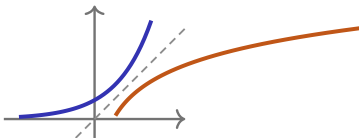


Exponential & Logarithmic Functions

AP Precalculus ■ Unit 2

AP Precalculus



What we will cover

- 1 Arithmetic & geometric sequences
- 2 Linear vs exponential change
- 3 Exponential functions
- 4 Composition & inverses
- 5 Logarithms
- 6 Solving equations
- 7 Modelling & semi-log plots

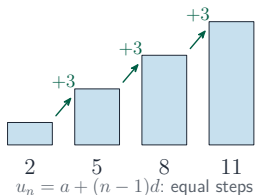
1

Sequences

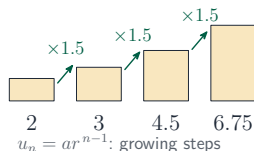
Arithmetic and geometric sequences

A **sequence** 数列 maps whole numbers to reals: its graph is **discrete points**.
 Arithmetic **adds** the same amount; geometric **multiplies** by the same ratio.

arithmetic: add d each step



geometric: multiply by r each step



arithmetic 等差数列: $a_n = a_0 + dn$ with **common difference** 公差 d

geometric 等比数列: $g_n = g_0 r^n$ with **common ratio** 公比 r

Worked example – addition versus multiplication

Compare the fourth terms

Arithmetic, $a_0 = 3$, $d = 5$:

$$a_4 = 3 + 5(4) = 23$$

Geometric, $g_0 = 2$, $r = 3$:

$$g_4 = 2 \cdot 3^4 = 162$$

Multiplication makes the geometric one pull far ahead.

The continuous cousins: linear $f(x) = b + mx$ mirrors arithmetic (repeated addition); exponential $f(x) = ab^x$ mirrors geometric (repeated multiplication).

2

Exponentials

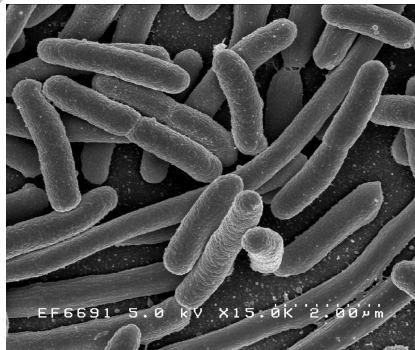
Exponential functions

$$f(x) = ab^x$$

$$a \neq 0, b > 0, b \neq 1$$

initial value 初始值 a , **base** 底数 b ;
domain: all reals.

- $b > 1$: **growth** 指数增长
- $0 < b < 1$: **decay** 指数衰减



Bacteria double each generation: exponential growth

What makes a function exponential

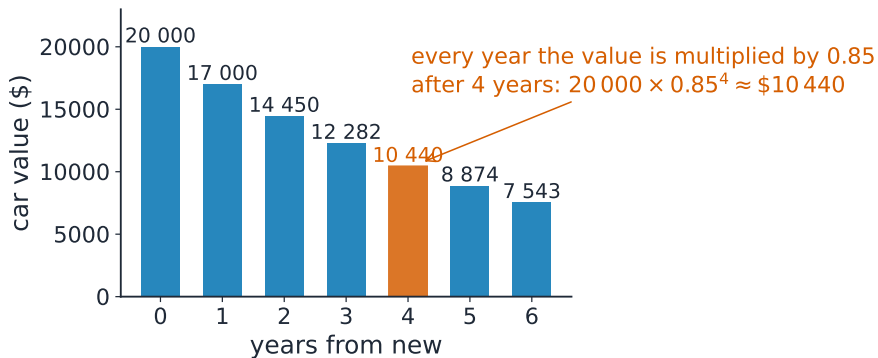
exponential over **equal-length** input intervals, outputs change by a **constant ratio**

- $b > 1$: **exponential growth** 指数增长
- $0 < b < 1$: exponential decay, towards 0 but never reaching it
- always increasing or decreasing; never changes concavity

x	linear		exponential	
0	2		2	
1	5	+3	6	$\times 3$
2	8	+3	18	$\times 3$
3	11	+3	54	$\times 3$

Test data by **dividing** consecutive outputs: a constant ratio is exponential, a constant difference is linear.

Exponential decay



Exponential decay: a fixed percentage is lost each period

Exponent rules are graph transformations

Product $b^m b^n = b^{m+n}$

a horizontal **shift** b^{x+k}
is a vertical **stretch**

Power $(b^m)^n = b^{mn}$

a horizontal **stretch** b^{cx}
is a **base change** $(b^c)^x$

Negative $b^{-n} = \frac{1}{b^n}$

Unit fraction $b^{1/k} = \sqrt[k]{b}$
(the k th **root** 方根)

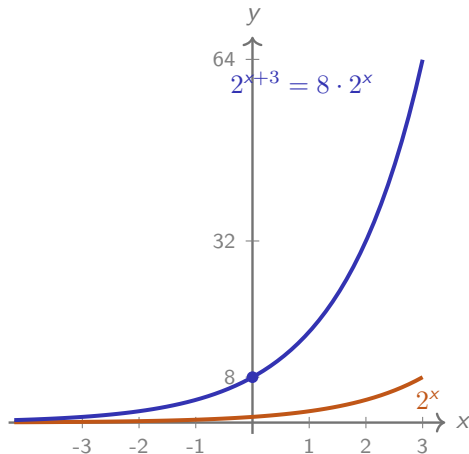
A shift is a stretch

Rewrite $f(x) = 2^{x+3}$

Product rule: $2^{x+3} = 2^3 \cdot 2^x = 8 \cdot 2^x$.

Shifting 2^x **left 3** gives the same graph as stretching it **vertically by 8**.

Any horizontal shift of b^x is a vertical stretch by b^k .



Modelling, and choosing between models

Exponentials model growth by a **constant proportion** over equal intervals.

- build from a ratio + an initial value, or from **two points**
- **exponential regression** 指数回归 fits a data set
- the **natural base** 自然底数 $e \approx 2.718$ is the standard

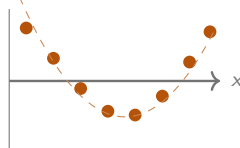
Choosing between competing models

Linear, quadratic and exponential can all *seem* to fit.

residual 残差 actual — predicted: the gap the model leaves at each point

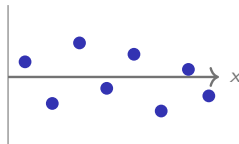
A model is **appropriate** when its **residual plot** 残差图 shows **no pattern**.

line fitted to a curve



curve: **wrong**

good model



random: **fine**

A high R^2 with a **patterned** residual plot is still the wrong model.

3

Composition & inverses

Composition of functions

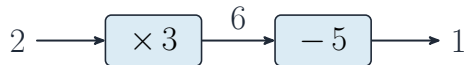
$$(f \circ g)(x) = f(g(x))$$

the **composite** 复合函数:
 g first, then f

Substitute $g(x)$ for every x in f .

Not commutative: $f(g(x))$ and $g(f(x))$ usually differ.

$f(x) = 3x - 5$: do $\times 3$, then -5



$$f^{-1}(x) = \frac{x + 5}{3}$$



reverse the order, undo each step

the inverse runs it backwards: $f(2) = 1$, so $f^{-1}(1) = 2$

Composing, decomposing, and inverting

decomposing 分解 $h(x) = (2x + 1)^2$

inner $g(x) = 2x + 1$, outer $f(x) = x^2$
so $h = f \circ g$.

Domain: x allowed in g and $g(x)$
allowed in f .

Every shift and stretch is a composition
with a simple function.

Invert $f(x) = 3x - 5$

swap: $x = 3y - 5$

solve: $y = \frac{x+5}{3}$, so $f^{-1}(x) = \frac{x+5}{3}$

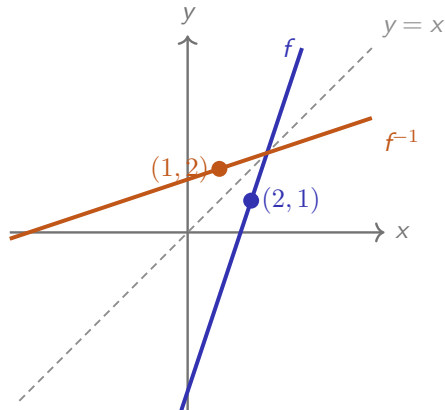
Check: $f(f^{-1}(x)) = 3 \cdot \frac{x+5}{3} - 5 = x$
✓

Inverse functions

f is **invertible** 可逆 where each output comes from a **unique** input. Then $f(a) = b \Rightarrow f^{-1}(b) = a$, and

$$f(f^{-1}(x)) = f^{-1}(f(x)) = x$$

Domain and range **swap**; the graph reflects over $y = x$.



$f(x) = 3x - 5$: **restrict the domain** when f is not one-to-one.

4

Logarithms

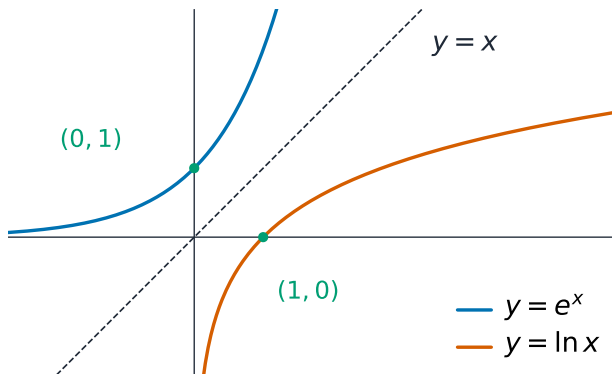
The logarithm answers “what exponent?”

$$\log_b c = a \iff b^a = c$$

The **logarithm** 对数 is the inverse of the exponential, so

$$\log_b(b^x) = x, \quad b^{\log_b x} = x.$$

$\ln x = \log_e x$ is the **natural logarithm** 自然对数, inverse of e^x .



The logarithmic function, and its properties

$f(x) = \log_b x$: domain $x > 0$, range all reals, a **vertical asymptote at $x = 0$** – the mirror of the exponential's horizontal one.

It grows **very slowly** for large x .

$$\log_b(xy) = \log_b x + \log_b y$$

$$\log_b \frac{x}{y} = \log_b x - \log_b y$$

$$\log_b(x^n) = n \log_b x$$

change of base 换底: $\log_b x = \frac{\ln x}{\ln b}$

Solving exponential and logarithmic equations

Use the fact that they are **inverses**:

- isolate the exponential, then **take a log** of both sides
- isolate the log, then **exponentiate** both sides

Check for **extraneous solutions** – a log's argument must stay **positive**.

Solve $2 \cdot 3^x = 54$

Divide by 2: $3^x = 27 = 3^3$, so $x = 3$.

When the sides are not tidy powers, take logs:

$$5^x = 20 \Rightarrow x = \frac{\ln 20}{\ln 5} \approx 1.86$$

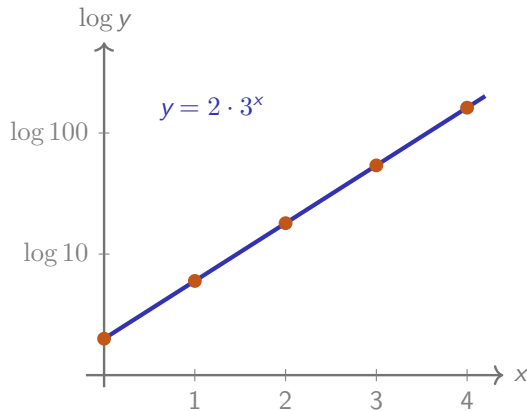
Semi-log plots

On a **logarithmic scale**, each unit is a **multiplicative** step ($10^0, 10^1, 10^2, \dots$).

$$y = ab^x \Rightarrow \log y = \log a + x \log b$$

slope $\log b$, intercept $\log a$

An exponential becomes a **straight line**.



Exam tips

- Distinguish **arithmetic** (add a common difference) from **geometric** (multiply a common ratio) sequences and their linear/exponential function cousins.
- Exponential = repeated **multiplication**, so it eventually outgrows any linear or polynomial model.
- The **logarithm is the inverse** of the exponential ($\log_b c = a \Leftrightarrow b^a = c$); use it to solve $b^x = k$.
- Apply the log laws (product, quotient, power) and the change-of-base formula; the argument of a log must be **positive**.
- On a **semi-log plot** an exponential model becomes a straight line.

5

Recap

Unit 2 – key takeaways

1

Two kinds of change

linear repeats addition;
exponential repeats multiplication

2

Exponentials

$f = ab^x$: no extrema, no
inflection, proportional growth

3

Inverses

swap domain and range; reflect
over $y = x$

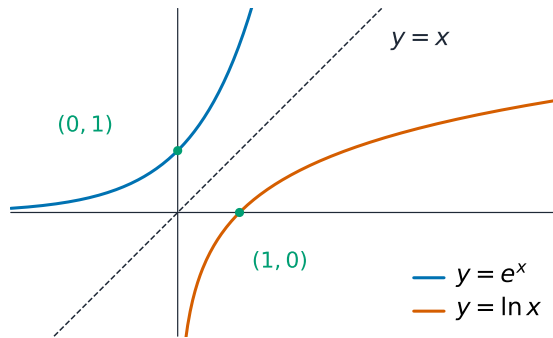
4

Logarithms

the inverse of b^x ; log rules reverse
the exponent rules

Check yourself

- 1 Solve $2 \cdot 3^x = 54$.
- 2 What does $\log_b c = a$ mean, as an exponential?
- 3 Does an exponential function have inflection points?
- 4 On a semi-log plot, what shape is an exponential?



Answers 1. $x = 3$ 2. $b^a = c$ 3. no 4. a straight line