------------------|
| <b>periodic</b> 周期性             | the output pattern repeats at regular input steps                                                       |
| <b>the period</b> 周期            | the <b>smallest</b> positive $k$ with $f(x + k) = f(x)$ for all $x$                                     |
| <b>a cycle</b> 周期循环             | one complete repetition of the pattern                                                                  |
| <b>standard position</b> 标准位置   | vertex at the origin, initial ray along the <b>positive x-axis</b> ; positive angles turn anticlockwise |
| <b>radian</b> 弧度 <b>measure</b> | the arc length on a <b>unit</b> circle – so a full turn is $2\pi$ , and $\pi$ radians is $180^\circ$    |
| <b>why radians</b>              | the arc-length and rate formulas only work in radians                                                   |

Anything that **oscillates** 振荡 – tides, daylight hours, a wheel – is modelled here, and always by fixing period, amplitude and midline first.

# Sine, cosine and tangent on the unit circle

For a point  $(x, y)$  at distance  $r$  along the terminal ray:

$$\cos \theta = \frac{x}{r}, \quad \sin \theta = \frac{y}{r}, \quad \tan \theta = \frac{y}{x}$$

$\tan \theta$  is the slope of the terminal ray – which is why it is undefined at  $\theta = \frac{\pi}{2}$ .



*A Ferris wheel seat traces sine and cosine*

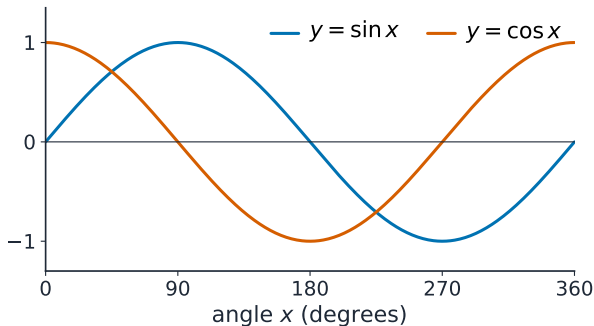
2

# Sinusoids

---

# Amplitude, midline and period

- **Amplitude** 振幅 = half the distance between max and min
- **Midline** 中线  $y = d$  = the average of max and min
- **Period**  $= \frac{2\pi}{b}$



## Worked example – read a sinusoid

$$f(\theta) = 3 \sin(2\theta) + 1$$

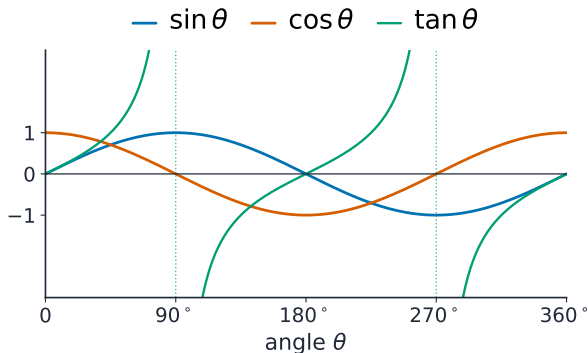
Amplitude **3**; period  $\frac{2\pi}{2} = \pi$ ; midline  $y = 1$ . So the maximum is  $1 + 3 = 4$  and the minimum is  $1 - 3 = -2$ .

The general form is

$$a \sin(b(\theta + c)) + d$$

$a$  sets amplitude,  $b$  the period,  $-c$  the **phase shift** ( $a + c$  shifts left),  $d$  the midline.

# Transformations and data modelling



To model periodic data (tides, daylight, temperature): read the **period** from where the pattern repeats, get the **amplitude** and **midline** from the max and min, and set the **phase shift** so a peak of the model lines up with a peak in the data.

Technology fits a sinusoidal regression.

# 3

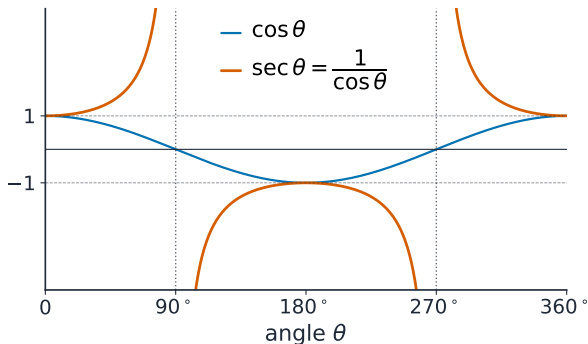
## Inverse trig & equations

---

# Tangent, and the reciprocal functions

$$\sec \theta = \frac{1}{\cos \theta}, \quad \csc \theta = \frac{1}{\sin \theta}, \quad \cot \theta = \frac{1}{\tan \theta}.$$

$\tan \theta$  has period  $\pi$  (not  $2\pi$ ) and a **vertical asymptote** wherever  $\cos \theta = 0$ .



# Inverse trigonometric functions

Sine, cosine and tangent are **not** one-to-one, so each inverse needs a **restricted domain**.

An inverse trig function **returns an angle** – and only the one in its restricted range.

arcsin: range  $[-\frac{\pi}{2}, \frac{\pi}{2}]$

arccos: range  $[0, \pi]$

arctan: range  $(-\frac{\pi}{2}, \frac{\pi}{2})$

## Worked example – a trigonometric equation

**Solve**  $2 \sin \theta = 1$  **on**  $[0, 2\pi)$

$\sin \theta = \frac{1}{2}$ . On the unit circle sine is positive in the **first and second** quadrants:

$$\theta = \frac{\pi}{6} \quad \text{or} \quad \theta = \frac{5\pi}{6}.$$

Over all reals, add  $2\pi k$  to each.

A trig equation has **infinitely many** solutions – the calculator gives you **one**. Add the period, and find the **second** quadrant solution yourself.

## Equivalent representations – the identities

**Pythagorean:**  $\sin^2 \theta + \cos^2 \theta = 1$

**Sum:**  $\sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta$        $\cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta$

**Double angle:**  $\sin 2\theta = 2 \sin \theta \cos \theta$

An **identity** 恒等式 is true for **every**  $\theta$  – use one to rewrite an expression into a form you can solve.

# The identities the exam expects

**Pythagorean identity**  
毕达哥拉斯恒等式

$\sin^2 \theta + \cos^2 \theta = 1$ , and its two derived forms with  $\tan$  and  $\cot$

**sum identities** 和角公式

$\sin(a \pm b) = \sin a \cos b \pm \cos a \sin b$ ;  
 $\cos(a \pm b) = \cos a \cos b \mp \sin a \sin b$

**double-angle** 倍角公式

$\sin 2\theta = 2 \sin \theta \cos \theta$ ;  $\cos 2\theta = \cos^2 \theta - \sin^2 \theta = 1 - 2 \sin^2 \theta$

**arcsine** 反正弦 **and friends**

inverse trig functions need a **restricted domain** to be functions at all

**the special triangles**

the **equilateral triangle** 等边三角形 halved gives  $30^\circ$ – $60^\circ$ ; the square halved gives  $45^\circ$

Every one of these lets you rewrite an expression into a form you can solve. Identities are **tools for rewriting**, not facts to recite.

4

Polar

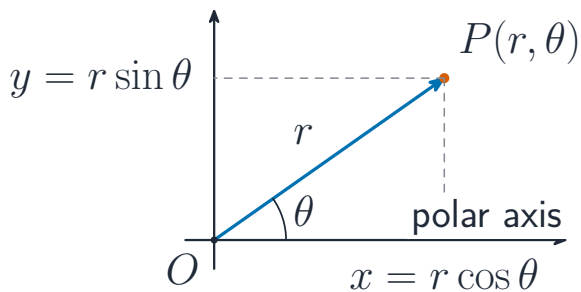
---

# Polar coordinates

**Polar coordinates** 极坐标 give a point by distance  $r$  and angle  $\theta$ :

$$x = r \cos \theta, \quad y = r \sin \theta$$

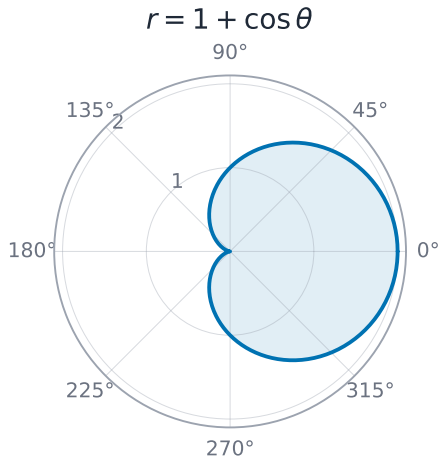
$$r = \sqrt{x^2 + y^2}, \quad \tan \theta = \frac{y}{x}.$$



# Polar graphs and their rates of change

A polar function  $r = f(\theta)$  sweeps a curve as  $\theta$  turns.

$r$  **increasing** means the curve moves **away** from the origin;  $r$  decreasing, toward it. A **negative**  $r$  plots **opposite** the angle.

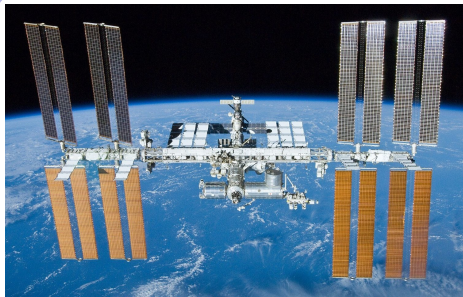


# Reading a polar graph

|                      |                                                                                                               |
|----------------------|---------------------------------------------------------------------------------------------------------------|
| what it is           | the graph of $r = f(\theta)$ is all points $(r, \theta)$ as $\theta$ sweeps around                            |
| restricting $\theta$ | draws only part of the curve – so the domain is part of the answer                                            |
| $r$ increasing       | the curve moves away from the origin as $\theta$ grows                                                        |
| $r$ decreasing       | it moves toward the origin – and a negative $r$ plots in the opposite direction                               |
| the extremes         | where $r$ switches between increasing and decreasing, the distance from the origin reaches a relative extreme |
| average rate         | $\frac{\Delta r}{\Delta \theta}$ – how fast the distance changes per radian turned                            |

Track distance from the origin against angle, separately. Trying to read a polar graph as if it were Cartesian is what makes it confusing.

## Periodic and circular motion in orbit



The ISS in orbit: trigonometric and polar functions describe periodic and circular motion

## Exam tips

- Use radians and the **unit circle**:  $\cos \theta$  and  $\sin \theta$  are the coordinates, always between  $-1$  and  $1$ .
- For a sinusoid  $a \sin(b(\theta + c)) + d$ :  $|a|$  is the amplitude,  $\frac{2\pi}{|b|}$  the period,  $d$  the midline, and the phase shift is  $-c$  ( $a + c$  shifts left).
- Model periodic data by reading period, amplitude, and midline from the max and min.
- **Inverse trig** functions need a restricted domain and return an angle; trig equations have infinitely many solutions (add the period).
- Convert polar  $\leftrightarrow$  rectangular with  $x = r \cos \theta$ ,  $y = r \sin \theta$ .

5

Recap

---

## Unit 3 – key takeaways

1

### The unit circle

$\cos = x/r$ ,  $\sin = y/r$ ,  $\tan =$  the ray's slope

2

### Sinusoids

$a \sin(b(\theta + c)) + d$ : amplitude, period, shift, midline

3

### Equations

infinitely many solutions – add the period

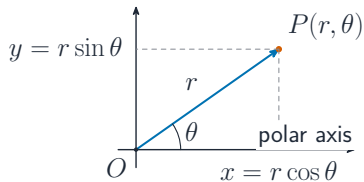
4

### Polar

$x = r \cos \theta$ ,  $y = r \sin \theta$ ;  $r$  rising moves away from O

## Check yourself

- 1 Give the amplitude, period and midline of  $3 \sin(2\theta) + 1$ .
- 2 Solve  $2 \sin \theta = 1$  on  $[0, 2\pi)$ .
- 3 Why does arcsin need a restricted domain?
- 4 Convert: what are  $x$  and  $y$  in terms of  $r$  and  $\theta$ ?



**Answers** 1. 3,  $\pi$ ,  $y = 1$     2.  $\frac{\pi}{6}$ ,  $\frac{5\pi}{6}$     3. sine is not one-to-one    4.  $x = r \cos \theta$ ,  $y = r \sin \theta$