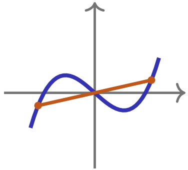
Polynomial & Rational Functions

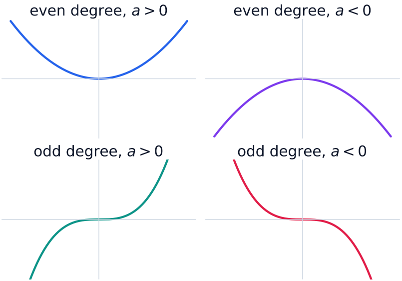
AP Precalculus ▪ Unit 1

AP Precalculus



What we will cover

1. Change in tandem
2. Rates of change
3. Polynomials: zeros & end behavior
4. Rational functions
5. Asymptotes & holes
6. Transformations
7. Modelling



Functions, and change in tandem

A **function** 函数 maps each input to exactly one output. Inputs form the **domain** 定义域; outputs the **range** 值域.

- **increasing** 递增: $a < b \Rightarrow f(a) < f(b)$
- **decreasing** 递减: $a < b \Rightarrow f(a) > f(b)$

bridge Curve

Average rate of change

The **average rate of change** 平均变化率 over $[a, b]$ is the slope of the secant line 割线:

$$\text{avg rate} = \frac{f(b) - f(a)}{b - a}.$$

> Positive rate: both quantities move the same way. Negative: opposite ways.

$f(x) = x^2$

Over $[1, 4]$: $\dfrac{16 - 1}{3} = 5$.

Over $[1, 2]$: $\dfrac{4 - 1}{1} = 3$. The rate itself changes – which is exactly why f is not linear.

The rate of the rate

Linear 线性

average rate is constant – so
the rate of the rate is zero

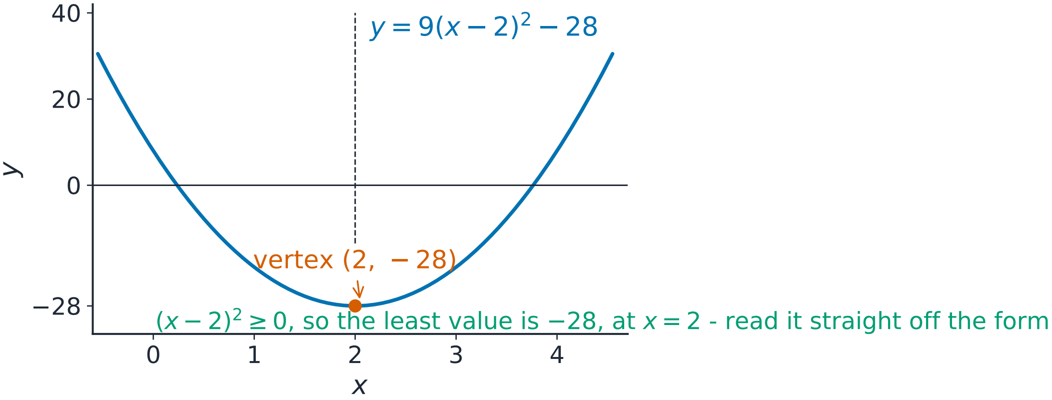
Quadratic 二次

rates over equal intervals form
a linear pattern – so they
change at a constant rate

parabola

Bridge

The rate of the rate



Polynomials – degree, leading term, extrema

$$p(x) = a_n x^n + \cdots + a_1 x + a_0, \quad a_n \neq 0$$

Degree 次数 n; **leading term** 首项 $a_n x^n$; **leading coefficient** 首项系数 a_n.

- where it switches increasing/decreasing: a **local extremum** 局部极值
- between two distinct real zeros there is at least one extremum
- an even-degree polynomial has a global max or a global min

> A **point of inflection** 拐点 is where concavity changes – **concave up** 上凹 to **concave down** 下凹 or back.

That is where the rate of change switches between increasing and decreasing.

Zeros, multiplicity and complex roots

a is a **zero** 零点 when $p(a) = 0$; then $(x - a)$ is a factor. Repeated n times $\Rightarrow$ **multiplicity** 重数 n.

> Counting multiplicities, a degree-n polynomial has exactly n **complex** 复数 zeros.

Non-real zeros come in **conjugate** 共轭 pairs: $a + bi$ and $a - bi$.

Even multiplicity

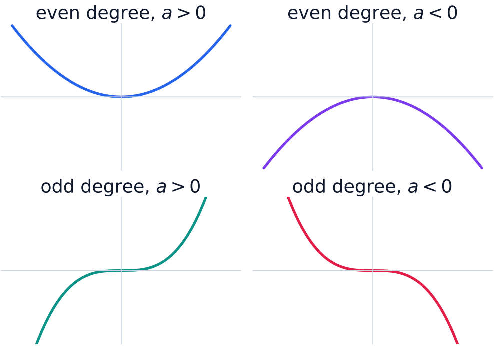
the graph is tangent – it touches, does not cross

Odd multiplicity

the graph crosses the axis

End behaviour is decided by the leading term

For large $|x|$, the leading term $a_n x^n$ **dominates** 主导 everything else – so the sign of a_n and the parity of n fix both ends:



Rational end behaviour – compare the degrees

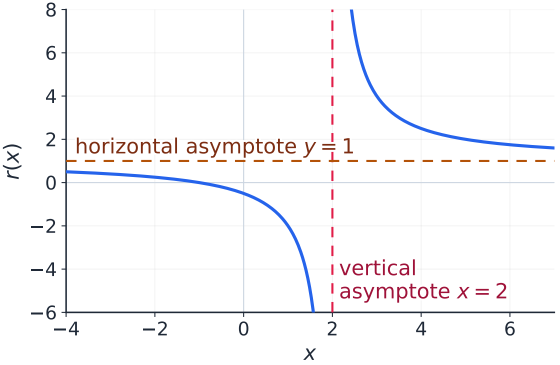
top > bottom	**equal** degrees	top < bottom
follows the quotient polynomial; if linear, a **slant asymptote** 斜渐近线	**horizontal asymptote** 水平渐近线 at the ratio of leading coefficients	the quotient $\to 0$, so $y = 0$

At a horizontal asymptote $y = b$, the outputs get and stay arbitrarily close to b.

Vertical asymptotes and holes

- a **vertical asymptote** 垂直渐近线 at $x = a$: a is a zero of the denominator that does not cancel
- a **hole** 空洞 at $x = c$: the factor cancels – a single missing point at (c, L)

> Check each side of an asymptote – the two sides can go opposite ways.



Worked example – describe a rational function

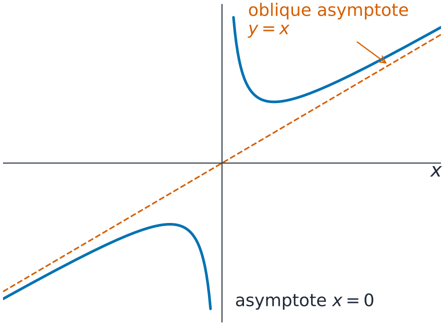
$$r(x) = \frac{2x^2+3}{x^2-1}$$

Equal degrees, so the horizontal asymptote is

$$y = \frac{2}{1} = 2.$$

$x^2 - 1 = 0$ at $x = \pm 1$, and neither cancels, so there are vertical asymptotes at

$$x = 1 \quad \text{and} \quad x = -1.$$



When the top's degree is one more than the bottom's, long division gives a slant asymptote.

Equivalent forms reveal different features

Factored form

shows real zeros – and hence intercepts, holes, vertical asymptotes, domain and range

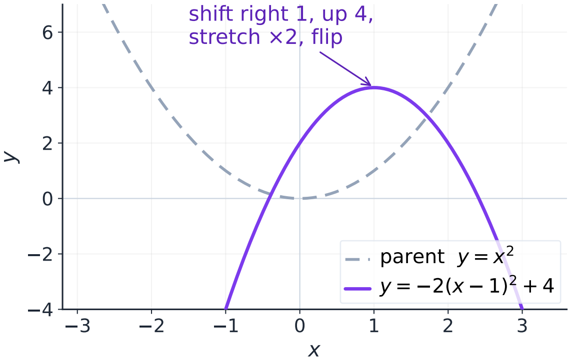
Standard form

shows the degree and leading term – and hence end behaviour

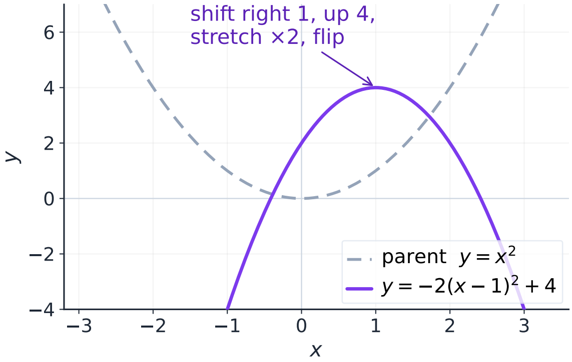
Long division 多项式长除法 rewrites $f = gq + r$; the **quotient** 商 q is the slant asymptote when the degrees differ by one.

Transformations of functions

- **Translations** 平移: $f(x) + k$ up/down; $f(x - h)$ right/left
- **Dilations** 伸缩: $a\,f(x)$ vertical; $f(bx)$ horizontal
- **Reflections** 反射: $-f(x)$ over the x-axis; $f(-x)$ over the y-axis



Combining shift, stretch and reflect



Combine additive shifts and multiplicative stretches to model a shifted, scaled version of a known shape.

> Read the general form $a\,f\big(b(x-h)\big)+k$ from the inside out.

Choosing and building a model

Match the **model** 模型 to how the quantity changes:

- constant rate → linear
- constant second difference → quadratic
- several turns → a higher-degree polynomial

State the **assumptions** 假设 – a model is only valid where they hold.

To construct one, use the given features (points, zeros with multiplicity, end behaviour), then check the answer against the domain that makes sense and interpret with real-world units.

Unit 1 – key takeaways

1 Rates

avg rate $=$ secant slope; constant $\to$ linear, linear $\to$ quadratic

2 Zeros

even multiplicity touches, odd crosses; n complex zeros

3 End behaviour

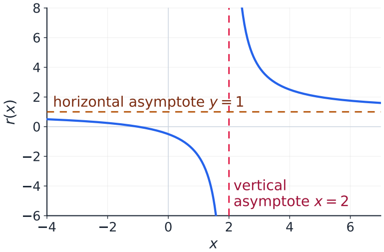
the leading term decides both ends

4 Rational

degrees give the asymptote; a cancelled factor is a hole

Check yourself

1. At a zero of even multiplicity, does the graph cross or touch?
2. Give the horizontal asymptote of $\frac{2x^2+3}{x^2-1}$.
3. Which term decides a polynomial's end behaviour?
4. A factor cancels in a rational function. Asymptote or hole?



Answers **1.** touches (tangent) **2.** $y = 2$ **3.** the leading term **4.** a hole