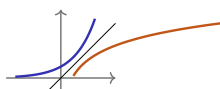


Exponential & Logarithmic Functions

AP Precalculus • Unit 2

AP Precalculus



Exponential & Logarithmic Functions

What we will cover

- 1 Arithmetic & geometric sequences
- 2 Linear vs exponential change
- 3 Exponential functions
- 4 Composition & inverses
- 5 Logarithms
- 6 Solving equations
- 7 Modelling & semi-log plots

1 Sequences

Exponential & Logarithmic Functions

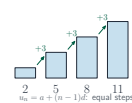
Sequences

Arithmetic and geometric sequences

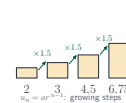
A **sequence** 数列 maps whole numbers to reals – so its graph is **discrete points**, not a curve.

- **arithmetic** 等差数列: **common difference** 公差 d , $a_n = a_0 + dn$
- **geometric** 等比数列: **common ratio** 公比 r , $g_n = g_0 r^n$

arithmetic: add d each step



geometric: multiply by r each step



Arithmetic grows by the **same amount** each step; geometric by a **larger amount** each step.

Exponential & Logarithmic Functions

Sequences

Worked example – addition versus multiplication

Compare the fourth terms

Arithmetic, $a_0 = 3$, $d = 5$:

$$a_4 = 3 + 5(4) = 23$$

Geometric, $g_0 = 2$, $r = 3$:

$$g_4 = 2 \cdot 3^4 = 162$$

Multiplication makes the geometric one pull far ahead.

The continuous cousins: **linear** $f(x) = b + mx$ mirrors arithmetic (repeated **addition**); **exponential** $f(x) = ab^x$ mirrors geometric (repeated **multiplication**).

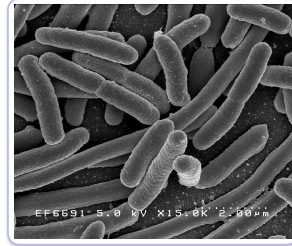
2 Exponentials

Exponential functions

$$f(x) = ab^x, \quad a \neq 0, b > 0, b \neq 1$$

initial value 初始值 a , **base** 底数 b ;
domain: all reals.

- $b > 1$: **growth** 指数增长
- $0 < b < 1$: **decay** 指数衰减



Bacteria exponential

Exponent rules are graph transformations

Product $b^m b^n = b^{m+n}$
a horizontal **shift** b^{x+k}
is a vertical **stretch**

Power $(b^m)^n = b^{mn}$
a horizontal **stretch** b^{cx}
is a **base change** $(b^c)^x$

Negative $b^{-n} = \frac{1}{b^n}$

Unit fraction $b^{1/k} = \sqrt[k]{b}$
(the k th **root** 方根)



Composition matryoshka

Modelling, and choosing between models

Exponentials model growth by a **constant proportion** over equal intervals.

- build from a ratio + an initial value, or from **two points**
- **exponential regression** 指数回归 fits a data set
- the **natural base** 自然底数 $e \approx 2.718$ is the standard



Exponential sprint

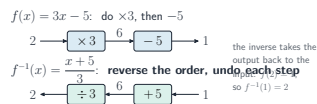
3 Composition & inverses

Composition of functions

The **composite** 复合函数
 $(f \circ g)(x) = f(g(x))$ feeds g 's output into f .

Composition is **not commutative**:
 $f(g(x))$ and $g(f(x))$ usually differ.

To build it, **substitute** $g(x)$ for every x in f .
The **identity** 恒等函数 $f(x) = x$ changes nothing.



Inverse functions

f is **invertible** 可逆 where each output comes from a **unique** input. Then
 $f(a) = b \Rightarrow f^{-1}(b) = a$, and

$$f(f^{-1}(x)) = f^{-1}(f(x)) = x.$$

Domain and range **swap**; the graph reflects over $y = x$.

swap x and y in $y = f(x)$

solve for y

You may **restrict the domain** to force invertibility.

4 Logarithms

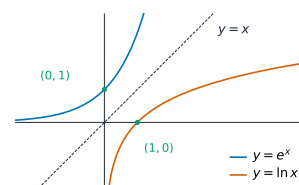
The logarithm answers "what exponent?"

$$\log_b c = a \iff b^a = c$$

The **logarithm** 对数 is the **inverse** of the **exponential**, so

$$\log_b(b^x) = x, \quad b^{\log_b x} = x.$$

$\ln x = \log_e x$ is the **natural logarithm** 自然对数, inverse of e^x .



The logarithmic function, and its properties

$f(x) = \log_b x$: domain $x > 0$, range all reals, a **vertical asymptote** at $x = 0$ – the mirror of the exponential's horizontal one.

It grows **very slowly** for large x .

$$\log_b(xy) = \log_b x + \log_b y$$

$$\log_b \frac{x}{y} = \log_b x - \log_b y$$

$$\log_b(x^n) = n \log_b x$$

change of base 换底: $\log_b x = \frac{\ln x}{\ln b}$

Solving exponential and logarithmic equations

Use the fact that they are **inverses**:

- isolate the exponential, then **take a log** of both sides
- isolate the log, then **exponentiate** both sides

Check for **extraneous solutions** – a log's argument must stay **positive**.

Solve $2 \cdot 3^x = 54$

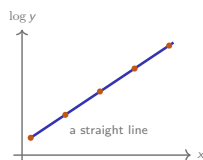
Divide by 2: $3^x = 27 = 3^3$, so $x = 3$.
When the sides are not tidy powers, take logs:

$$5^x = 20 \Rightarrow x = \frac{\ln 20}{\ln 5} \approx 1.86.$$

Semi-log plots

On a **logarithmic scale**, each unit is a **multiplicative step** ($10^0, 10^1, 10^2, \dots$).

Plot $\log y$ against x : an **exponential** becomes a **straight line**, whose slope and intercept recover the model.



5 Recap

Unit 2 – key takeaways

1 Two kinds of change

linear repeats addition;
exponential repeats multiplication

2 Exponentials

$f = ab^x$: no extrema, no
inflection, proportional growth

3 Inverses

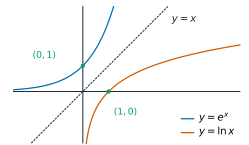
swap domain and range; reflect
over $y = x$

4 Logarithms

the inverse of b^x ; log rules reverse
the exponent rules

Check yourself

- 1 Solve $2 \cdot 3^x = 54$.
- 2 What does $\log_b c = a$ mean, as an exponential?
- 3 Does an exponential function have inflection points?
- 4 On a semi-log plot, what shape is an exponential?



Answers 1. $x = 3$ 2. $b^a = c$ 3. no 4. a straight line