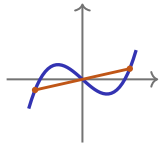


Polynomial & Rational Functions

AP Precalculus ■ Unit 1

AP Precalculus



Polynomial & Rational Functions

What we will cover

- 1 Change in tandem
- 2 Rates of change
- 3 Polynomials: zeros & end behavior
- 4 Rational functions
- 5 Asymptotes & holes
- 6 Transformations
- 7 Modelling

1

Change in tandem

Functions, and change in tandem

A **function** 函数 maps each input to **exactly one** output. Inputs form the **domain** 定义域; outputs the **range** 值域.

- **increasing** 递增:
 $a < b \Rightarrow f(a) < f(b)$
- **decreasing** 递减:
 $a < b \Rightarrow f(a) > f(b)$



Bridge curve

Average rate of change

The **average rate of change** 平均变化率 over $[a, b]$ is the **slope of the secant line** 割线:

$$\text{avg rate} = \frac{f(b) - f(a)}{b - a}.$$

Positive rate: both quantities move the same way. **Negative**: opposite ways.

$$f(x) = x^2$$

Over $[1, 4]$: $\frac{16 - 1}{3} = 5.$

Over $[1, 2]$: $\frac{4 - 1}{1} = 3.$ The rate **itself changes** – which is exactly why f is not linear.

The rate of the rate

Linear 线性

average rate is **constant** – so the rate of the rate is **zero**

Quadratic 二次

rates over equal intervals form a **linear** pattern – so they change at a **constant** rate



Parabola

bridge

2 Polynomials

Polynomial & Rational Functions

└ Polynomials

Polynomials – degree, leading term, extrema

$$p(x) = a_n x^n + \cdots + a_1 x + a_0, \quad a_n \neq 0$$

Degree 次数 n ; **leading term** 首项 $a_n x^n$;
leading coefficient 首项系数 a_n .

- where it switches increasing/decreasing:
a **local extremum** 局部极值
- between two distinct real zeros there is
at least one extremum
- an **even-degree** polynomial has a global
max or a global min

A **point of inflection** 拐点
is where concavity changes –
concave up 上凹 to **concave
down** 下凹 or back.

That is where the rate of change switches
between increasing and decreasing.

Polynomial & Rational Functions

└ Polynomials

Zeros, multiplicity and complex roots

a is a **zero** 零点 when $p(a) = 0$; then
 $(x - a)$ is a factor. Repeated n times $\Rightarrow$
multiplicity 重数 n .

Counting multiplicities, a degree- n poly-
nomial has exactly n **complex** 复数 ze-
ros.

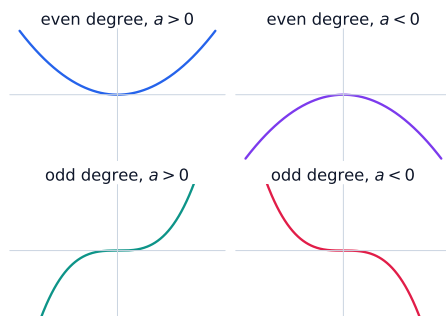
Non-real zeros come in **conjugate** 共轭 pairs:
 $a + bi$ and $a - bi$.

Even multiplicity
the graph is **tangent** – it
touches, does not cross

Odd multiplicity
the graph **crosses** the axis

End behaviour is decided by the leading term

For large $|x|$, the leading term $a_n x^n$ **dominates** 主导 everything else – so the **sign of a_n** and the **parity of n** fix both ends:



3 Rational functions

Rational end behaviour – compare the degrees

top $>$ bottom
follows the **quo-**
tient polynomial;
if linear, a **slant**
asymptote 斜渐近线

equal degrees
horizontal asymp-
tote 水平渐近线
at the ratio of lead-
ing coefficients

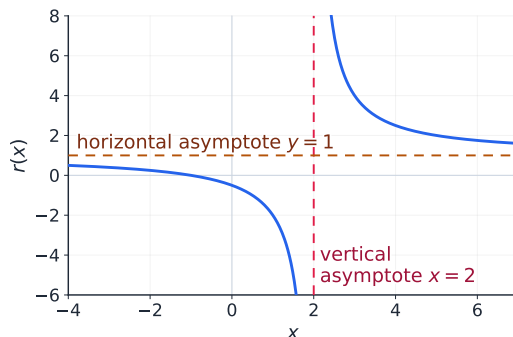
top $<$ bottom
the quotient
 $\rightarrow 0$, so $y = 0$

At a horizontal asymptote $y = b$, the outputs get **and stay** arbitrarily close to b .

Vertical asymptotes and holes

- a **vertical asymptote** 垂直渐近线 at $x = a$: a is a zero of the **denominator** that does **not** cancel
- a **hole** 空洞 at $x = c$: the factor **cancels** – a single missing point at (c, L)

Check **each side** of an asymptote – the two sides can go opposite ways.



Worked example – describe a rational function

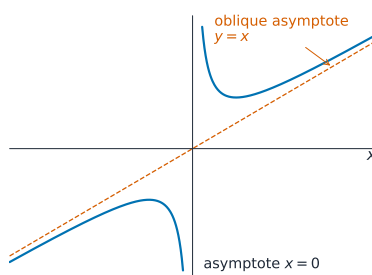
$$r(x) = \frac{2x^2 + 3}{x^2 - 1}$$

Equal degrees, so the horizontal asymptote is

$$y = \frac{2}{1} = 2.$$

$x^2 - 1 = 0$ at $x = \pm 1$, and neither cancels, so there are vertical asymptotes at

$$x = 1 \quad \text{and} \quad x = -1.$$



When the top's degree is **one more** than the bottom's, long division gives a **slant asymptote**.

Equivalent forms reveal different features

Factored form

shows real **zeros** – and hence intercepts, holes, vertical asymptotes, domain and range

Standard form

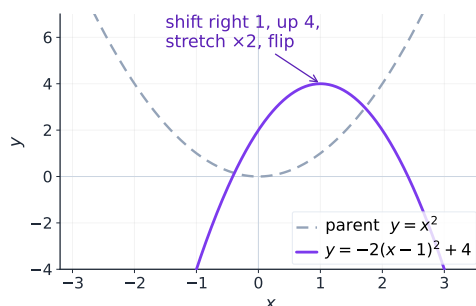
shows the **degree** and **leading term** – and hence end behaviour

Long division 多项式长除法 rewrites $f = gq + r$, the **quotient** 商 q is the **slant asymptote** when the degrees differ by one.

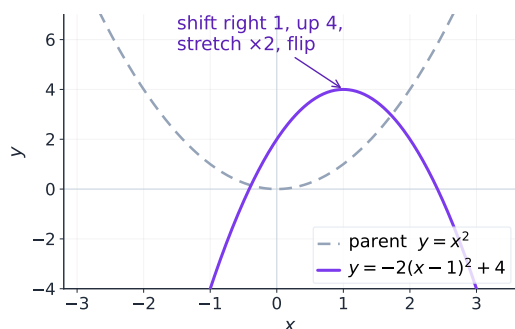
4 Transformations

Transformations of functions

- **Translations** 平移: $f(x) + k$ up/down; $f(x - h)$ right/left
- **Dilations** 伸缩: $a f(x)$ vertical; $f(bx)$ horizontal
- **Reflections** 反射: $-f(x)$ over the x -axis; $f(-x)$ over the y -axis



Combining shift, stretch and reflect



Combine **additive** shifts and **multiplicative** stretches to model a shifted, scaled version of a known shape.

Read the general form $a f(b(x - h)) + k$ from the inside out.

5 Modelling

Polynomial & Rational Functions

└ Modelling

Choosing and building a model

Match the **model** 模型 to how the quantity changes:

- constant rate → linear
- constant second difference → quadratic
- several turns → a higher-degree polynomial

State the **assumptions** 假设 – a model is only valid where they hold.

To **construct** one, use the given features (points, zeros with multiplicity, end behaviour), then check the answer against the **domain that makes sense** and interpret with real-world units.

6 Recap

Unit 1 – key takeaways

1 Rates

avg rate = secant slope; constant
→ linear, linear → quadratic

2 Zeros

even multiplicity touches, odd crosses; n complex zeros

3 End behaviour

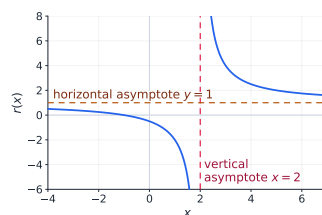
the leading term decides both ends

4 Rational

degrees give the asymptote; a cancelled factor is a hole

Check yourself

- 1 At a zero of even multiplicity, does the graph cross or touch?
- 2 Give the horizontal asymptote of $\frac{2x^2+3}{x^2-1}$.
- 3 Which term decides a polynomial's end behaviour?
- 4 A factor cancels in a rational function. Asymptote or hole?



Answers 1. touches (tangent) 2. $y = 2$ 3. the leading term 4. a hole