

Most of this exam asks you to build or interpret a model, and the credit sits in the sentences around the algebra: which family fits and why, what the parameters mean, and over what domain the model is reasonable. Each solution below writes those out. 得分点多在“选择理由、参数含义、适用范围”的表述, 而非计算本身。

Choosing and building a model

I A table gives values (1, 6), (2, 12), (3, 24), (4, 48). Decide whether a linear or an exponential model fits, justify the choice, and write the model.

Exponential: $f(x) = 3 \cdot 2^x$.

Test both properties. The differences are 6, 12, 24 – not constant, so it is not linear. The ratios are $12/6 = 2$, $24/12 = 2$, $48/24 = 2$ – constant, so the function changes by a constant FACTOR over equal intervals, which is the defining property of an exponential. So $f(x) = ab^{\hat{x}}$ with $b = 2$. Using the point (1, 6): $6 = a \cdot 2$, so $a = 3$, and $f(x) = 3 \cdot 2^x$. Check against (4, 48): $3 \cdot 16 = 48$. Stating the ratio test explicitly is what earns the justification point; a fitted equation with no reason does not.

I A ferris wheel of radius 20 m has its centre 25 m above the ground and completes one revolution every 40 seconds. A rider boards at the lowest point at $t = 0$. Write a model for the rider's height.

$h(t) = 25 - 20 \cos(\pi t/20)$, with t in seconds and h in metres.

Identify the four parameters from the description. The midline is the centre height, 25 m. The amplitude is the radius, 20 m. The period is 40 s, so $b = 2\pi/40 = \pi/20$. At $t = 0$ the rider is at the MINIMUM, and a negative cosine starts at its minimum, so use $-20 \cos$ rather than a sine with a phase shift. Hence $h(t) = 25 - 20 \cos(\pi t/20)$. Check: at $t = 0$, $h = 25 - 20 = 5$ m, the lowest point; at $t = 20$, $h = 45$ m, the top. Define the variables and their units in the answer – that is scored separately from the equation.

Rates of change and function behaviour

I For $f(x) = x^3 - 6x^2 + 9x$, find the average rate of change on $[0, 2]$ and describe the concavity there using average rates of change.

Average rate of change is 1; the function is concave down on that interval.

$f(0) = 0$ and $f(2) = 8 - 24 + 18 = 2$, so the average rate of change is $(2 - 0)/(2 - 0) = 1$. For concavity in this course's language, compare average rates of change over successive equal sub-intervals. On $[0, 1]$ it is $(4 - 0)/1 = 4$; on $[1, 2]$ it is $(2 - 4)/1 = -2$. The average rates are DECREASING, so the graph is concave down on $[0, 2]$. Phrasing it through decreasing average rates rather than a second derivative is what this exam expects, and it is the wording the rubric matches.

I A rational function has $f(x) = (x^2 - 4)/(x^2 - x - 6)$. Identify every asymptote and hole, with justification.

Hole at $x = -2$; vertical asymptote at $x = 3$; horizontal asymptote $y = 1$.

Factor both before doing anything else: the numerator is $(x-2)(x+2)$ and the denominator $x^2 - x - 6 = (x-3)(x+2)$. The factor $(x+2)$ is common and cancels, so at $x = -2$ the function has a HOLE, not an asymptote – the graph is undefined there but approaches a finite value. The factor $(x-3)$ does not cancel, so $x = 3$ is a vertical asymptote. The degrees of numerator and denominator are equal, so the horizontal asymptote is the ratio of the leading coefficients, $y = 1$. The distinction that earns the marks is exactly that one: a factor that cancels gives a hole, a factor that does not gives a vertical asymptote.

Exponentials, logarithms and inverses

| Solve $3^{(2x-1)} = 40$, giving an exact answer and a decimal to three places.

$$x = (\log_3 40 + 1)/2 \quad 2.179.$$

Take logarithms of both sides: $(2x - 1)\ln 3 = \ln 40$, so $2x - 1 = \ln 40 / \ln 3 = 3.358$, giving $2x = 4.358$ and $x = 2.179$. Exactly, $x = (\ln 40 / \ln 3 + 1)/2$. Check by substituting back: $3^{(3.358)} \approx 40$. Present the exact form first – the rubric asks for it and a decimal alone will not earn that point.

| A quantity decays exponentially with a half-life of 12 years. Write the model with an initial amount of 500, and find when 80 units remain.

$$A(t) = 500(1/2)^{(t/12)}; \text{ about 31.7 years.}$$

Half-life form is the fastest route: $A(t) = A_0(1/2)^{(t/H)}$ with $H = 12$, so $A(t) = 500(1/2)^{(t/12)}$. Set $A = 80$: $(1/2)^{(t/12)} = 0.16$. Taking logs, $(t/12)\ln(0.5) = \ln(0.16)$, so $t/12 = \ln 0.16 / \ln 0.5 = 2.644$ and $t = 31.7$ years. Note the check: 31.7 years is between two and three half-lives (24 and 36 years), and after two half-lives 125 remain while after three, 62.5 – so a value of 80 in between is consistent. That order-of-magnitude check costs one line and catches sign errors.

| Given $f(x) = 2x/(x - 3)$, find the inverse function and state its domain.

$$f^{-1}(x) = 3x/(x - 2), \text{ with domain all real } x \text{ except } 2.$$

Set $y = 2x/(x - 3)$ and solve for x : $y(x - 3) = 2x$, so $yx - 3y = 2x$, so $x(y - 2) = 3y$ and $x = 3y/(y - 2)$. Swapping names, $f^{-1}(x) = 3x/(x - 2)$. The domain of the inverse is the RANGE of the original, and the original has a horizontal asymptote at $y = 2$ which it never reaches – consistent with the inverse being undefined at $x = 2$. Making that connection explicitly, rather than only reading the new denominator, is what the justification asks for.