Course Companion

AP Precalculus

Every topic handout in one volume

全部专题讲义合集

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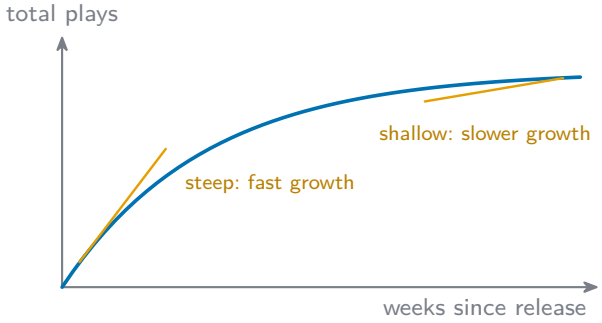
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Polynomial and Rational Functions

AP Precalculus

Change in Tandem

Why it matters. When a new song is released, its total number of plays climbs fast in the first weeks and then more slowly. A record label that wants to predict next month's total, or decide when the song has peaked, needs a function that matches how the numbers change. This unit is about reading that change, through rates of change, and choosing polynomial and rational functions that model it.



Increasing throughout, but the rate of increase falls.

A **function** 函数 is a rule that maps each input to **exactly one** output. The set of allowed inputs is the **domain** 定义域; the set of outputs is the **range** 值域. The input variable is the **independent variable** 自变量 and the output variable is the **dependent variable** 因变量. A function rule can be shown graphically, numerically, analytically (a formula), or verbally.

As the input changes, the output changes "in tandem" – together. Over an interval, a function is:

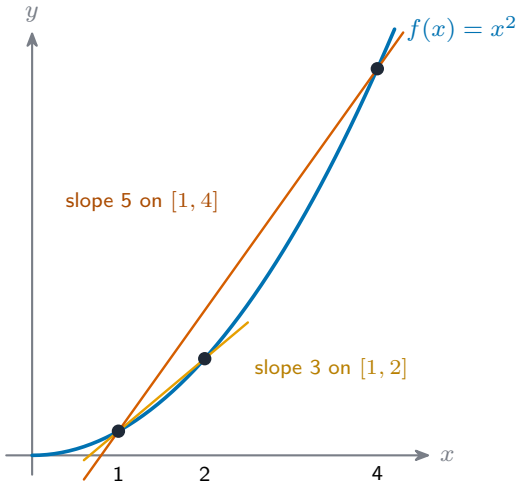
- **increasing** 递增 if, whenever $a < b$, then $f(a) < f(b)$ (bigger input, bigger output);
- **decreasing** 递减 if, whenever $a < b$, then $f(a) > f(b)$ (bigger input, smaller output).

A graph of a function shows all its input–output pairs, so you can read this behavior straight off the picture.

Rates of Change

The **average rate of change** 平均变化率 of a function over an interval is the change in output divided by the change in input – the single constant rate that would give the same total change. Over $[a, b]$:

$$\text{avg rate} = \frac{f(b) - f(a)}{b - a}.$$



The average rate changes with the interval, so f is not linear.

The **rate of change at a point** measures how fast the output changes right at that input. You **approximate** it with average rates over small intervals around the point. Comparing two points, the one with the larger average rate (over small enough intervals) is changing faster. A **positive** rate means both quantities move the same way; a **negative** rate means they move in opposite ways.

> **Worked example.** For $f(x) = x^2$, the average rate of change over $[1, 4]$ is $\frac{f(4) - f(1)}{4 - 1} = \frac{16 - 1}{3} = 5$. Over $[1, 2]$ it is $\frac{4 - 1}{1} = 3$ – the rate itself changes, which is exactly why f is not linear.

Rates of change and concavity. If the rate of change of f is increasing over an interval, the graph is **concave up** 上凹 there; if the rate is decreasing, the graph is **concave down** 下凹. From a table with equal input steps, compare the changes in output.

> **Worked example.** A tank's volume $f(t)$, in litres, is recorded every 2 minutes:

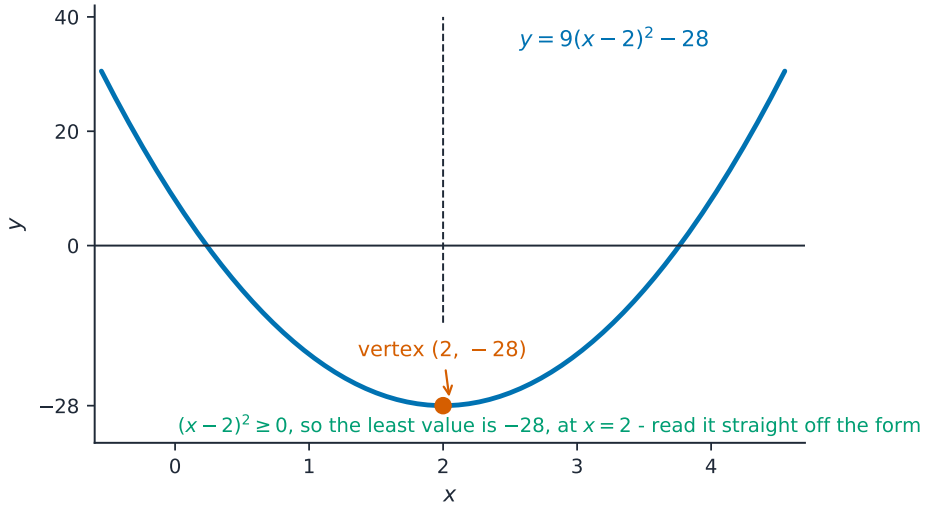
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t (min)	0	2	4	6	8
$f(t)$ (L)	10	14	20	28	38

Over each 2-minute interval the volume increases by 4, 6, 8, then 10 litres. The output increases by **increasing** amounts over equal-length input intervals, so the average rates of change (2, 3, 4, 5 L/min) are increasing. f is increasing and its graph is concave up on $[0, 8]$. Free-response questions ask for exactly this kind of reason: refer to the values in the table.

Rates of Change in Linear and Quadratic Functions

The average rate of change over $[a, b]$ is the slope of the **secant line** 割线 from $(a, f(a))$ to $(b, f(b))$.



Completing the square gives the vertex of a parabola

- For a **linear** 线性 function, the average rate of change over any interval is **constant** – so the rate at which the rate changes is zero.
- For a **quadratic** 二次 function, the average rates of change over equal-length intervals themselves form a **linear** pattern – so those rates change at a **constant** rate.

This "rate of the rate" idea distinguishes function types: a constant second difference signals a quadratic.

> **Worked example.** Which kind of function fits this table (equally spaced inputs)?

x	0	1	2	3	4
$f(x)$	1	3	9	19	33
1st differences		2	6	10	14
2nd differences			4	4	4

The first differences (the average rates over each step) are not constant, so f is not linear. They increase by the same amount, 4, each time, so the second differences are constant: a **quadratic** model fits. (In fact $f(x) = 2x^2 + 1$.)



A hanging bridge cable takes the shape of a parabola-like curve – a quadratic you can see in steel

Polynomial Functions and Rates of Change

A nonconstant **polynomial** 多项式 has the form

$$p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0, \quad a_n \neq 0.$$

Its **degree** 次数 is n, its **leading term** 首项 is $a_n x^n$, and its **leading coefficient** 首项系数 is a_n. A nonzero constant is a polynomial of degree 0; the zero polynomial has no defined degree.

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A smooth bridge curve: polynomial and rational graphs model real shapes with asymptotes and turns

Key features:

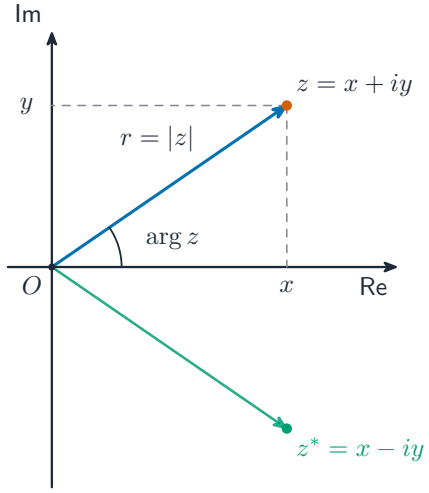
- Where a polynomial switches between increasing and decreasing, it has a **local (relative) extremum** 局部极值. The greatest local maximum is the **global (absolute) maximum** 全局最大值 *only when the polynomial has one* – every odd-degree polynomial is unbounded both above and below, so it has neither a global maximum nor a global minimum.
- Between any two distinct real zeros there is at least one local maximum or minimum.
- An **even-degree** polynomial has either a global maximum or a global minimum.
- A **point of inflection** 拐点 is where the graph changes concavity – from concave up to concave down or the reverse – i.e. where the rate of change switches between increasing and decreasing.
- A function has **symmetry** when it is **even** 偶函数 – $f(-x) = f(x)$, graph symmetric across the y-axis (like x^2, x^4) – or **odd** 奇函数 – $f(-x) = -f(x)$, graph symmetric by rotation about the origin (like x^3, x^5). Test by substituting $-x$ and simplifying; later, $\cos\theta$ is even and $\sin\theta$ is odd.



A satellite dish is a parabola of revolution: every parallel ray reflects to the same focus

Polynomial Functions and Complex Zeros

A **zero** 零点 (or **root** 根) of p is a number a with $p(a) = 0$. For a real a, $(x - a)$ is a factor of p exactly when a is a zero. If the factor $(x - a)$ is repeated n times, that zero has **multiplicity** 重数 n. Counting multiplicities, a degree-n polynomial has exactly n **complex** 复数 zeros.



A complex number on an Argand diagram: modulus is the distance, argument the angle

- A real zero a gives an x**-intercept** at $(a, 0)$; real zeros are the endpoints of the intervals where $p(x) \geq 0$ or ≤ 0.
- Non-real zeros come in **conjugate** 共轭 pairs: if $a + bi$ is a zero, so is $a - bi$.
- At a real zero of **even** multiplicity the graph is **tangent** to the x-axis (it touches but does not cross); at odd multiplicity it crosses.

> **Worked example.** A cubic p has a zero of multiplicity 2 at $x = -1$, a zero at $x = 3$, and $p(0) = 6$. Find $p(x)$ and describe its graph.

Each zero gives a factor, so $p(x) = a(x+1)^2(x-3)$. Use the point: $p(0) = a(1)^2(-3) = -3a = 6$, so $a = -2$ and

$$p(x) = -2(x + 1)^2(x - 3).$$

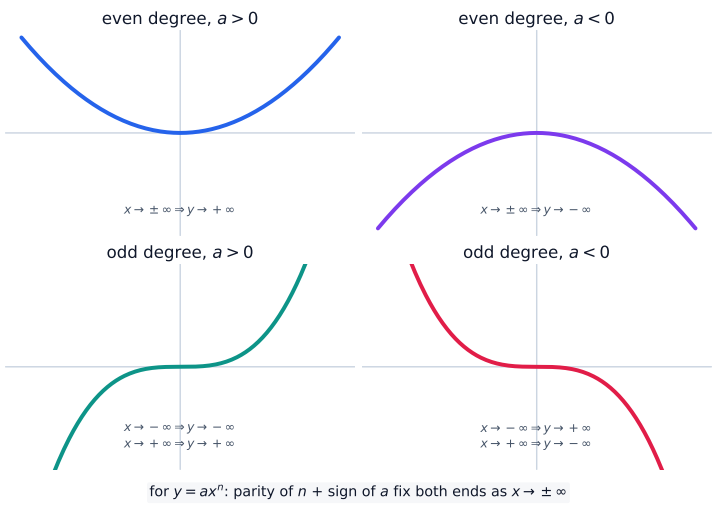
The graph touches the x-axis at $x = -1$ (even multiplicity) and crosses it at $x = 3$. The leading term is $-2x^3$ (negative, odd degree), so $\lim_{x\to\infty} p(x) = -\infty$ and $\lim_{x\to-\infty} p(x) = \infty$.

- The degree equals the least n for which the nth **successive differences** of equally-spaced outputs become constant.

Polynomial Functions and End Behavior

End behavior 末端行为 describes where a function heads as the input grows without bound. For a nonconstant polynomial, as $x \to \pm\infty$ the output goes to $+\infty$ or $-\infty$, written e.g. $\lim_{x\to\infty} p(x) = \infty$. Which way depends entirely on the **leading term** $a_n x^n$, because for large $|x|$ it **dominates** all lower-degree terms: the sign of a_n and whether n is even or odd fix both ends.

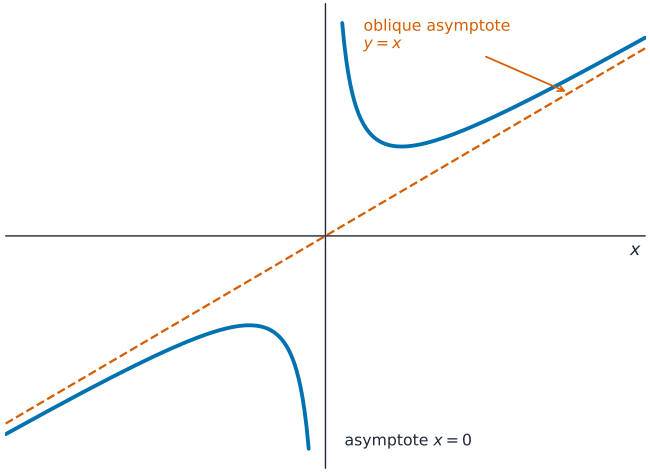
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Every polynomial's ends are decided by its degree's parity and the sign of its leading coefficient

Rational Functions and End Behavior

A **rational function** 有理函数 is a quotient of two polynomials, $r(x) = \dfrac{\text{numerator}}{\text{denominator}}$. Its end behavior is governed by the **quotient of the leading terms**:



When the numerator degree is one more than the denominator degree, the curve approaches a slanting asymptote

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- **Numerator degree > denominator degree:** the quotient is a nonconstant polynomial, and r follows that polynomial's end behavior. If that quotient is linear, the graph has a **slant asymptote** 斜渐近线.
- **Equal degrees:** the quotient is a constant, giving a **horizontal asymptote** 水平渐近线 $y =$ the ratio of leading coefficients.
- **Numerator degree < denominator degree:** the quotient tends to 0, so the horizontal asymptote is $y = 0$.

At a horizontal asymptote $y = b$, the outputs get and stay arbitrarily close to b: $\lim_{x\to\pm\infty} r(x) = b$.

> **Worked example: limits and an equation.** Let $r(x) = \dfrac{3x^2 - 5}{x^2 + 2}$.

- End behaviour: equal degrees, leading coefficients 3 and 1, so $\lim\limits_{x\to\infty} r(x) = 3$ and $\lim\limits_{x\to-\infty} r(x) = 3$.
- Solve $r(x) = 1$. The denominator $x^2 + 2$ is never zero, so multiply both sides by it:

$$3x^2 - 5 = x^2 + 2 \quad \Rightarrow \quad 2x^2 = 7 \quad \Rightarrow \quad x = \pm\sqrt{3.5} \approx \pm 1.871$$

For $g(x) = \dfrac{x^3}{x^2 + 1}$, the numerator's degree is higher, so g is unbounded: $\lim\limits_{x\to\infty} g(x) = \infty$ and $\lim\limits_{x\to-\infty} g(x) = -\infty$. Write "the limit is infinity", not "the limit is a number".

Rational Functions and Zeros

The real **zeros** of a rational function are the real zeros of its **numerator** that are still in the domain. These zeros, together with the zeros of the denominator, split the number line into the intervals you test when solving inequalities $r(x) \geq 0$ or $r(x) \leq 0$.

> **Worked example.** Solve $\dfrac{x - 2}{x + 1} \geq 0$. The numerator is zero at $x = 2$ and the denominator at $x = -1$. Test one value in each interval:

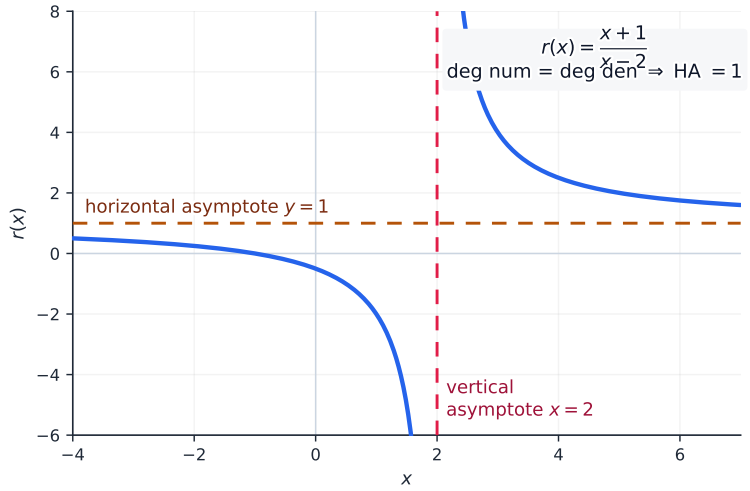
Interval	$x < -1$	$-1 < x < 2$	$x > 2$
Test value	$x = -2$: $\frac{-4}{-1} = 4$	$x = 0$: $\frac{-2}{1} = -2$	$x = 3$: $\frac{1}{4}$
Sign	$+$	$-$	$+$

So $x < -1$ or $x \geq 2$. Include $x = 2$ (the value 0 is allowed) but never $x = -1$, where r is undefined.

Rational Functions and Vertical Asymptotes

A **vertical asymptote** 垂直渐近线 occurs at $x = a$ when a is a zero of the **denominator** but not cancelled by the numerator – more precisely, when its multiplicity in the denominator exceeds its multiplicity in the numerator. Near it, the denominator is nearly zero, so r shoots to $\pm\infty$: $\lim_{x\to a^{\pm}} r(x) = \pm\infty$. Check each side, since the two sides can go opposite ways.

> **Worked example.** Describe $r(x) = \dfrac{2x^2+3}{x^2-1}$. The numerator and denominator have **equal degree**, so the horizontal asymptote is $y = \dfrac{2}{1} = 2$. The denominator $x^2 - 1$ is zero at $x = \pm 1$ and neither cancels, so there are **vertical asymptotes** at $x = 1$ and $x = -1$.



A rational function approaching a vertical asymptote and a horizontal asymptote

Rational Functions and Holes

A **hole** 空洞 (removable point) occurs at $x = c$ when a factor cancels – the multiplicity of the zero c in the numerator is at least its multiplicity in the denominator. The graph is missing a single point. Its height is the limit of the simplified function: if inputs near c give outputs near L, the hole is at (c, L), and $\lim_{x\to c} r(x) = L$.

> **Worked example.** Find all the key features of $r(x) = \dfrac{x^2-4}{x^2-x-2}$. Factor first:

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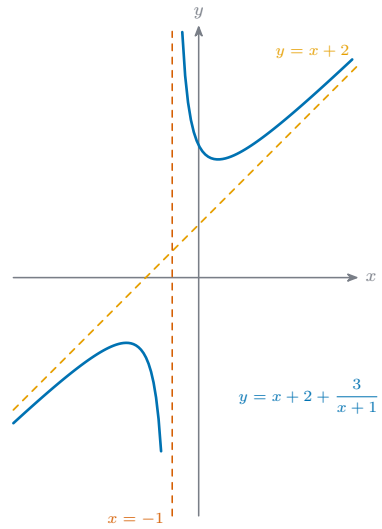
$$r(x) = \frac{(x-2)(x+2)}{(x-2)(x+1)} = \frac{x+2}{x+1} \quad (x \neq 2).$$

- **Hole** at $x = 2$: the factor $(x-2)$ cancels. Its height comes from the simplified form, $\frac{2+2}{2+1} = \frac{4}{3}$, so the hole is at $\left(2, \frac{4}{3}\right)$.
- **Vertical asymptote** $x = -1$: a zero of the denominator that does not cancel.
- **Zero** at $x = -2$, so the x-intercept is $(-2, 0)$.
- **Horizontal asymptote** $y = 1$: equal degrees, leading coefficients $\frac{1}{1}$.

A common error is to call $x = 2$ a vertical asymptote because it makes the original denominator zero: check whether the factor cancels first.

Equivalent Representations of Polynomial and Rational Expressions

The same expression, written differently, reveals different features:



Long division exposes the slant asymptote.

- **Factored form** shows real zeros, and hence x-intercepts, holes, vertical asymptotes, and domain.
- **Standard form** (expanded) shows the degree and leading term, and hence end behavior.

Polynomial long division 多项式长除法 rewrites $f(x) = g(x)\,q(x) + r(x)$, where q is the **quotient** 商 and r the **remainder** 余数 (of smaller degree than g). The quotient

gives the equation of a **slant asymptote** when the numerator's degree is one more than the denominator's.

Worked example. Divide: $x^2+3x+5=(x+1)(x+2)+3$, so

$$\frac{x^2+3x+5}{x+1}=x+2+\frac{3}{x+1}.$$

As $x\to\pm\infty$ the remainder term $\frac{3}{x+1}$ tends to 0, so the graph approaches the slant asymptote $y=x+2$.

The binomial theorem. Multiplying out $(a+b)^n$ term by term is slow. The **binomial theorem** 二项式定理 uses row n of **Pascal's triangle** 杨辉三角 as the coefficients. Each entry is the sum of the two entries above it:

n	**coefficients**
2	1, 2, 1
3	1, 3, 3, 1
4	1, 4, 6, 4, 1
5	1, 5, 10, 10, 5, 1

The powers of a fall from n to 0 while the powers of b rise from 0 to n.

Worked example. Expand $(x+2)^4$. Use row 4:

$$(x+2)^4=x^4+4x^3(2)+6x^2(2)^2+4x(2)^3+2^4=x^4+8x^3+24x^2+32x+16$$

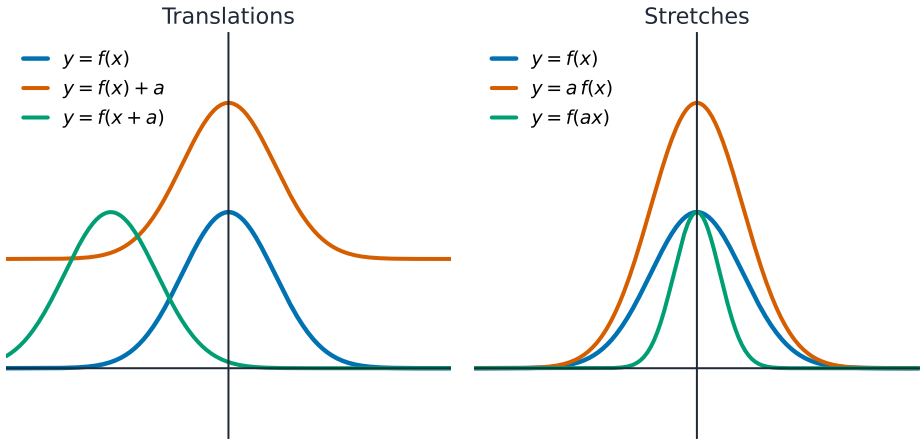
For $(2x-1)^3$, keep the bracket on each term: $(2x)^3+3(2x)^2(-1)+3(2x)(-1)^2+(-1)^3=8x^3-12x^2+6x-1$.

Transformations of Functions

A **transformation** 变换 builds a new function from an old one f:

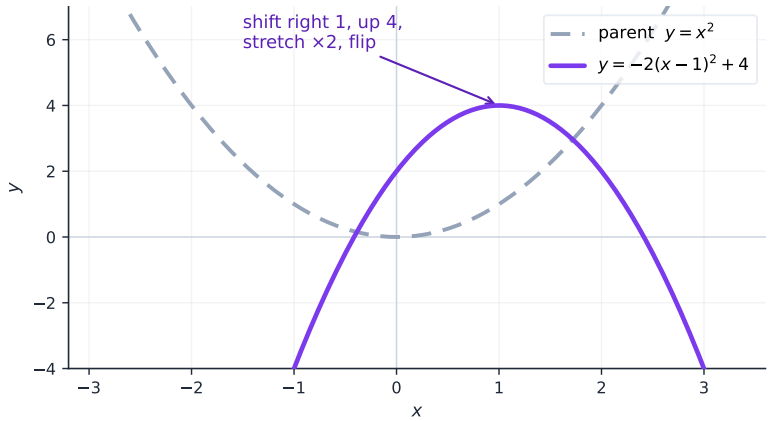
- **Translations** 平移 (shifts): $f(x)+k$ moves up/down; $f(x-h)$ moves right/left.
- **Dilations** 伸缩 (stretches): $a\,f(x)$ stretches vertically by a factor of $|a|$; $f(bx)$ stretches horizontally by a factor of $\frac{1}{|b|}$ (so $f(2x)$ squeezes the graph to half its width).
- **Reflections** 反射: $-f(x)$ flips over the x-axis; $f(-x)$ flips over the y-axis.

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Adding to the output or input shifts the curve; a multiplier stretches it

Combine additive shifts and multiplicative stretches to model shifted, scaled versions of a known shape.



$y=a\,f(x-h)+k$ · $|a|$ stretch, $a<0$ flip, h horizontal, k vertical

Transforming the parent parabola: shift, stretch, and reflect

Function Model Selection and Assumption Articulation

Choosing a **model** 模型 means matching a function type to how a quantity changes. A linear model fits a constant rate of change; a quadratic fits a constant second difference; a polynomial fits data with several turns. In geometric situations the number of dimensions is a strong hint: quantities about **area** (two dimensions) are typically **quadratic**, and quantities about **volume** (three dimensions) are typically **cubic** – for example a box's volume as a function of a cut length. State the **assumptions** 假设 your model relies on (for example, that the pattern continues), because a model is only valid where those assumptions hold.

Function Model Construction and Application

To **construct** a model, use given features – points, intercepts, zeros with multiplicities, end behavior – to write the function, then use it to answer questions in context. Always check the answer against the **domain** that makes sense for the situation, and interpret outputs with their real-world units.

A **piecewise-defined function** 分段函数 uses different rules over non-overlapping intervals of the domain. To evaluate it, pick the branch whose interval contains the input. For example,

$$f(x) = \begin{cases} x^2, & x < 0 \\ 2x, & x \geq 0 \end{cases}$$

gives $f(-3) = 9$ from the first branch but $f(3) = 6$ from the second – useful when behaviour changes at a threshold (a tiered price, a speed limit that changes).

A polynomial of degree n or less can pass through any $n + 1$ points with different inputs, so three data points fix a quadratic.

■ Worked example: a quadratic model from three data points

This follows the AP free-response pattern. The total plays of a song, in thousands, are 0 at release ($t = 0$), 36 after 2 months and 56 after 4 months. The model is $D(t) = at^2 + bt + c$.

(a) Three equations. Substitute each data point:

$$D(0) = c = 0$$

$$D(2) = 4a + 2b + c = 36$$

$$D(4) = 16a + 4b + c = 56$$

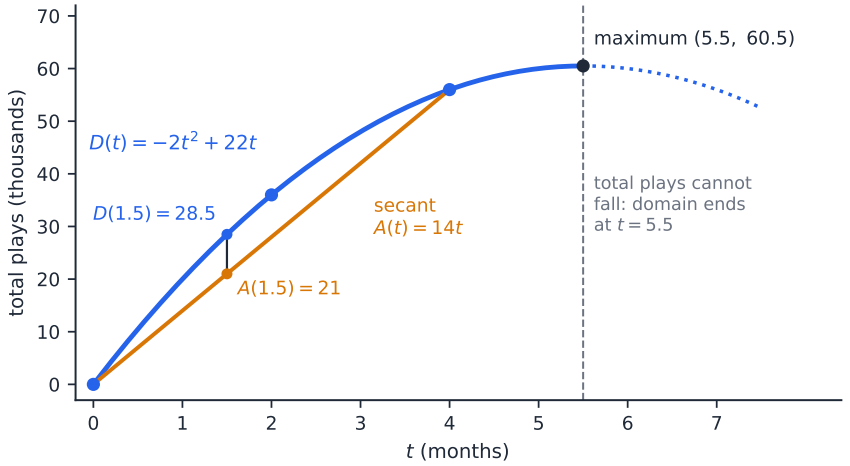
(b) Solve. With $c = 0$: $4a + 2b = 36$ and $16a + 4b = 56$. Doubling the first gives $8a + 4b = 72$. Subtracting, $8a = -16$, so $a = -2$ and then $b = 22$. The model is $D(t) = -2t^2 + 22t$.

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(c) Average rate of change and a secant estimate. From $t = 0$ to $t = 4$:

$$\frac{D(4) - D(0)}{4 - 0} = \frac{56 - 0}{4} = 14 \text{ thousand plays per month}$$

The secant line is $A(t) = 0 + 14t$, so the estimate for $t = 1.5$ is $A(1.5) = 21$ thousand plays. The model gives $D(1.5) = -2(2.25) + 33 = 28.5$.



On $0 < t < 4$ the concave-down graph of D lies above its secant line, so every secant estimate is too low. The model's maximum marks where it stops making sense.

(d) Why the estimate is too low. D is concave down ($a < 0$), so on $0 < t < 4$ its graph lies **above** the secant line through $(0, 0)$ and $(4, 56)$. Every secant estimate there is below the model: $A(t) < D(t)$. For a concave-up model the secant lies above the graph, and the estimates are too high.

(e) A domain restriction. The vertex is at $t = -\frac{b}{2a} = \frac{22}{4} = 5.5$, where $D(5.5) = 60.5$ thousand. Total plays can never decrease, but the model decreases after $t = 5.5$. So the model can only apply for $0 \leq t \leq 5.5$: its maximum gives the boundary of a sensible domain.

Exam tips

- Read a function through its **rate of change**: average rate = secant slope; a constant rate means linear, a linear-changing rate means quadratic.
- Factor a polynomial to find **zeros** (x-intercepts) and their multiplicity; even multiplicity touches, odd crosses the axis.
- End behaviour is set by the **leading term**; a rational function's asymptotes come from the degrees of top and bottom.

- Build models from key features (points, zeros, end behaviour) and state your assumptions.
- Describe transformations precisely: shift $f(x-h)+k$, stretch $af(bx)$, reflect $-f(x)$ / $f(-x)$.

Common mistakes.

- Calling a function concave up because it is increasing. Concavity is about whether the **rate** of change increases.
- Explaining concavity from a table without quoting the changes in output over equal input intervals.
- Calling every original denominator zero a vertical asymptote. Factor and cancel first: a hole occurs only if no factor for that zero remains in the denominator.
- Writing $\lim_{x\to\infty} r(x)$ for a rational function with a higher-degree numerator as a number. It is ∞ or $-\infty$.
- Dropping the sign or coefficient inside a binomial: in $(2x-1)^3$ every term uses $(2x)$ and (-1).
- Using a model outside its sensible domain, for example past a maximum when the quantity cannot decrease.

Practice.

1. Values of f are $5, 8, 13, 20, 29$ at $x = 0, 1, 2, 3, 4$. (a) Is a linear or a quadratic model more appropriate? Give a reason using the table. (b) Is the graph concave up or concave down? (c) Find the quadratic through these points.
2. A car's value, in thousands of dollars, is 24 when new ($t = 0$), 16 after 2 years and 10 after 4 years. It is modeled by $V(t) = at^2 + bt + c$. (a) Write three equations for a, b and c. (b) Solve them. (c) Find the average rate of change from $t = 0$ to $t = 4$. (d) Use it to estimate the value at $t = 1$, and compare with $V(1)$. Explain why the secant estimate is too high or too low for $0 < t < 4$. (e) Explain how the model's minimum limits its domain.
3. For $r(x) = \dfrac{x^2 - 9}{x^2 - 2x - 3}$, find any holes, vertical asymptotes, zeros and horizontal asymptote. Then solve $r(x) = 2$.
4. Expand $(x - 3)^4$.
5. Write the end behaviour with limit notation: (a) $p(x) = -2x^5 + x^2$; (b) $q(x) = \dfrac{4x - 1}{2x + 5}$.
6. A cubic p has a zero of multiplicity 2 at $x = 1$, a zero at $x = -2$ and $p(0) = 4$. Find $p(x)$.
7. Solve $\dfrac{x + 3}{x - 1} < 0$.

Answers.

1. (a) Quadratic: the first differences 3, 5, 7, 9 are not constant, but the second differences are all 2. (b) Concave up: outputs increase by increasing amounts over equal steps. (c) $c = 5$; $a + b = 3$ and $4a + 2b = 8$ give $a = 1$, $b = 2$: $f(x) = x^2 + 2x + 5$.
2. (a) $c = 24$; $4a+2b+c = 16$; $16a+4b+c = 10$. (b) $4a+2b = -8$ and $16a+4b = -14$ give $8a = 2$: $a = 0.25$, $b = -4.5$, $c = 24$. (c) $\dfrac{10 - 24}{4} = -3.5$ thousand dollars per year. (d) $A(1) = 24 - 3.5 = 20.5$; $V(1) = 0.25 - 4.5 + 24 = 19.75$. V is concave up ($a > 0$), so its graph lies below the secant on $0 < t < 4$ and the secant estimate is too high. (e) The vertex is at $t = \dfrac{4.5}{0.5} = 9$. After that the model's value rises, which does not fit an ageing car, so the model applies only for $0 \le t \le 9$.
3. $r(x) = \dfrac{(x-3)(x+3)}{(x-3)(x+1)} = \dfrac{x+3}{x+1}$ for $x \neq 3$. Hole at $\left(3, \frac{6}{4}\right) = (3, 1.5)$; vertical asymptote $x = -1$; zero $x = -3$; horizontal asymptote $y = 1$. $\dfrac{x+3}{x+1} = 2$ gives $x + 3 = 2x + 2$, so $x = 1$ (allowed, since $x \neq 3, -1$).
4. Row 4 is 1, 4, 6, 4, 1: $x^4 - 12x^3 + 54x^2 - 108x + 81$.
5. (a) $\lim\limits_{x\to\infty} p(x) = -\infty$ and $\lim\limits_{x\to-\infty} p(x) = \infty$. (b) $\lim\limits_{x\to\pm\infty} q(x) = \frac{4}{2} = 2$.
6. $p(x) = a(x-1)^2(x+2)$ and $p(0) = 2a = 4$, so $a = 2$: $p(x) = 2(x-1)^2(x+2)$.
7. The sign changes at $x = -3$ and $x = 1$; testing $x = 0$ gives $\frac{3}{-1} < 0$. Solution: $-3 < x < 1$.

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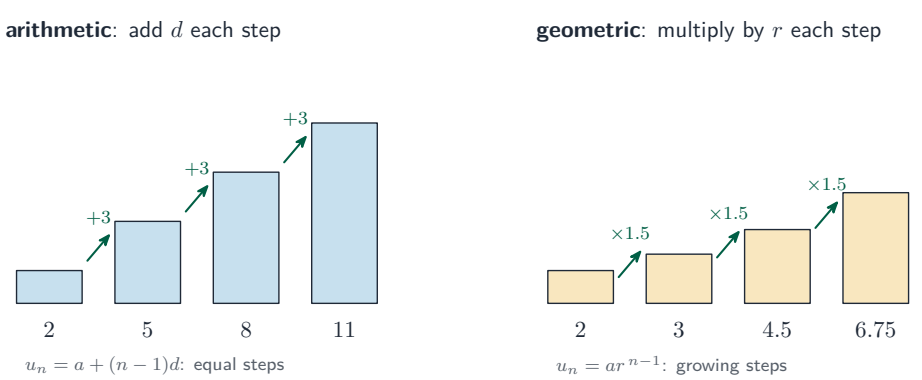
Exponential and Logarithmic Functions

AP Precalculus

Change in Arithmetic and Geometric Sequences

Why it matters. A video that gets 1,000 views on its first day and then 50% more views each day than the day before would pass 1 million daily views in under three weeks. Growth that multiplies, rather than adds, surprises people again and again: in savings, in viral posts, in an epidemic's early days. This unit compares adding (arithmetic, linear) with multiplying (geometric, exponential), then uses logarithms to undo exponential growth and answer "how long until…?" questions.

A **sequence** 数列 is a function from the whole numbers to the real numbers, so its graph is **discrete points**, not a curve.



An arithmetic sequence climbs in equal steps; a geometric one multiplies by a ratio

- An **arithmetic sequence** 等差数列 has a **common difference** 公差 d (a constant rate of change): $a_n = a_0 + dn$, or from a known term, $a_n = a_k + d(n-k)$.
- A **geometric sequence** 等比数列 has a **common ratio** 公比 r (a constant proportional change): $g_n = g_0\, r^{\,n}$, or $g_n = g_k\, r^{\,(n-k)}$.

An increasing arithmetic sequence changes by the **same positive amount** each step. For a **positive geometric starting value** and $r > 1$, successive increases get larger. A negative starting value with $r > 1$ instead becomes more negative; with a positive start and $0 < r < 1$, the sequence decreases toward zero.

> **Worked example.** An arithmetic sequence with $a_0 = 3$ and $d = 5$ has $a_4 = 3 + 5(4) = 23$. A geometric sequence with $g_0 = 2$ and $r = 3$ has $g_4 = 2 \cdot 3^4 = 162$ – addition versus multiplication makes the geometric one pull far ahead.

Check the starting index. The example above numbers its first term 0. If 3 is instead called a_1, the same arithmetic list is $a_n = 3+5(n-1)$ and its fourth numbered term is $a_4 = 18$. The original $a_4 = 23$ is the fifth listed value. Count the number of changes from the known term, rather than memorising one exponent.

An **explicit sequence rule** 数列显式公式 is a formula that gives a term directly from its index. A **recursive sequence rule** 数列递推公式 is a starting value together with a formula for the next term.

> **Worked example: recursive form.** Start with $g_0 = 2$, then use $g_n = 3g_{n-1}$ for $n \geq 1$. This gives the same $g_n = 2 \cdot 3^n$ as the original worked example.

A second comparison. With first terms numbered 1, consider $u_n = 2+2(n-1)$ and $v_n = 2^{n-1}$. At term 5 the values are 10 and 16. Their first five terms are $2, 4, 6, 8, 10$ and $1, 2, 4, 8, 16$. With a positive geometric starting value and $r > 1$, eventual dominance over a fixed arithmetic rule follows from the growth rates, not from seeing a few plotted terms.

> **Try it.** An arithmetic sequence has $a_4 = 17$ and $d = 3$. Find a_9. A geometric sequence has $g_2 = 6$ and $r = 2$. Find g_6. Use $n - k$ changes in each case.

Check. $a_9 = 17 + 3(9-4) = 32$; $g_6 = 6 \cdot 2^{6-2} = 96$.

Change in Linear and Exponential Functions

A quantity is 12 at input 2. A linear model adds 3 per input unit, while an exponential model multiplies by 1.5 per unit. At input 4 they predict $12 + 3(4-2) = 18$ and $12(1.5)^{4-2} = 27$. Both models pass through the known point, but one observation does not choose between them.



Distance covered, elapsed time and speed are different quantities; a model must identify its input, output and units.

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- A **linear function** 线性函数 $f(x) = y_0 + m(x - x_0)$ has constant output differences over equal input intervals. From two distinct inputs, $m = (y_2 - y_1)/(x_2 - x_1)$.
- For positive outputs, an exponential model $f(x) = y_0 b^{x-x_0}$ has constant output ratios over equal input intervals. From two observations, $b = (y_2/y_1)^{1/(x_2-x_1)}$. Attach the input unit to this factor.

Compare equal intervals. At inputs $0, 2, 4$, outputs $5, 9, 13$ have a constant two-unit difference of 4, so the linear slope is 2 per unit. Outputs $5, 20, 80$ have a constant two-unit factor of 4, so the positive per-unit factor is 2. Ratios or differences from unequal input intervals cannot be compared directly.

- **Linear functions** $f(x) = b + mx$ mirror arithmetic sequences – an initial value plus repeated addition of the slope m. Point form: $f(x) = y_i + m(x - x_i)$.
- **Exponential functions** $f(x) = ab^x$ mirror geometric sequences – an initial value times repeated multiplication by the base b. Point form: $f(x) = y_i\, b^{(x-x_i)}$.

A **domain** 定义域 states the permitted inputs. Sequences use discrete indices; a continuous function model can also use intermediate inputs, but that requires a modeling assumption. A 20% increase followed by a 20% decrease multiplies by $1.2(0.8) = 0.96$, leaving a 4% decrease overall. Both percentages refer to the value just before their change, not to a common fixed amount.

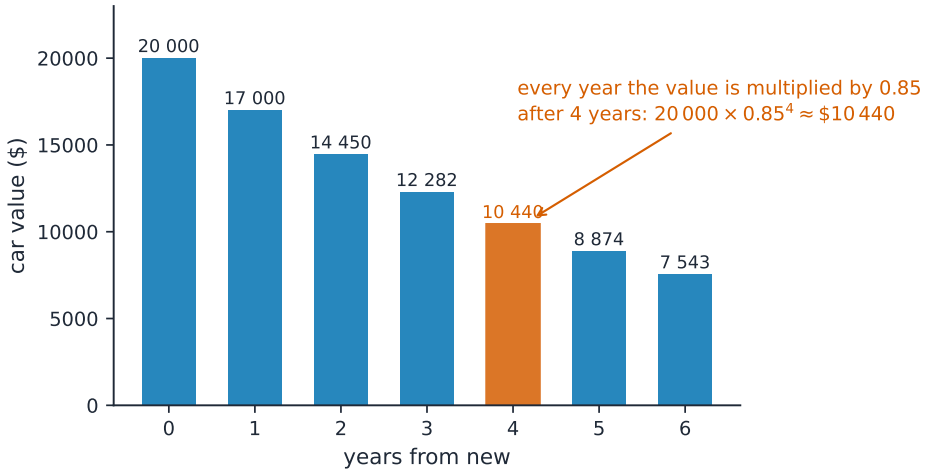
> **Try it.** A positive exponential model passes through $(2, 18)$ and $(5, 144)$. Find its per-unit factor and value at input 0. A linear model passes through the same points; find its slope.

Check. $b^3 = 144/18 = 8$, so $b = 2$ and $f(0) = 18/2^2 = 4.5$. The linear slope is $(144 - 18)/(5 - 2) = 42$ per input unit.

Exponential Functions

The general **exponential function** 指数函数 is $f(x) = ab^x$, with **initial value** 初始值 $a \neq 0$ and **base** 底数 $b > 0$, $b \neq 1$. Its mathematical domain is all real numbers, although a context may use only part of it.

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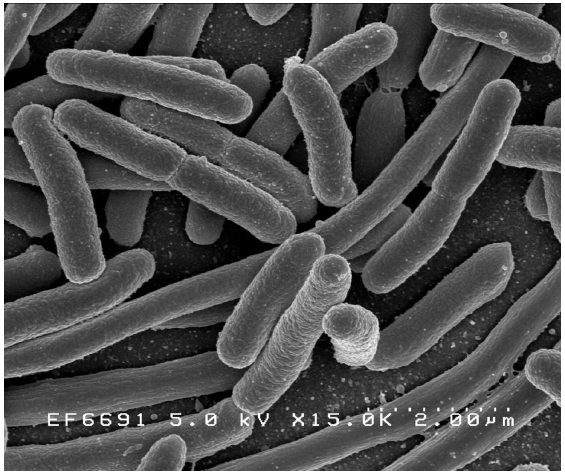
A positive decay model keeps a constant factor over equal time intervals; its output approaches zero without reaching it.

For $a > 0$, $b > 1$ gives **exponential growth** 指数增长 and $0 < b < 1$ gives **exponential decay** 指数衰减. A negative a reflects the graph across the horizontal axis. The four cases are:

Start	Base	Direction	Concavity
$a > 0$	$b > 1$	Increasing	Up
$a > 0$	$0 < b < 1$	Decreasing	Up
$a < 0$	$b > 1$	Decreasing	Down
$a < 0$	$0 < b < 1$	Increasing	Down

The outputs never change sign. The range is $y > 0$ for $a > 0$ and $y < 0$ for $a < 0$. The horizontal asymptote is $y = 0$: for $b > 1$ it is approached as $x \to -\infty$, while for $0 < b < 1$ it is approached as $x \to \infty$. At the opposite end, the magnitude increases without bound.

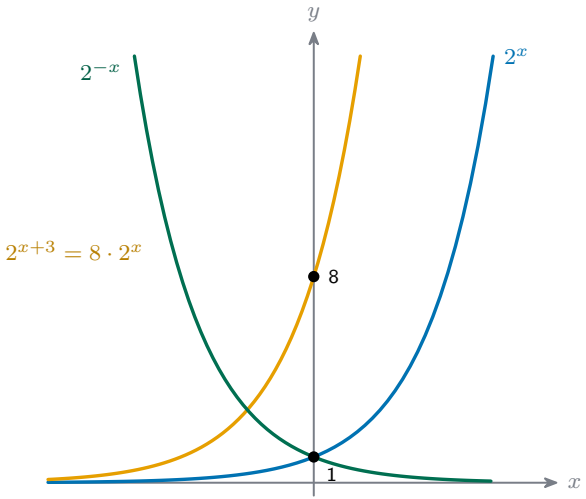
A nonconstant exponential is strictly monotonic and has no turning points or inflection points. On a closed interval its maximum and minimum occur at the endpoints. The cases $a = 0$ or $b = 1$ give constants and are separate from this nonconstant family. For a shifted model $f(x) = k + ab^{x-h}$, apply these tests to $f(x) - k$; the asymptote is $y = k$.



Population growth is approximately exponential only while a constant-factor assumption remains suitable.

Exponential Function Manipulation

For a positive base, exponent rules relate equivalent expressions. Keep a negative coefficient separate from a base: -2^x means $-(2^x)$, whereas $(-2)^x$ is not real for every real input.



A shift left by 3 is a stretch by 8.

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- **Product:** $b^u b^v = b^{u+v}$. For example, $2^{x+3} = 8(2^x)$ is both a shift left by 3 of 2^x and a vertical stretch by 8.
- **Power:** $(b^u)^v = b^{uv}$. Thus $2^{3x} = 8^x$ compresses the graph horizontally by a factor of 3; $2^{x/3}$ stretches it horizontally by 3.
- **Negative exponent:** $b^{-x} = 1/b^x = (1/b)^x$. This reflects the graph of b^x across the vertical axis; it is not a negative output.
- **Unit-fraction exponent:** $b^{1/k} = \sqrt[k]{b}$ for a positive integer k; this is the kth **root** 方根. More generally, $b^{m/k} = \sqrt[k]{b^m}$.

> **Worked example: preserve the order of transformations.** For $F(x) = -3 \cdot 2^{2(x-1)} + 5$, use the adjusted input $2(x-1)$, multiply the exponential output by -3, then add 5. The asymptote is $y = 5$, the range is $y < 5$, and $F(1) = -3 + 5 = 2$. Doubling the adjusted input is not the same as doubling the whole function output.

Change the input unit. Since $b^x = e^{x \ln b}$, $45(2)^t = 45e^{(\ln 2)t}$. If u counts minutes instead of hours, $t = u/60$ and the exponent is $(\ln 2)u/60$. An equivalent form changes the coefficient in the exponent with the input units.

> **Try it.** Rewrite $3 \cdot 4^{x-2}$ as $A \cdot 4^x$, and rewrite $2^{x/2}$ with exponent x.

Check. $A = 3/4^2 = 3/16$; $2^{x/2} = (\sqrt{2})^x$.

> **Worked example: one power.** Rewrite $k(x) = 5^{(2x-1)} \cdot 5^x$ in the form $a \cdot b^x$.

$$k(x) = 5^{(2x-1)+x} = 5^{3x-1} = 5^{-1} \cdot (5^3)^x = \frac{1}{5} \cdot 125^x$$

So $a = \frac{1}{5}$ and $b = 125$.

Exponential Function Context and Data Modeling

A culture has 180 cells at hour 2 and 1440 cells at hour 5. An exponential model assumes a **constant proportion** of change over equal-length input intervals. The observations span three hours, so their ratio is a three-hour factor, not an hourly factor.

Worked example: construct a model from two observations

For positive counts, write $N(t) = ab^t$ with $a > 0$ and $b > 0$. Substitute the observations and divide to remove a:

$$\begin{aligned} 180 &= ab^2, \\ 1440 &= ab^5, \\ b^{5-2} &= \frac{1440}{180} = 8, \\ b &= \sqrt[3]{8} = 2, \\ a &= \frac{180}{2^2} = 45. \end{aligned}$$

Thus $N(t) = 45(2)^t$. Check both observations: $N(2) = 180$ and $N(5) = 1440$. At hour 4 it predicts $N(4) = 720$. The initial value is 45 at the chosen hour 0, not the first recorded count of 180. It is **inferred**, rather than measured.

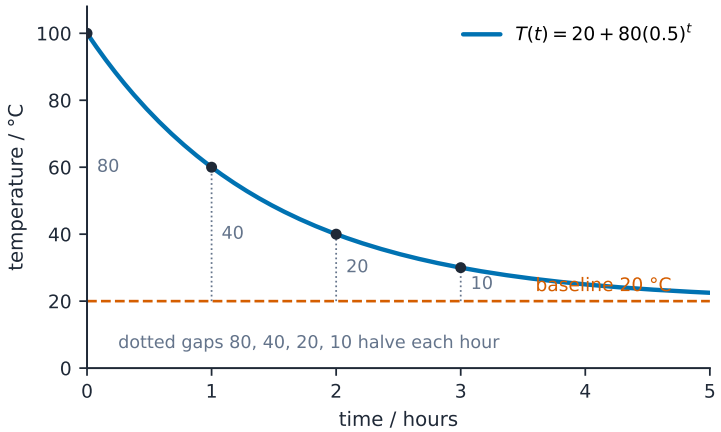
Keep the input interval attached to the factor

The hourly factor 2 means a 100% increase per hour. The three-hour factor 8 means a 700% increase across three hours. If $u = t - 2$ counts hours since the first observation, use $N = 180(2)^u$. If $s = t/3$ counts three-hour periods from the original origin, use $N = 45(8)^s$. These are descriptions of the same model with different input labels or units.

Worked example: adjust for a nonzero background. A drink cools in a room held at 20°C. The raw temperature ratios are not constant, but subtracting the **baseline** 基准值 of 20 reveals a proportional pattern:

Time (hours)	Temperature (°C)	Temperature minus 20 (°C)
0	100	80
1	60	40
2	40	20
3	30	10

A baseline is the fixed background value from which the modeled difference is measured. The differences halve each hour, giving $T(t) = 20 + 80(0.5)^t$. Using just hours 0 and 2, the adjusted ratio gives



Subtract the 20 °C baseline: the gaps above it halve each hour

$$\begin{aligned} b^2 &= \frac{40 - 20}{100 - 20} = \frac{1}{4}, \\ b &= \frac{1}{2}, \\ T(0) &= 20 + 80 = 100. \end{aligned}$$

Check the distinction: For a shifted model $y = c + ab^t$, the initial output is $c + a$, not a. The baseline c must be known or separately estimated: two readings alone do not determine all three parameters. Subtract a positive background, or equivalently add its negative, before looking for constant ratios. Here the model approaches 20°C rather than zero. Counting time from the hour-2 reading gives the equivalent form $T = 20 + 20(0.5)^u$, with $u = t - 2$.

Natural-base form and noisy data. The **natural base** 自然底数 is $e \approx 2.718$. Since $b = e^{\ln b}$, the same culture model is $N = 45e^{(\ln 2)t}$. In ae^{kt}, $k = \ln b$; it is not the ordinary fractional increase $b - 1$. Here $k \approx 0.693$ per hour, while the one-hour increase is 100%.

Use **exponential regression** 指数回归 with technology when measurements are noisy, then check the predictions against all the data. A log-linear fit regresses $\ln y$ on the input for positive y; with a known baseline, apply it to $\ln(y - c)$ only when $y - c > 0$. A two-point fit can be exact even when an exponential is unsuitable. Population forecasts need continued growth conditions; the cooling model needs a stable room temperature. Justify predictions beyond the observations.

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Try it. A culture has 81 cells at hour 3 and 729 cells at hour 5. Assuming $N = ab^t$, find the hourly factor, the inferred count at hour 0 and the prediction at hour 4. For $T(t) = 18 + 62(0.5)^t$, find the initial temperature and the temperature at hour 2.

Check. For the culture, $b^2 = 729/81 = 9$, so $b = 3$ and $a = 81/3^3 = 3$. Hence $N(4) = 3(3)^4 = 243$ cells. For the drink, $T(0) = 80°\text{C}$ and $T(2) = 18 + 62/4 = 33.5°\text{C}$.

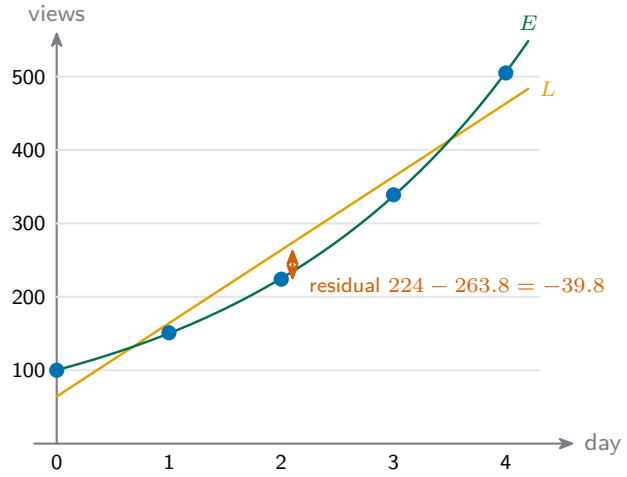
Worked example: two points. An exponential model $f(x) = ab^x$ passes through $(1, 6)$ and $(3, 54)$. Divide the two equations so that a cancels:

$$\frac{ab^3}{ab^1} = \frac{54}{6} \;\Rightarrow\; b^2 = 9 \;\Rightarrow\; b = 3 \;(b > 0).$$

Then $ab = 6$ gives $a = 2$, so $f(x) = 2 \cdot 3^x$. Check: $f(3) = 2 \cdot 27 = 54$.

Competing Function Model Validation

Several families can appear close to the same data over a short interval. Construct candidates on the **same observations**, compare their errors, and use context to judge their applicability.



A residual is observed minus predicted.

Worked example: compare three regressions. A video has 100, 151, 224, 339 and 505 views on days $x = 0, 1, 2, 3, 4$. Technology gives these fitted models, with displayed coefficients rounded:

$$\begin{aligned} L(x) &= 64.2 + 99.8x, \\ Q(x) &\approx 103.0571 \\ &\quad + 22.0857x \\ &\quad + 19.4286x^2, \\ E(x) &\approx 100.2264(1.498927)^x. \end{aligned}$$

The linear and quadratic regressions minimise squared errors in the view counts. This exponential fit uses linear regression of $\ln y$ on x, then converts back; it minimises squared log errors. Different calculator fitting methods can give slightly different exponential coefficients. Keep full precision for residual calculations.

A **residual** 残差 is **observed minus predicted**. On day 2 the line predicts $L(2) = 263.8$, while the observed count is 224:

$$e = 224 - 263.8 = -39.8.$$

The negative residual means the line **overestimates** by 39.8 views. A positive residual means that an observation lies above the model and the model **underestimates** it. A **residual plot** 残差图 plots these signed differences against the input, with zero as a reference.

Day	Observed	Linear residual	Quadratic residual	Exponential residual
0	100	35.80	-3.06	-0.23
1	151	-13.00	6.43	0.77
2	224	-39.80	-0.94	-1.19
3	339	-24.60	-5.17	1.46
4	505	41.60	2.74	-0.95

Read size and pattern together. The line's residuals are positive at the ends and negative in the middle: a curved pattern suggests it misses the data's shape. Their signed total is zero, although individual errors are large. Adding absolute residuals avoids this cancellation. The totals are 154.8 views for the line, about 18.34 for the quadratic and about 4.59 for the exponential, using full-precision values.

Here the exponential has smaller errors and no clear residual pattern, consistent with roughly constant proportional growth. Pattern-free residuals **support** a model; five observations do not prove that it is the true rule or that it will forecast well far beyond days 0–4. Sketching the three residual plots from the table makes the comparison clearer than the fitted curves alone.

Use the consequence of an error. If a bus-demand model predicts 48 passengers and 55 arrive, the residual is $55 - 48 = 7$: it underestimates demand. A cautious overestimate may be preferable when too few seats would leave people behind. A useful

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recommendation states the error's direction and size, the interval and the contextual assumptions.

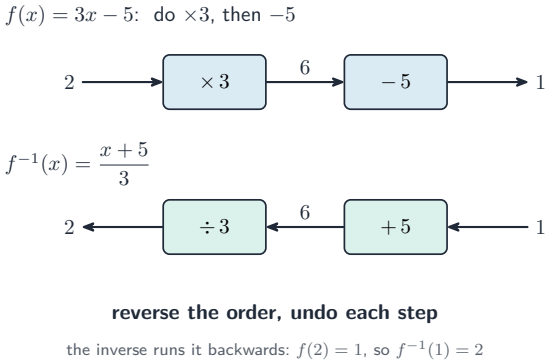
Keep eventual growth separate from present fit. A positive-start exponential with base greater than 1 eventually exceeds any fixed polynomial as the input increases without bound. This fact does not establish which family fits a particular data set best. Decay does not have that positive-growth conclusion, and a polynomial passing through every point can still give poor forecasts outside the data.

> **Try it.** A model predicts 210 when the observed value is 195. Find and interpret the residual. Model A has residuals $6, -4, 2$ and model B has $5, 5, 5$ on the same three observations. Compare their totals of absolute residuals, then name one further validity check.

Check. The residual is $195 - 210 = -15$, so the model overestimates by 15. A totals $6 + 4 + 2 = 12$; B totals $5 + 5 + 5 = 15$. A is better **by this measure on these observations**. Inspect the residual plot for a pattern and check that the family and prediction interval make sense in context before recommending it.

Composition of Functions

The **composite function** 复合函数 $(f \circ g)(x) = f(g(x))$ feeds g's output into f. An input is allowed only if it lies in g's domain and the resulting $g(x)$ lies in f's domain.



An inverse must recover the original input; an ordinary composition need not reverse a function.

> **Worked example: order and domain.** Let $f(u) = \sqrt{u}$ and $g(x) = x - 3$. Then $(f \circ g)(x) = \sqrt{x-3}$ needs $x \geq 3$, while $(g \circ f)(x) = \sqrt{x} - 3$ needs $x \geq 0$. At input 4 their outputs are 1 and -1, so composition is not generally commutative.

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- Composition is **not commutative**: $f(g(x))$ and $g(f(x))$ usually differ. For $f(x) = x^2$ and $g(x) = x + 3$: $f(g(x)) = (x+3)^2$ but $g(f(x)) = x^2 + 3$, so $f(g(2)) = 25$ while $g(f(2)) = 7$.
- To build $f(g(x))$ analytically, **substitute** $g(x)$ for every x in f.
- The **identity function** 恒等函数 $f(x) = x$ leaves any function unchanged under composition.
- A function can also be **decomposed** into simpler pieces – useful for seeing an additive shift as composing with $x + k$, or a dilation as composing with kx.

Do not lose a restriction by simplifying. Let $p(u) = u^2$ and $q(x) = \sqrt{x}$. Then $p(q(x)) = x$ only for $x \geq 0$, the original domain of q. The simplified expression alone would appear to allow every real input. Conversely, $q(p(x)) = \sqrt{x^2} = |x|$ for all real x, not x for negative inputs.

The **identity function** returns its input unchanged. On a specified domain, $I(x) = x$ and $f \circ I = f$; $I \circ f = f$ when I accepts every output of f. A **decomposition** 分解 writes a function as a composition of simpler functions. For $H(x) = \sqrt{2x+1}$, take $g(x) = 2x + 1$ and $f(u) = \sqrt{u}$, retaining $x \geq -1/2$. A different decomposition is possible, so name the inner and outer functions.



Nesting represents the order: the inner function acts before the outer function.

> **Try it.** For $f(u) = 1/u$ and $g(x) = x^2 - 4$, find $(f \circ g)(x)$ and its domain.

Check. $(f \circ g)(x) = 1/(x^2 - 4)$, with $x \neq -2, 2$. The inner expression exists for every real input, but the outer reciprocal rejects its zero outputs.

Inverse Functions

A function is **invertible** 可逆 on a domain where each output has exactly one input. Its **inverse function** 反函数 f^{-1} reverses the mapping: if $f(a) = b$, then $f^{-1}(b) = a$. The notation f^{-1} means inverse, not the reciprocal $1/f$.

- Domain and **range** 值域 swap. Reverse (a, b) to (b, a) in a table; reflect the graph across $y = x$ using equal axis units.
- Check both compositions on their proper domains: $f^{-1}(f(x)) = x$ for inputs of f, and $f(f^{-1}(y)) = y$ for inputs of f^{-1}.
- To find a formula, write $y = f(x)$, solve for x, then rename the inverse input. Retain the original domain and range while solving.

Worked example: select a branch. The function x^2 on all real inputs is not one-to-one, since 2 and -2 both give 4. Restricting to $x \geq 0$ gives inverse $\sqrt{x}$, while restricting to $x \leq 0$ gives inverse $-\sqrt{x}$; both inverse domains are $x \geq 0$. Writing $\pm\sqrt{x}$ would not define a single-valued inverse function.

Worked example: a contextual inverse. If $T(t) = 20 + 80(0.5)^t$ for $t \geq 0$, its outputs are $20 < T \leq 100$. Solving for elapsed time gives

$$T - 20 = 80(0.5)^t,$$
$$t = \frac{\ln((T-20)/80)}{\ln(0.5)}.$$

The contextual inverse accepts $20 < T \leq 100$ and returns $t \geq 0$. A temperature above 100 would give a negative time in the extended formula, outside this context; 20 is never reached at a finite time.

Try it. Find the inverse of $f(x) = 3x - 2$ on all real inputs, and verify one composition.

Check. $f^{-1}(x) = (x + 2)/3$; $f^{-1}(f(x)) = (3x - 2 + 2)/3 = x$. Both domains and ranges are all real numbers.

Worked example. Find the inverse of $f(x) = 2e^x + 1$. Write $x = 2e^y + 1$ and solve for y:

$$x - 1 = 2e^y \;\Rightarrow\; e^y = \frac{x-1}{2} \;\Rightarrow\; y = \ln\left(\frac{x-1}{2}\right).$$

So $f^{-1}(x) = \ln\left(\frac{x-1}{2}\right)$, with domain $x > 1$: the range of f (it is always above its asymptote $y = 1$) becomes the domain of f^{-1}.

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Logarithmic Expressions

The **logarithm** 对数 answers "what exponent?": $\log_b c = a$ means $b^a = c$. Real logarithms require $b > 0$, $b \neq 1$ and $c > 0$. The output can be positive, zero or negative: $\log_2 8 = 3$, $\log_2 1 = 0$ and $\log_2(1/8) = -3$.

An unwritten base in this chapter means the **common logarithm** 常用对数, base 10. The **natural logarithm** 自然对数 is $\ln x = \log_e x$, where $e \approx 2.718$. State other bases explicitly: $\log_2 16 = 4$ differs from $\log_{10} 16 \approx 1.204$.

A **logarithmic scale** 对数刻度 is a scale with equal spacing for equal factors. The values $1, 10, 100, 1000$ are equally spaced on a base-10 scale. Its midpoint between 10 and 100 is $\sqrt{1000} \approx 31.623$, not 55. Zero and negative values have no finite position on a standard real log scale.

Estimate before using technology. Since $2^3 < 12 < 2^4$, $3 < \log_2 12 < 4$. With a base between zero and one, powers decrease: $(1/2)^2 = 1/4$ and $(1/2)^3 = 1/8$, so $2 < \log_{1/2}(1/5) < 3$.

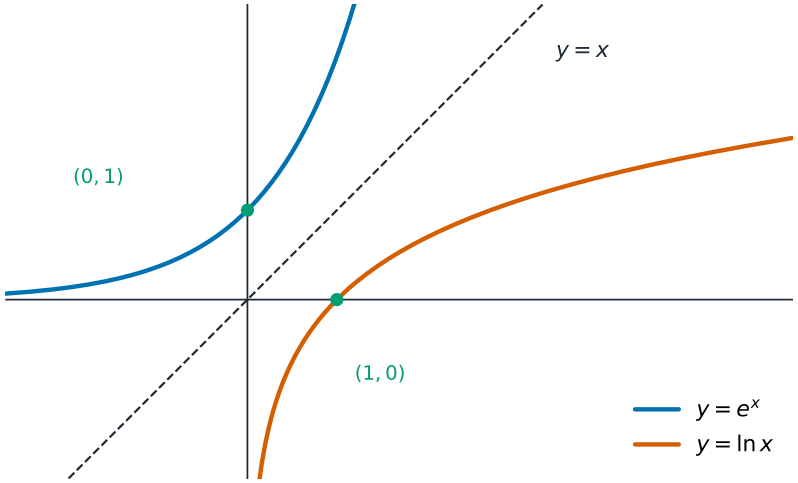
Try it. Evaluate $\log_3(1/9)$ and $\log_5 1$. Is $\log_2(-4)$ a real number?

Check. The values are -2 and 0. The last expression has no real value because a positive-base exponential cannot output -4.

Inverses of Exponential Functions

For $b > 0$, $b \neq 1$, the functions $y = b^x$ and $y = \log_b x$ undo each other. Their domains and ranges swap: the exponential accepts all real inputs and outputs positive values, while the logarithm accepts positive inputs and outputs all real values. In the identities below, the first allows every real input; the second requires $x > 0$.

$$\log_b(b^x) = x,$$
$$b^{\log_b x} = x.$$



Reflection swaps coordinates: an exponential point (u, v) becomes the logarithmic point (v, u).

Worked example: retain the scale factor. For $f(x) = 2^x$, $f(3) = 8$ gives $f^{-1}(8) = 3$. For $F(x) = 2 \cdot 2^x$, solving $y = 2 \cdot 2^x$ gives $F^{-1}(y) = \log_2(y/2)$, not $2\log_2 y$. In particular, $F(3) = 16$ and $F^{-1}(16) = 3$.

Shifted inverse forms. For $F(x) = k + ab^{x-h}$ with $a \neq 0$, isolate $(y-k)/a = b^{x-h}$:

$$F^{-1}(y) = h + \frac{\ln((y-k)/a)}{\ln b}.$$

The inverse needs $(y-k)/a > 0$, the original function's range. It accepts $y > k$ when $a > 0$ and $y < k$ when $a < 0$. Its outputs are all real numbers unless the original input was restricted by context.

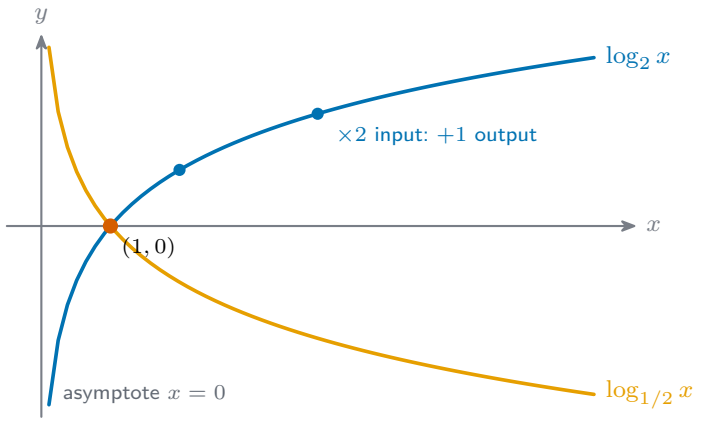
Try it. For $G(x) = 3 + 2^x$, find $G^{-1}(x)$ and its domain.

Check. $G^{-1}(x) = \log_2(x-3)$, with $x > 3$.

Logarithmic Functions

The **logarithmic function** 对数函数 $f(x) = \log_b x$, with $b > 0$, $b \neq 1$, has domain $x > 0$ and range all real numbers. It passes through $(1, 0)$ and has a vertical asymptote at $x = 0$.

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Multiplying the input by the base adds 1.

- For $b > 1$, it is increasing and concave down. As $x \to 0^+$, $f(x) \to -\infty$; as $x \to \infty$, $f(x) \to \infty$.
- For $0 < b < 1$, it is decreasing and concave up. The two limits reverse: $f(x) \to \infty$ as $x \to 0^+$ and $f(x) \to -\infty$ as $x \to \infty$.

There are no turning or inflection points. An unbounded function can still change slowly: an increasing logarithm does not have a horizontal ceiling. On a closed interval within its domain, the largest and smallest values occur at endpoints.

Read proportional inputs. Multiplying a positive input by b adds 1 to its logarithm. For $H(x) = a\log_b(x-h) + k$, multiply the adjusted input $x - h$ by b to add a to the output. Equal ordinary input increments generally do not produce equal output increments.

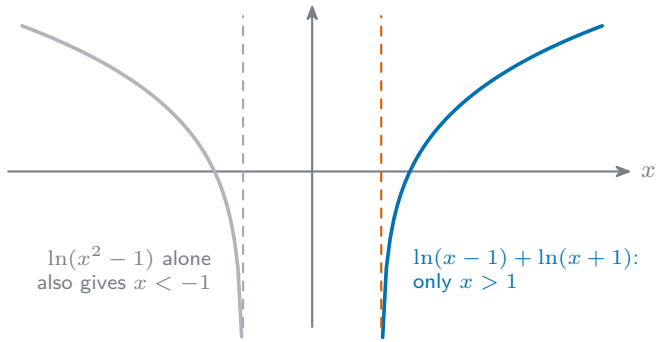
Worked example: a reflected logarithm. For $H(x) = -2\log_2(x-3) + 1$, the domain is $x > 3$, the asymptote is $x = 3$, and $H(4) = 1$. A negative coefficient reverses the direction and concavity of the increasing base-2 logarithm: this function decreases and is concave up. Its values tend to $+\infty$ near 3^+ and to $-\infty$ as $x \to \infty$.

Try it. Find the domain and value at $x = 5$ of $K(x) = \log_{1/2}(x-1)$.

Check. The domain is $x > 1$; $K(5) = \log_{1/2} 4 = -2$.

Logarithmic Function Manipulation

For $b > 0$, $b \neq 1$, $x > 0$ and $y > 0$, the log laws are



Combining logs can widen the domain: keep the original restriction.

$$\log_b(xy) = \log_b x + \log_b y,$$
$$\log_b(x/y) = \log_b x - \log_b y,$$
$$\log_b(x^r) = r\log_b x.$$

The positive-input conditions matter. For example, $\ln(x^2) = 2\ln|x|$ for $x \neq 0$; the form $2\ln x$ only covers $x > 0$. Combining $\ln(x-1)+\ln(x+1)$ gives $\ln(x^2-1)$ **with the original restriction** $x > 1$. The combined expression alone also exists for $x < -1$, which the original sum does not permit.

There is no corresponding sum law: $\log_b(x+y)$ is not generally $\log_b x + \log_b y$. At $x = y = 1$, base 10 gives $\log_{10} 2 \neq 0 + 0$.

The **change-of-base** 换底 formula uses any valid base q:

$$\log_b x = \frac{\log_q x}{\log_q b}, \qquad = \frac{\ln x}{\ln b}, \qquad x > 0.$$

Changing a base rescales the output: $\log_{b^r} x = (1/r)\log_b x$ for $r \neq 0$. Multiplying the input by $c > 0$ instead shifts the output: $\log_b(cx) = \log_b x + \log_b c$.

> **Try it.** Expand $\ln(x^2/y)$ for $x \neq 0$, $y > 0$, without losing the negative values of x.

Check. $\ln(x^2/y) = 2\ln|x| - \ln y$, retaining $x \neq 0$, $y > 0$.

> **Worked example: rewrite as a single term.** AP free-response questions ask for a simplified equivalent form. Rewrite $j(x) = \log_2(4x^3) + \log_2(8x) - 3\log_2 x$ for $x > 0$.

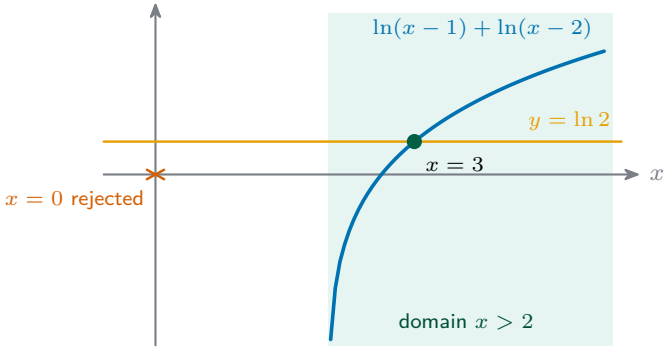
$$j(x) = \log_2\left(\frac{4x^3 \cdot 8x}{x^3}\right) = \log_2(32x) = \log_2 32 + \log_2 x = 5 + \log_2 x$$

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Combine with the product, quotient and power laws first, then evaluate any exact logarithm ($\log_2 32 = 5$). The domain $x > 0$ comes from the original expression.

Exponential and Logarithmic Equations and Inequalities

First state any input restrictions. Isolate an exponential before taking logs, or isolate a logarithm before exponentiating. **Extraneous solutions** 增根 are the candidate answers that fail the original relation or its domain conditions. Substitute candidates back into the original relation to reject them.



Check candidates against the original domain.

> **Worked example: an equation.** Solve $2 \cdot 3^x = 54$. Dividing by 2 gives $3^x = 27 = 3^3$, so $x = 3$. If powers are not tidy, $5^x = 20$ gives $x = \ln 20/\ln 5 \approx 1.861$. A relation such as $2^x = -3$ has no real solution because its left side is positive.

> **Worked example: a rejected root.** Solve $\ln(x-1) + \ln(x-2) = \ln 2$. The original domain is $x > 2$. Combining the logs gives

$$(x-1)(x-2) = 2,$$
$$x^2 - 3x = 0,$$
$$x(x-3) = 0.$$

Candidates 0 and 3 satisfy the combined algebraic equation. Only $x = 3$ lies in the original domain; $x = 0$ makes both original log arguments negative.

> **Worked example: inequality direction and endpoints.** For $\log_2(x-1) \geq 2$, the domain is $x > 1$. Base 2 is increasing, so $x - 1 \geq 4$, giving $x \geq 5$, including the endpoint. For $\log_{1/2}(x-1) \geq 2$, the logarithm decreases, so $x - 1 \leq 1/4$. Intersecting with the domain gives $1 < x \leq 5/4$: the domain boundary is excluded and the equality boundary is included.

Ordinary algebra can also reverse an inequality. For $-2 \cdot 3^x \geq -18$, dividing by -2 gives $3^x \leq 9$, hence $x \leq 2$. Keep this sign reversal separate from a reversal caused by a decreasing function.

> **Try it.** Solve $\log_2(x-2) < 1$ and $2^x \leq 0$. If a time equation gives $t = -1$, can it answer a question restricted to $t \geq 0$?

Check. The first solution is $2 < x < 4$; the second has no real solutions. A negative time must be rejected in that stated context, even when it solves an extended mathematical equation.

> **Worked example: an extraneous solution.** Solve $\log_2 x + \log_2(x-2) = 3$. Combine with the product property, then exponentiate:

$$\log_2\big(x(x-2)\big) = 3 \;\Rightarrow\; x(x-2) = 2^3 = 8 \;\Rightarrow\; x^2 - 2x - 8 = 0 \;\Rightarrow\; (x-4)(x+2) = 0.$$

So $x = 4$ or $x = -2$. Check each in the **original** equation: $x = -2$ makes $\log_2(-2)$ undefined, so reject it. The only solution is $x = 4$ (check: $\log_2 4 + \log_2 2 = 2 + 1 = 3$).

> **Worked example: an exponential equation of quadratic type.** Solve $e^{2x} - 3e^x - 10 = 0$. Since $e^{2x} = (e^x)^2$, let $u = e^x$:

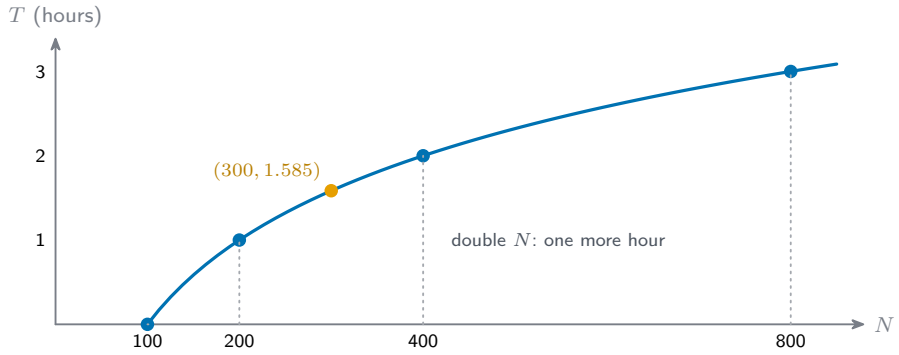
$$u^2 - 3u - 10 = 0 \quad\Rightarrow\quad (u-5)(u+2) = 0$$

So $e^x = 5$ or $e^x = -2$. An exponential output is always positive, so reject $e^x = -2$. Then $x = \ln 5 \approx 1.609$.

Logarithmic Function Context and Data Modeling

A **logarithmic model** 对数模型 is a model whose output is linear in the logarithm of its input. A doubling process can be described in two directions. If $N(t) = 100(2)^t$, amount is an exponential function of time. Solving for elapsed time instead gives a logarithmic model:

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Each doubling of the amount adds one hour.

$$\begin{aligned} T(N) &= \log_2(N/100), \\ &= \frac{\ln(N/100)}{\ln 2}. \end{aligned}$$

Here $T(100) = 0$ and each doubling of the input amount adds one hour. The mathematical domain is $N > 0$; a process starting at time zero uses $N \geq 100$. At amount 300, the model predicts about 1.585 hours, assuming continuous growth between observations.

> **Worked example: two observations.** Fix the unshifted form $y = a\log_{10} x + c$. Observations $(1, 1)$ and $(10, 3)$ give $c = 1$, $a = 2$, hence $y = 2\log_{10} x + 1$. For any fixed valid base, two distinct positive inputs give

$$\begin{aligned} a &= \frac{y_2 - y_1}{\log_b x_2 - \log_b x_1}, \\ c &= y_1 - a\log_b x_1. \end{aligned}$$

The **parameters** 参数 have specific meanings: c is the model output at input 1, and multiplying the input by b adds a to the output. The logarithm is undefined at input zero. An unknown input shift adds another parameter and needs additional information.

Fit and check noisy data. A **logarithmic regression** 对数回归 is a least-squares fit of an output against a logarithm of its positive input. It does not log the output. For illustrative observations $(1, 1.2)$, $(10, 2.6)$ and $(100, 5.2)$, the least-squares fit is $\hat{y} = 2\log_{10} x + 1$. Its residuals are $0.2, -0.4, 0.2$, with squared sum 0.24. A regression can pass through some or all observations; fitting within this family does not prove that the family is appropriate.

Separate domain from support. Interpolation 内插 is a prediction at an input between the observed input limits. **Extrapolation** 外推 is a prediction beyond the

observed input limits. With observations only at inputs 1 and 10, predicting at 5 interpolates, while predicting at 100 extrapolates. The formula gives about 2.398 at 5 and 5 at 100, but both predictions require modeling assumptions, especially beyond the observations. A score model $S(h) = 20\log_{10} h + 40$ can be a local fit; it cannot describe a score capped at 100 for every positive h, since $S(10000) = 120$.

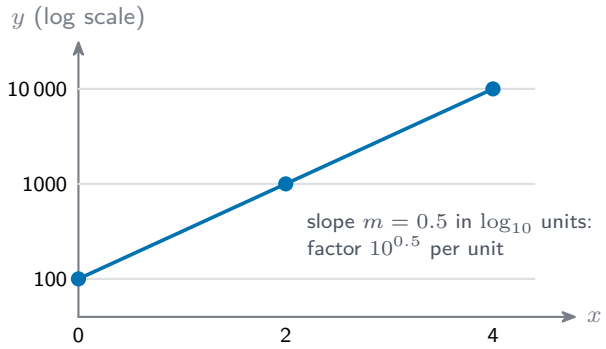
For a shifted example $P(t) = 10\ln(t+1) + 5$, the mathematical domain is $t > -1$, with $P(0) = 5$ and $P(e-1) = 15$. Proportional changes belong to the adjusted input $t+1$. Choose or estimate a shift for a contextual reason, not just to hide an invalid log argument.

> **Try it.** For $y = 3\log_{10} x + 2$, find the output at 100 and the output change when a positive input is multiplied by 10. Does a decreasing curve that flattens always justify a logarithmic model?

Check. The output is 8 and the increment is 3. Shape alone is insufficient: an exponential decay or a bounded process can also flatten. Check which variable is the input, ratios, residuals and context.

Semi-log Plots

A **semi-log plot** 半对数图 has exactly one logarithmic axis. With a linear horizontal axis and a logarithmic vertical axis, a positive exponential $y = ab^x$ gives a straight pattern:



An exponential becomes a line on a semi-log plot.

$$z = \log_q y = mx + c,$$
$$m = \log_q b,$$
$$c = \log_q a,$$
$$a = q^c,$$
$$b = q^m.$$

Here $a > 0$, $b > 0$ and the scale base $q > 1$. The transformed **slope** 斜率 describes log change per input unit. The **intercept** 截距 is the log of the original initial value, not that value itself. A log-scaled axis can label ticks with the original y values; an ordinary axis labelled $z = \log_q y$ instead displays their logged coordinates. Do not log a value twice.

> **Worked example: recover a model.** Observations $(0, 100)$ and $(4, 10000)$ become $(0, 2)$ and $(4, 4)$ when base-10 outputs are logged. Then $m = (4-2)/4 = 0.5$, $c = 2$, so $y = 100 \cdot 10^{0.5x}$. At $x = 2$, the output is 1000. The per-unit factor is $10^{0.5}$, not 0.5.

Growth and decay. For $q > 1$, a positive slope means growth, a negative slope means decay, and zero slope gives a positive constant. If $\log_{10} y = 2 - 0.25t$, the initial amount is 100 and the factor per time unit is $10^{-0.25} \approx 0.5623$, a decrease of about 43.77%. Half the initial amount is reached at $t = \log_{10} 2/0.25 \approx 1.204$ time units.

Changing the log base changes the numerical slope, not the original model. For $y = 4(2)^x$ with x in hours, the base-2 line is $z = x + 2$; base 10 gives slope $\log_{10} 2$ and intercept $\log_{10} 4$. With time in minutes, the base-2 slope becomes $1/60$ per minute. Compare slopes only with a common base and input units.

Read the orientation. If the horizontal axis is logarithmic and the vertical axis is linear, a logarithmic form $y = a\log_q x + c$ gives a line instead. A standard real log axis excludes nonpositive values. With both axes logarithmic, the graph is not semi-log.

Use evidence carefully. Two transformed points always determine a line if their inputs differ; they do not establish a reliable exponential model. Noisy exponential observations may deviate from a line. Fitting a line to $\log y$ minimises log errors rather than errors in the original output, so it is not automatically more accurate for the intended prediction. A vertical offset such as $y = 4(2)^x + 4$ generally prevents a straight log-output pattern; subtract a baseline only when justified.

> **Try it.** For $y = 4(2)^x$, find the original output and its base-2 log coordinate at $x = 3$. Can a positive constant also give a straight semi-log pattern?

Check. The original output is 32 and its transformed coordinate is 5. The axis labels distinguish them. A positive constant gives a horizontal straight line, with factor 1 and transformed slope zero.

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Worked example. On a semi-log plot (base 10), data lie on the line $\log y = 0.301 + 0.477x$. Undo each logarithm: $a = 10^{0.301} \approx 2$ and $b = 10^{0.477} \approx 3$, so the model is $y \approx 2 \cdot 3^x$. A slope of 0.477 means y is multiplied by $10^{0.477} \approx 3$ for every increase of 1 in x.

Common mistakes.

- Testing for an exponential model with differences. Exponential outputs have a constant **ratio** over equal input intervals; constant differences mean linear.
- Reading the base as a percentage directly: $y = 500(0.92)^t$ decreases by 8% per unit, not 92%.
- Writing $\log(a+b) = \log a + \log b$. The product law needs a product: $\log(ab) = \log a + \log b$.
- Forgetting that $f^{-1}(x)$ means the inverse, not $\dfrac{1}{f(x)}$.
- Composing in the wrong order: $f(g(x))$ applies g first.
- Dropping the domain check after combining logs, so an extraneous root such as $x = -3$ in $\ln(x-1) + \ln(x+1) = \ln 8$ survives.
- Keeping the inequality sign when dividing by a negative log, or when the log base lies between 0 and 1.
- Reading a semi-log slope as the growth factor. With base-10 logs, the factor is 10^{slope}.

Exam tips

- State domains and input units; compare differences or ratios over equal input intervals.
- Track the input/output order in composition, and preserve restrictions after simplifying. An inverse reverses a one-to-one mapping on its stated domain.
- Check signs, bases, shifts and contextual restrictions. Negative coefficients and bases below 1 can reverse direction in different ways.
- Use logarithm laws only where their arguments are valid; keep absolute values when needed and reject extraneous roots in the original relation.
- Distinguish a fitted model from a reliable prediction. Inspect residuals and the observed range before extrapolating.
- On semi-log axes, identify which axis is logarithmic, the scale base and the input unit before interpreting a straight line or its slope.

■ Cumulative checkpoint

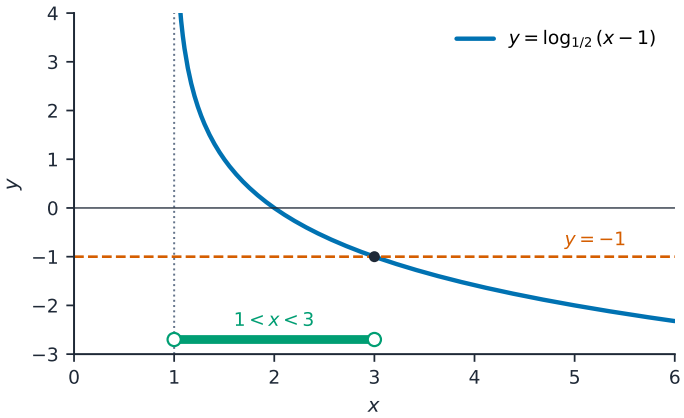
1. Let $H(x) = \sqrt{2x-4}$. State its domain, then find the inverse and the inverse's domain.
2. Solve $\ln(x-1) + \ln(x+1) = \ln 8$. State the original domain before combining logs.

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3. Solve $\log_{1/2}(x-1) > -1$. Explain both endpoints of the solution interval.
4. A base-10 semi-log model is $\log_{10} y = 0.2t + 1$, with t in hours. Recover the initial value, hourly factor and prediction at hour 5. If observations cover hours 0–5, classify a prediction at hour 6.
5. Solve $e^{2x} - e^x - 12 = 0$.
6. Rewrite $\log_3(9x^4) - \log_3(x^2) + 2\log_3 x$ as an expression in which $\log_3 x$ appears once, for $x > 0$.

Answers and reasoning.

1. H needs $2x - 4 \geq 0$, so $x \geq 2$, and has range $y \geq 0$. Solving $y^2 = 2x - 4$ gives $H^{-1}(x) = (x^2+4)/2$ for $x \geq 0$. Its domain is the original range, not all real inputs.
2. The original domain is $x > 1$. Combining gives $x^2 - 1 = 8$, hence candidates $x = \pm 3$. Only $x = 3$ is allowed.
3. The log argument requires $x > 1$. The decreasing base reverses the inequality, giving $x - 1 < 2$, so $1 < x < 3$. Input 1 is undefined and input 3 gives equality, which the strict inequality excludes.
4. $y = 10(10^{0.2})^t$: initial value 10, hourly factor $10^{0.2} \approx 1.585$, and $y(5) = 10(10) = 100$. Hour 6 is extrapolation and needs further justification.
5. With $u = e^x$: $u^2 - u - 12 = (u-4)(u+3) = 0$. Reject $u = -3$; $e^x = 4$ gives $x = \ln 4 \approx 1.386$.
6. $\log_3\left(\dfrac{9x^4 \cdot x^2}{x^2}\right) = \log_3(9x^4) = 2 + 4\log_3 x$.

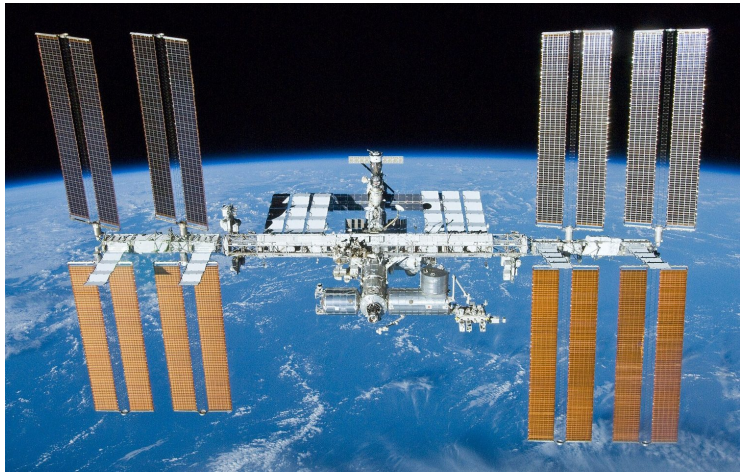


Checkpoint 3: the decreasing log lies above −1 only for $1 < x < 3$

Trigonometric and Polar Functions

AP Precalculus

Periodic Phenomena



The ISS in orbit: trigonometric and polar functions describe periodic and circular motion

Why it matters. In Beijing, the longest day of the year (around 21 June) has about 15 hours of daylight; the shortest (around 22 December) has only about 9 hours and 20 minutes. Every year the pattern repeats. Tides, a guitar string, a turning wheel and your heartbeat all repeat in the same way. This unit builds the functions that model repeating change, then extends them to angles, identities and polar graphs.

A relationship is **periodic** 周期性 if its output pattern repeats at regular input steps. The **period** 周期 is the smallest positive k with $f(x+k) = f(x)$ for all x. You can build the whole graph by copying a single **cycle** 周期段, and estimate the period by finding how far apart the pattern repeats. Within each cycle a periodic function still has intervals of increase/decrease and maxima/minima.

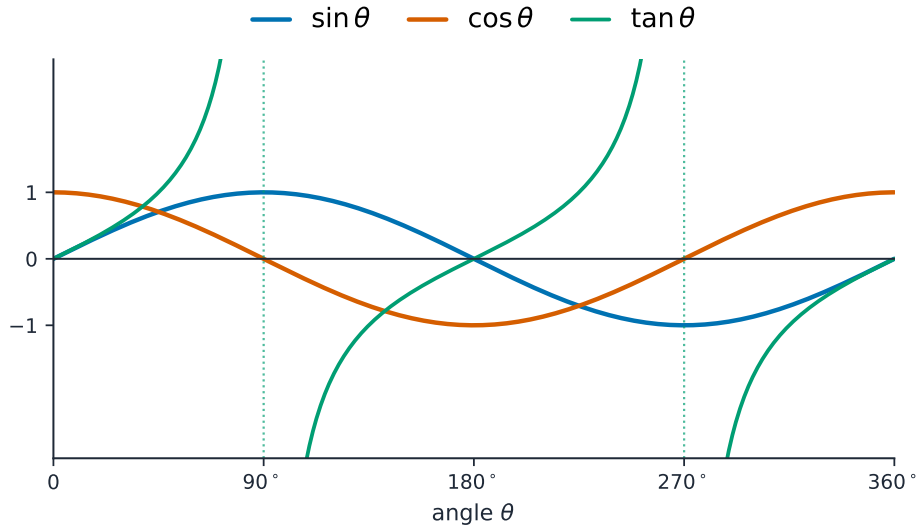
> **Worked example: build a graph from words.** A buoy rides steady waves. Its height above the sea floor is 2.0 m on each crest and 0.5 m in each trough, and it takes 4 seconds to go from one crest to the next. At $t = 0$ the buoy is on a crest.

- One cycle: high at $t = 0$, low at $t = 2$, high again at $t = 4$. The pattern from $t = 0$ to $t = 4$ is the whole story.
- To graph 12 seconds, copy that single cycle three times: highs at 0, 4, 8, 12; lows at 2, 6, 10.
- The period is 4 seconds: $f(t+4) = f(t)$ for every t.

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- Height decreases on $(0, 2)$ and increases on $(2, 4)$, and the same pattern holds on every later interval of width 4.

Estimating a period from data. Outputs measured every hour are 3, 5, 6, 5, 3, 1, 0, 1, 3, 5, 6, … The value 6 appears at hours 2 and 10, and the sequence of outputs starts to repeat from hour 8. The period is about 8 hours.



sin and cos wave between -1 and 1; tan breaks at 90 and 270 degrees



As a Ferris wheel turns steadily, your height rises and falls over and over – a real sine wave in time

Sine, Cosine, and Tangent

An angle is in **standard position** 标准位置 when its vertex is at the origin and its initial ray lies on the positive x-axis. Its **radian** 弧度 measure is the arc length on a unit circle. For the point P where the terminal ray meets a circle of radius r:

$$\cos\theta = \frac{x}{r}, \qquad \sin\theta = \frac{y}{r}, \qquad \tan\theta = \frac{y}{x} \text{ (the slope of the terminal ray).}$$

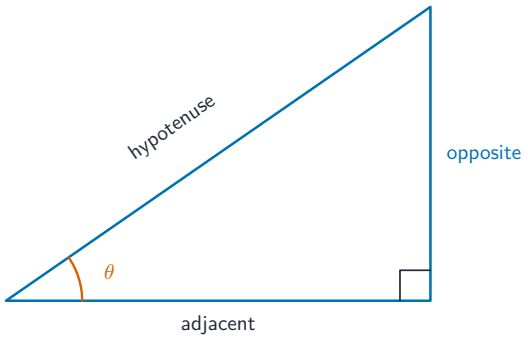
Positive angles turn **counterclockwise** from the positive x-axis; negative angles turn clockwise. One full turn is 2π radians, so $180^\circ = \pi$ radians.

> **Worked example.** The terminal ray of θ passes through $P(-3, 4)$. Find $\sin\theta$, $\cos\theta$ and $\tan\theta$.

$$r = \sqrt{(-3)^2 + 4^2} = 5$$

$$\sin\theta = \frac{4}{5}, \qquad \cos\theta = \frac{-3}{5}, \qquad \tan\theta = \frac{4}{-3} = -\frac{4}{3}$$

The point is in quadrant II, so cosine is negative and sine is positive. The tangent is the slope of the ray from $(0, 0)$ to $(-3, 4)$.



Naming the sides of a right triangle from the angle

Sine and Cosine Function Values

On a circle of radius r, $x = r\cos\theta$ and $y = r\sin\theta$. Exact values at the special angles $(0, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2}, \dots)$ come from **isosceles right** and **equilateral triangle** 等边三角形 geometry, with signs set by the **quadrant** 象限 of the angle.

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
$\sin\theta$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0

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For another angle, find its **reference angle** 参考角 (the acute angle to the x-axis), take the value from the table, then attach the sign of its quadrant: x (cosine) is negative in quadrants II and III; y (sine) is negative in quadrants III and IV.

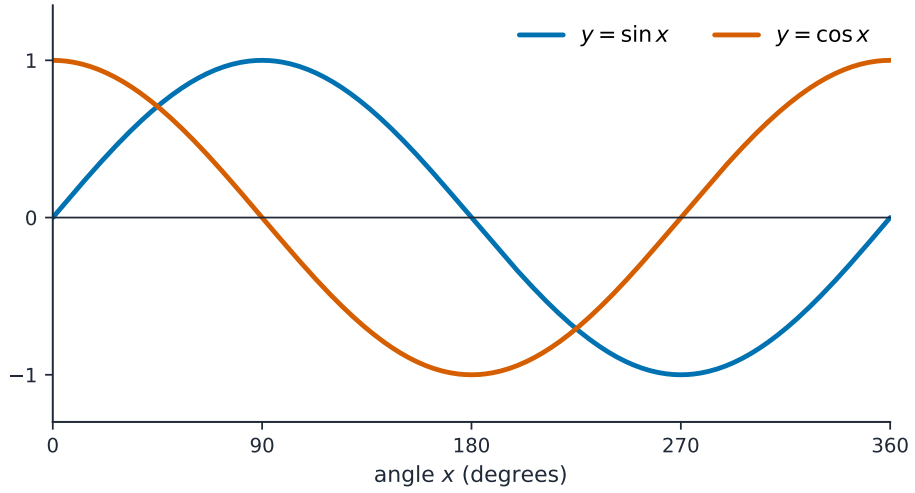
> **Worked example.** Find $\cos\frac{5\pi}{6}$, $\sin\frac{7\pi}{4}$ and the point at angle $\frac{4\pi}{3}$ on a circle of radius 6.

- $\frac{5\pi}{6}$ is in quadrant II with reference angle $\frac{\pi}{6}$: $\cos\frac{5\pi}{6} = -\frac{\sqrt{3}}{2}$.
- $\frac{7\pi}{4}$ is in quadrant IV with reference angle $\frac{\pi}{4}$: $\sin\frac{7\pi}{4} = -\frac{\sqrt{2}}{2}$.
- $\frac{4\pi}{3}$ is in quadrant III with reference angle $\frac{\pi}{3}$: $P = (6\cos\frac{4\pi}{3}, 6\sin\frac{4\pi}{3}) = \left(-3, -3\sqrt{3}\right)$.

Sine and Cosine Function Graphs

On the **unit circle** 单位圆 ($r = 1$), $\sin\theta$ is the y-coordinate and $\cos\theta$ is the x-coordinate. As θ increases, both **oscillate** 振荡 smoothly between -1 and 1 with period 2π. Sine starts at 0 (rising); cosine starts at 1. They are the same wave shifted by $\frac{\pi}{2}$.

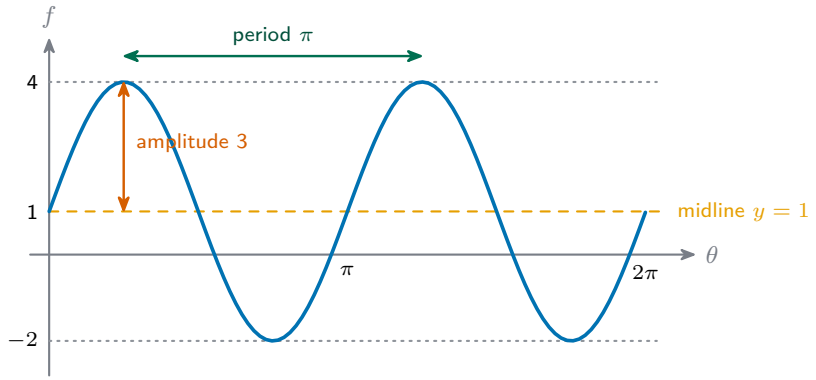
Symmetry. $\sin(-\theta) = -\sin\theta$: the sine graph has rotational symmetry about the origin, so sine is an **odd function** 奇函数. $\cos(-\theta) = \cos\theta$: the cosine graph is symmetric about the y-axis, so cosine is an **even function** 偶函数. For example, $\sin\left(-\frac{\pi}{6}\right) = -\frac{1}{2}$ but $\cos\left(-\frac{\pi}{3}\right) = \frac{1}{2}$.



$y = \sin x$ and $y = \cos x$ are smooth waves between -1 and 1

Sinusoidal Functions

A **sinusoidal function** 正弦型函数 is any transformation of sine (or cosine). Its key features:



Read amplitude, midline and period off the graph.

- **Period** and **frequency** 频率 are reciprocals; $\sin\theta$ has period 2π.
- **Amplitude** 振幅 = half the distance between the maximum and minimum output.
- **Midline** 中线 $y = d$ = the average of the maximum and minimum (the horizontal center line).

The graph alternates concave up and concave down each half-cycle.

> **Worked example.** For $f(\theta) = 3\sin(2\theta) + 1$: the amplitude is 3, the period is $\frac{2\pi}{2} = \pi$, and the midline is $y = 1$. So the maximum output is $1 + 3 = 4$ and the minimum is $1 - 3 = -2$.

Read a quarter-cycle at a time. AP questions name two labelled points and ask about the function between them. Split one cycle at the maxima, minima and midline crossings. Starting at a maximum:

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Quarter-cycle	From → to	Sign of $h - d$	h is	Concavity	Rate of change is
1	maximum → midline	above midline	decreasing	concave down	decreasing
2	midline → minimum	below midline	decreasing	concave up	increasing
3	minimum → midline	below midline	increasing	concave up	increasing
4	midline → maximum	above midline	increasing	concave down	decreasing

Concavity changes at every midline crossing. Concave up always means the rate of change is increasing; concave down means it is decreasing. In quarter 2, h is falling, but it falls more and more slowly: the rate of change is negative and increasing toward 0.

Sinusoidal Function Transformations

The general sinusoid is

$$f(\theta) = a\sin\big(b(\theta + c)\big) + d \quad \text{or} \quad a\cos\big(b(\theta + c)\big) + d,$$

where $|a|$ is the **amplitude**, $\dfrac{2\pi}{|b|}$ is the **period**, $-c$ is the horizontal **phase shift** 相移 (a $+c$ inside the bracket shifts the graph **left** by c), and d is the **vertical shift** (midline). A negative a reflects the wave.

AP free-response questions ask for the **sine** form $a\sin(b(\theta + c)) + d$ even when the graph starts at a maximum. Use $\cos\theta = \sin\left(\theta + \frac{\pi}{2}\right)$: a cosine curve is a sine curve shifted left by a quarter period.

> **Worked example.** Write $g(\theta) = 5\cos(2\theta) - 1$ in the form $a\sin(b(\theta + c)) + d$.
>
> $$5\cos(2\theta) - 1 = 5\sin\left(2\theta + \frac{\pi}{2}\right) - 1 = 5\sin\left(2\left(\theta + \frac{\pi}{4}\right)\right) - 1$$
>
> So $a = 5$, $b = 2$, $c = \frac{\pi}{4}$ and $d = -1$. The phase shift $-\frac{\pi}{4}$ is left by a quarter of the period π, not left by $\frac{\pi}{2}$: factor b out before reading c. Other answers are also correct, such as $c = \frac{\pi}{4} + \pi$ or $a = -5$, $c = -\frac{\pi}{4}$.

Sinusoidal Function Context and Data Modeling

To model periodic data (tides, daylight, temperature): read the **period** from where the pattern repeats, get the **amplitude** and **midline** from the max and min, and set the phase shift so a peak lands at the right input. Technology fits a sinusoidal regression. Such a model is only trustworthy over its **contextual domain**.

Worked example: a Ferris wheel. A rider's height h (metres) on a wheel varies from 2 m at the bottom to 50 m at the top, and one turn takes 12 minutes. At $t = 0$ the rider is at the bottom. Build a model.

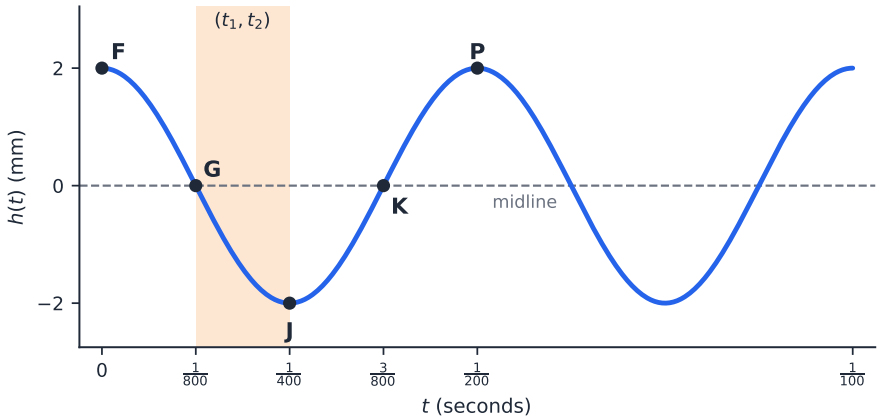
- Amplitude $= \frac{50-2}{2} = 24$; midline $y = \frac{50+2}{2} = 26$.
- Period 12, so $b = \frac{2\pi}{12} = \frac{\pi}{6}$.
- At $t = 0$ the height is a **minimum**. cos starts at a maximum, so $-\cos$ starts at a minimum and no phase shift is needed:

$$h(t) = -24\cos\left(\frac{\pi}{6}t\right) + 26.$$

Check: $h(6) = -24\cos\pi + 26 = 50$ (the top, after half a turn). After 2 minutes, $h(2) = -24\cos\frac{\pi}{3} + 26 = -12 + 26 = 14$ m.

■ Worked example: a vibrating guitar string

This follows the 2025 AP free-response question. Point X on a guitar string starts at its highest position, 2 mm above rest, at $t = 0$. It moves down through rest to 2 mm below rest and back up, completing this motion 200 times per second. $h(t)$ is the displacement in millimetres, positive above rest.



One cycle lasts 1/200 s, so each labelled point is a quarter-period (1/800 s) after the one before.

(a) Coordinates of the labelled points. 200 cycles per second means the period is $\frac{1}{200}$ s and a quarter-period is $\frac{1}{800}$ s. Midline $h = 0$, amplitude 2.

$$F\,(0,\,2)\,,\quad G\left(\tfrac{1}{800},\,0\right),\quad J\left(\tfrac{1}{400},\,-2\right),\quad K\left(\tfrac{3}{800},\,0\right),\quad P\left(\tfrac{1}{200},\,2\right)$$

(b) Constants for $h(t) = a\sin(b(t+c)) + d$. $a = 2$ and $d = 0$. Period $\frac{2\pi}{b} = \frac{1}{200}$ gives $b = 400\pi$. A sine curve starts on the midline, rising. Here the curve rises through the midline at K, a quarter-period **before** each maximum, and at $t = -\frac{1}{800}$, a quarter-period before F. So the graph is a sine curve shifted left by $\frac{1}{800}$: $c = \frac{1}{800}$. Check $h(0) = 2\sin\left(400\pi \cdot \frac{1}{800}\right) = 2\sin\frac{\pi}{2} = 2$. ✓

(c) Between G and J. On $(t_1, t_2) = \left(\frac{1}{800}, \frac{1}{400}\right)$, h is **negative and decreasing**. The graph is concave up, so the rate of change of h is increasing (from its most negative value at G to 0 at J).

The Tangent Function

$\tan\theta = \dfrac{\sin\theta}{\cos\theta}$ is the **slope** of the terminal ray. Its slope values repeat every half-revolution, so its period is π. It has **vertical asymptotes** where $\cos\theta = 0$ (at $\theta = \frac{\pi}{2} + k\pi$), and it is always increasing between consecutive asymptotes, switching concavity at each zero.

Worked example. For $f(x) = 3\tan(2x)$, find the period and the asymptotes.

The period of $\tan(bx)$ is $\frac{\pi}{|b|}$, so the period is $\frac{\pi}{2}$. Asymptotes occur where $\cos(2x) = 0$: $2x = \frac{\pi}{2} + k\pi$, so $x = \frac{\pi}{4} + \frac{k\pi}{2}$ for integers k. Between $x = -\frac{\pi}{4}$ and $x = \frac{\pi}{4}$ the graph increases through $(0, 0)$, concave down to the left of 0 and concave up to the right.

Inverse Trigonometric Functions

The **inverse trigonometric functions** 反三角函数 – **arcsine** 反正弦, **arccosine**, **arctangent** – reverse the trig functions, so they take a ratio and return an angle. Because sine, cosine, and tangent repeat, their domains must be **restricted** (e.g. sine to $[-\frac{\pi}{2}, \frac{\pi}{2}]$) before an inverse can exist.

So each inverse returns an angle in one fixed range: arcsin in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, arccos in $[0, \pi]$, arctan in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. For example, $\arcsin\left(-\frac{1}{2}\right) = -\frac{\pi}{6}$, but $\arccos\left(-\frac{1}{2}\right) = \frac{2\pi}{3}$ (not $-\frac{\pi}{3}$, which is outside $[0, \pi]$).

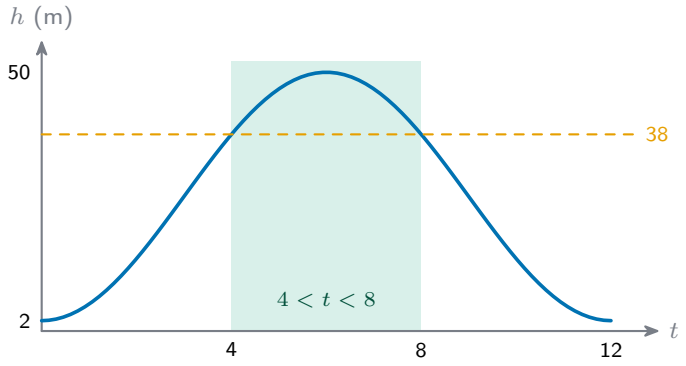
Composition with a triangle. Find $\cos\left(\arcsin\frac{3}{5}\right)$ exactly. Let $\theta = \arcsin\frac{3}{5}$, so $\sin\theta = \frac{3}{5}$ with θ in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. Here θ is in quadrant I. A right triangle with opposite side 3 and hypotenuse 5 has adjacent side 4, so $\cos\theta = \frac{4}{5}$.

Domain of a composite. Let $m(x) = \arccos(\tan x)$. The input to arccos must lie in $[-1, 1]$, so the domain of m needs $-1 \le \tan x \le 1$. To solve $m(x) = \frac{\pi}{2}$: $\arccos(\tan x) = \frac{\pi}{2}$ means $\tan x = \cos\frac{\pi}{2} = 0$, so $x = k\pi$ for integers k.

Trigonometric Equations and Inequalities

Use inverse trig functions to solve trig equations. Because the functions are **periodic**, there are usually **infinitely many** solutions – write the general solution by adding multiples of the period, then keep those in the required (often contextual) domain.

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Solve the equation, then read the inequality from the graph.

Worked example. Solve $2\sin\theta = 1$ on $[0, 2\pi)$. Then $\sin\theta = \frac{1}{2}$, and on the unit circle sine is positive in the first and second quadrants, so $\theta = \dfrac{\pi}{6}$ or $\dfrac{5\pi}{6}$. Over all reals you would add $2\pi k$ to each.

Worked example: a quadratic in cosine. Solve $2\cos^2 x - \cos x - 1 = 0$ on $[0, 2\pi)$.

$$(2\cos x + 1)(\cos x - 1) = 0$$

$$\cos x = -\tfrac{1}{2} \quad \text{or} \quad \cos x = 1$$

$\cos x = -\frac{1}{2}$ in quadrants II and III: $x = \frac{2\pi}{3}$ or $\frac{4\pi}{3}$. $\cos x = 1$ gives $x = 0$. The solutions are 0, $\frac{2\pi}{3}$, $\frac{4\pi}{3}$.

Worked example: an inequality in context. On the Ferris wheel above, $h(t) = -24\cos\left(\frac{\pi}{6}t\right) + 26$. During the first turn, $0 \le t \le 12$, when is the rider higher than 38 m?

$$-24\cos\left(\tfrac{\pi}{6}t\right) + 26 > 38 \quad \Rightarrow \quad \cos\left(\tfrac{\pi}{6}t\right) < -\tfrac{1}{2}$$

On one turn, $\cos u < -\frac{1}{2}$ for $\frac{2\pi}{3} < u < \frac{4\pi}{3}$. With $u = \frac{\pi}{6}t$: $4 < t < 8$. The rider is above 38 m from 4 to 8 minutes, a third of each turn. The context limits the answer to one turn; over all time you would add multiples of 12.

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The Secant, Cosecant, and Cotangent Functions

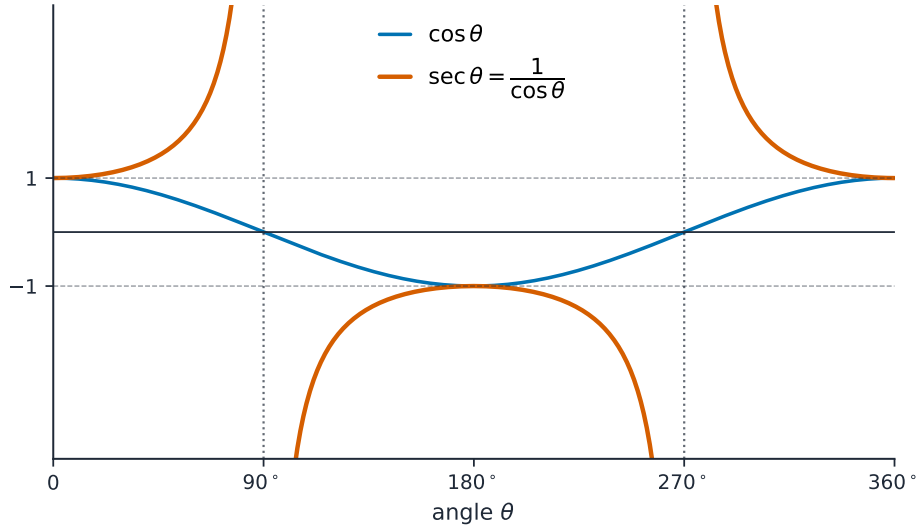
These are the **reciprocals** 倒数 of the main three:

$$\sec\theta = \frac{1}{\cos\theta}, \qquad \csc\theta = \frac{1}{\sin\theta}, \qquad \cot\theta = \frac{\cos\theta}{\sin\theta}.$$

The identity $\cot\theta = 1/\tan\theta$ holds where both sides are defined; cotangent is also defined when $\cos\theta = 0$. Secant and cosecant have vertical asymptotes where cosine and sine are zero; cotangent has them where tangent is zero (where sine is zero).

The range of secant and cosecant is $(-\infty, -1] \cup [1, \infty)$, because $|\cos\theta| \le 1$ makes $\left|\frac{1}{\cos\theta}\right| \ge 1$. Cotangent, $\cot\theta = \frac{\cos\theta}{\sin\theta}$, is decreasing between consecutive asymptotes.

Worked example. $\sec\frac{2\pi}{3} = \dfrac{1}{-1/2} = -2$, $\csc\frac{7\pi}{6} = \dfrac{1}{-1/2} = -2$, and $\cot\frac{\pi}{4} = \dfrac{\sqrt{2}/2}{\sqrt{2}/2} = 1$. $\csc\pi$ is undefined because $\sin\pi = 0$.



sec is the reciprocal of cos, with asymptotes where cos is zero

Equivalent Representations of Trigonometric Functions

The **Pythagorean identity** 毕达哥拉斯恒等式 comes from $x^2 + y^2 = 1$ on the unit circle:

$$\sin^2\theta + \cos^2\theta = 1,$$

which rearranges into forms with sec and tan. The **sum identities** 和角公式:

$$\sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta,\qquad \cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta,$$

also give the difference and **double-angle** 二倍角 identities (set $\beta=\alpha$).

> **Worked example.** Find the exact value of $\sin\frac{5\pi}{12}$ (that is, $\sin 75°$). Write $\frac{5\pi}{12}=\frac{\pi}{4}+\frac{\pi}{6}$:

$$\sin\frac{5\pi}{12}=\sin\frac{\pi}{4}\cos\frac{\pi}{6}+\cos\frac{\pi}{4}\sin\frac{\pi}{6}=\frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2}+\frac{\sqrt{2}}{2}\cdot\frac{1}{2}=\frac{\sqrt{6}+\sqrt{2}}{4}\approx 0.966.$$

Dividing $\sin^2\theta+\cos^2\theta=1$ by $\cos^2\theta$ gives $\tan^2\theta+1=\sec^2\theta$; dividing by $\sin^2\theta$ gives $1+\cot^2\theta=\csc^2\theta$. Setting $\beta=\alpha$ gives $\sin 2\alpha=2\sin\alpha\cos\alpha$ and $\cos 2\alpha=\cos^2\alpha-\sin^2\alpha$.

> **Worked example: one term in $\tan x$.** AP free-response questions ask you to rewrite an expression "as a single term involving $\tan x$". Rewrite $k(x)=\dfrac{\sin(2x)}{2\cos^2 x}$.

$$k(x)=\frac{2\sin x\cos x}{2\cos^2 x}=\frac{\sin x}{\cos x}=\tan x\quad(\cos x\neq 0)$$

> **Worked example: an identity to solve an equation.** Solve $\sin(2x)=\sin x$ on $[0,2\pi)$.

$$2\sin x\cos x-\sin x=0\quad\Rightarrow\quad\sin x\,(2\cos x-1)=0$$

$\sin x=0$ gives $x=0,\ \pi$. $\cos x=\frac{1}{2}$ gives $x=\frac{\pi}{3},\ \frac{5\pi}{3}$. Never divide both sides by $\sin x$: that loses the solutions 0 and π.

Trigonometry and Polar Coordinates

Polar coordinates 极坐标 locate a point by (r,θ) – signed radius r at angle θ; distance from the origin is $|r|$. Convert with

$$x=r\cos\theta,\quad y=r\sin\theta,\qquad r=\sqrt{x^2+y^2},\quad \tan\theta=\frac{y}{x}.$$

A **complex number** 复数 is the point $(x,y)=x+yi$ and can likewise be written with r and θ.

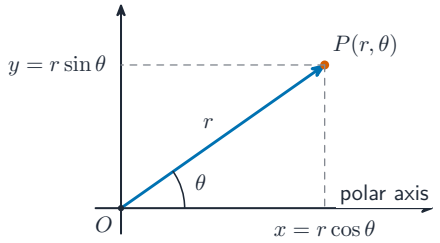
> **Worked example.** Write the point $(-1,\sqrt{3})$ in polar form. $r=\sqrt{(-1)^2+(\sqrt{3})^2}=2$ and $\tan\theta=\frac{\sqrt{3}}{-1}=-\sqrt{3}$. A calculator's $\arctan(-\sqrt{3})=-\frac{\pi}{3}$ points into the fourth quadrant, but the point is in the **second** quadrant ($x<0$, $y>0$), so add π: $\theta=\frac{2\pi}{3}$. The polar form is $\left(2,\frac{2\pi}{3}\right)$. Always check the quadrant before trusting arctan.

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> **Worked example: polar to rectangular.** Convert $\left(4,\frac{5\pi}{6}\right)$: $x=4\cos\frac{5\pi}{6}=-2\sqrt{3}$ and $y=4\sin\frac{5\pi}{6}=2$, giving $\left(-2\sqrt{3},\,2\right)$.

Negative r. The point $\left(-3,\frac{\pi}{4}\right)$ lies 3 units from the origin in the direction **opposite** to $\frac{\pi}{4}$, so it is the same point as $\left(3,\frac{5\pi}{4}\right)$. Each point has many polar names.

Complex numbers. The complex number $a+bi$ is the point (a,b). With $r=\sqrt{a^2+b^2}$ and angle θ, $a+bi=r(\cos\theta+i\sin\theta)$. For $-1+\sqrt{3}\,i$, the point above gives $-1+\sqrt{3}\,i=2\left(\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}\right)$.



Polar coordinates give a point by its distance r and angle theta

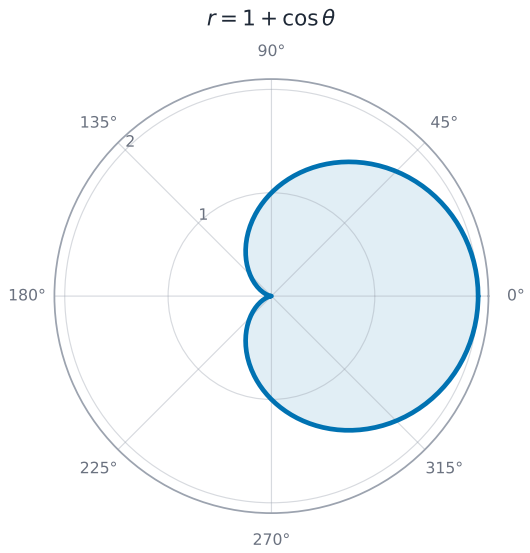
Polar Function Graphs

The graph of $r=f(\theta)$ is all points (r,θ) as θ sweeps around. Restricting the domain of θ draws only part of the curve. As θ increases, r grows or shrinks, tracing spirals, circles, roses, and limaçons.

> **Worked example: plot a rose from a table.** For $r=3\sin(2\theta)$ on $\left[0,\frac{\pi}{2}\right]$:

θ	0	$\frac{\pi}{12}$	$\frac{\pi}{4}$	$\frac{5\pi}{12}$	$\frac{\pi}{2}$
r	0	1.5	3	1.5	0

The curve leaves the origin, reaches distance 3 along the ray $\theta=\frac{\pi}{4}$, and returns: one petal in quadrant I. On $\left(\frac{\pi}{2},\pi\right)$, $r<0$, so those points are drawn in the opposite direction, in quadrant IV. The full graph on $[0,2\pi]$ is a four-petal rose.



A cardioid r = 1 + cos theta, sketched directly in polar form

Rates of Change in Polar Functions

As θ increases, the **distance from the origin** $|r|$ changes:

- r positive and increasing (or negative and decreasing) $\Rightarrow$ moving **away** from the origin;
- r positive and decreasing (or negative and increasing) $\Rightarrow$ moving **toward** the origin.

Where r switches between increasing and decreasing, the distance reaches a relative extreme. The **average rate of change** of r with respect to θ, $\dfrac{\Delta r}{\Delta \theta}$, estimates how fast the curve moves in or out over an interval.

Worked example. For $r = 3 + 2\cos\theta$ on $\left[0, \frac{\pi}{2}\right]$: $r(0) = 5$ and $r\left(\frac{\pi}{2}\right) = 3$. The average rate of change is $\dfrac{3-5}{\pi/2 - 0} = -\dfrac{4}{\pi} \approx -1.27$. Here r is positive and decreasing, so the curve moves **toward** the origin, on average by about 1.27 units of distance per radian.

Worked example: negative and decreasing. For $r = 1 + 2\cos\theta$ on $\left(\frac{2\pi}{3}, \pi\right)$: $r\left(\frac{2\pi}{3}\right) = 1 + 2\left(-\frac{1}{2}\right) = 0$ and $r(\pi) = 1 - 2 = -1$. Cosine is decreasing there, so r is **negative and decreasing**. Then $|r|$ grows from 0 to 1, and the point moves **away** from the origin. Distance is $|r|$, not r.

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Exam tips

- Use radians and the **unit circle**: $\cos\theta$ and $\sin\theta$ are the coordinates, always between -1 and 1.
- For a sinusoid $a\sin(b(\theta + c)) + d$: $|a|$ is the amplitude, $\frac{2\pi}{|b|}$ the period, d the midline, and the phase shift is $-c$ (a $+c$ shifts left).
- Model periodic data by reading period, amplitude, and midline from the max and min.
- **Inverse trig** functions need a restricted domain and return an angle; trig equations have infinitely many solutions (add the period).
- Convert polar↔rectangular with $x = r\cos\theta$, $y = r\sin\theta$.

Common mistakes.

- Reading the phase shift before factoring: in $\sin(2\theta + \frac{\pi}{2})$ the shift is $-\frac{\pi}{4}$ (left), not $-\frac{\pi}{2}$.
- Giving a cosine model when the question asks for $a\sin(b(t + c)) + d$.
- Linking "decreasing" with "concave down". Between a midline crossing and a minimum, the graph is decreasing **and** concave up.
- Dividing an equation by $\sin x$ or $\cos x$ and losing solutions where it is zero.
- Trusting $\arctan\frac{y}{x}$ without checking the quadrant of (x, y).
- Treating a negative, decreasing r as moving toward the origin. Distance is $|r|$.

Practice.

1. Find $\sin\theta$, $\cos\theta$ and $\tan\theta$ if the terminal ray of θ passes through $(5, -12)$.
2. Find exactly: (a) $\sin\frac{4\pi}{3}$, (b) $\cos\left(-\frac{3\pi}{4}\right)$, (c) $\sec\frac{\pi}{3}$, (d) $\arccos\left(-\frac{\sqrt{2}}{2}\right)$.
3. A waterwheel turns once every 10 seconds. At $t = 0$ point W is 6 feet directly above the centre. $h(t)$ is the height of W above the horizontal line through the centre. (a) Give coordinates of the first maximum F, the next midline crossing G, the minimum J, the next midline crossing K and the next maximum P. (b) Write $h(t) = a\sin(b(t + c)) + d$. (c) On the interval from K to P, is h positive or negative, increasing or decreasing? Is the rate of change increasing or decreasing?
4. Solve $2\sin^2 x = 1$ on $[0, 2\pi)$.
5. Solve $\cos(2x) = \cos x$ on $[0, 2\pi)$. (Use $\cos 2x = 2\cos^2 x - 1$.)
6. Rewrite $\dfrac{\sec^2 x - 1}{\tan x}$ as a single term involving $\tan x$, for $\tan x \neq 0$.
7. Convert $(-2, -2)$ to polar coordinates with $r > 0$ and $0 \leq \theta < 2\pi$. Then write $-2 - 2i$ in polar form.
8. For $r = 2 - 4\sin\theta$: (a) find r at $\theta = \frac{\pi}{6}$ and $\theta = \frac{\pi}{2}$; (b) on $\left(\frac{\pi}{6}, \frac{\pi}{2}\right)$, is the point moving toward or away from the origin? (c) Find the average rate of change of r over that interval.

Answers.

1. $r = 13$: $\sin\theta = -\frac{12}{13}$, $\cos\theta = \frac{5}{13}$, $\tan\theta = -\frac{12}{5}$.
2. (a) $-\frac{\sqrt{3}}{2}$. (b) $-\frac{\sqrt{2}}{2}$ (cosine is even). (c) 2. (d) $\frac{3\pi}{4}$.
3. (a) Quarter-period 2.5 s: $F(0,6)$, $G(2.5,0)$, $J(5,-6)$, $K(7.5,0)$, $P(10,6)$. (b) $a = 6$, $b = \frac{2\pi}{10} = \frac{\pi}{5}$, $c = 2.5$, $d = 0$: $h(t) = 6\sin\left(\frac{\pi}{5}(t+2.5)\right)$. (c) Positive and increasing; the graph is concave down there, so the rate of change is decreasing.
4. $\sin x = \pm\frac{\sqrt{2}}{2}$: $x = \frac{\pi}{4},\ \frac{3\pi}{4},\ \frac{5\pi}{4},\ \frac{7\pi}{4}$.
5. $2\cos^2 x - \cos x - 1 = 0$, so $\cos x = 1$ or $-\frac{1}{2}$: $x = 0,\ \frac{2\pi}{3},\ \frac{4\pi}{3}$.
6. $\sec^2 x - 1 = \tan^2 x$, so the expression is $\dfrac{\tan^2 x}{\tan x} = \tan x$.
7. $r = \sqrt{8} = 2\sqrt{2}$. The point is in quadrant III, so $\theta = \frac{5\pi}{4}$. $-2 - 2i = 2\sqrt{2}\left(\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}\right)$.
8. (a) $r\left(\frac{\pi}{6}\right) = 2 - 2 = 0$ and $r\left(\frac{\pi}{2}\right) = 2 - 4 = -2$. (b) r is negative and decreasing, so $|r|$ grows: moving away from the origin. (c) $\dfrac{-2-0}{\pi/2 - \pi/6} = \dfrac{-2}{\pi/3} = -\dfrac{6}{\pi} \approx -1.91$.

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Functions Involving Parameters, Vectors, and Matrices

AP Precalculus

Parametric Functions



A roller coaster: vectors and parametric motion describe path and velocity together

Unit 4 is part of the course but is not assessed on the AP Precalculus Exam (which covers only Units 1–3); it builds tools used in later courses.

Why it matters. A map app shows your position as a moving dot. Behind it, your phone reports an east–west coordinate and a north–south coordinate that **both** change with time. One function of time cannot describe a path that loops back on itself, but a pair of functions can. This unit describes motion and shapes with parameters, vectors and matrices, the tools of computer graphics, robotics and physics.

A **parametric function** 参数函数 describes a curve by giving both coordinates as functions of a third variable, the **parameter** 参数 t: $\big(x(t),\, y(t)\big)$. As t runs through its domain, the point traces a path – and, unlike $y = f(x)$, the path may loop or cross itself.

Parametric Functions and Planar Motion

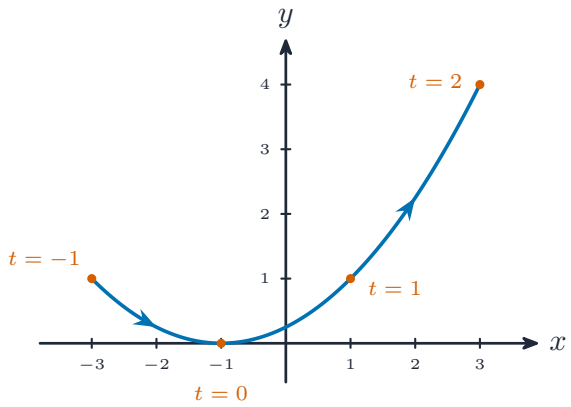
Reading t as **time**, a parametric function models motion in the plane. $x(t)$ and $y(t)$ give the horizontal and vertical position; the point's **direction** of travel is the order in which the curve is drawn as t increases (mark it with arrows).

Worked example. A particle moves with $x(t) = 2t-1$ and $y(t) = t^2$ for $-1 \le t \le 2$. Describe its path.

Make a table first:

t	-1	0	1	2
(x, y)	$(-3, 1)$	$(-1, 0)$	$(1, 1)$	$(3, 4)$

To **eliminate the parameter**, solve $x = 2t-1$ for $t = \frac{x+1}{2}$ and substitute: $y = \frac{(x+1)^2}{4}$. The path is the part of this parabola from $(-3, 1)$ to $(3, 4)$. Because x increases with t, the particle moves left to right: down to the vertex $(-1, 0)$ at $t = 0$, then up.



The same parabola could be traced in either direction; the arrows and t-values carry the information that $y = \frac{(x+1)^2}{4}$ alone loses.

Common mistake. Giving the whole parabola $y = \frac{(x+1)^2}{4}$ as the answer. The restriction $-1 \le t \le 2$ becomes $-3 \le x \le 3$, and the equation alone shows no direction.

Parametric Functions and Rates of Change

Over an interval of t, the average rate of change of x is $\dfrac{\Delta x}{\Delta t}$ and of y is $\dfrac{\Delta y}{\Delta t}$. Their signs tell you the net direction of displacement (right/left, up/down); together they give average velocity, not average speed along the path.

Worked example. Find the average rates of change for the same particle on $0 \le t \le 2$ and on $-1 \le t \le 0$.

$$0 \le t \le 2: \quad \frac{\Delta x}{\Delta t} = \frac{3-(-1)}{2} = 2, \quad \frac{\Delta y}{\Delta t} = \frac{4-0}{2} = 2; \qquad -1 \le t \le 0: \quad \frac{\Delta y}{\Delta t} = \frac{0-1}{1} = -1.$$

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On the first interval it moves right and up on average; on the second it moves down while it moves right.

Parametrically Defined Circles and Lines

A **circle** of radius R centered at (h, k) is $x = h + R\cos t$, $y = k + R\sin t$. A **line** through (x_0, y_0) with direction (a, b) is $x = x_0 + at$, $y = y_0 + bt$. Adjusting the coefficients changes the start point, speed, and direction of tracing.

Worked example. The circle with center $(3, -2)$ and radius 5 is $x = 3+5\cos t$, $y = -2+5\sin t$ for $0 \le t < 2\pi$. It starts at $(8, -2)$ and turns counterclockwise. Writing $y = -2-5\sin t$ instead traces the same circle clockwise.

Worked example. The segment from $(1, 4)$ to $(7, 1)$ has direction $(7-1,\ 1-4) = (6, -3)$, so $x = 1+6t$, $y = 4-3t$ for $0 \le t \le 1$. At $t = 0$ the point is at $(1, 4)$; at $t = 1$ it reaches $(7, 1)$.

Implicitly Defined Functions

An **implicit** 隐式 equation relates x and y without solving for either, such as $x^2+y^2 = 25$. Its graph may fail the vertical-line test (not a function), so it is often split into pieces or described parametrically.

Conic Sections

Conic sections 圆锥曲线 – circles, ellipses, parabolas, and hyperbolas – are the curves from slicing a cone, each given by a quadratic equation in x and y. Their standard forms reveal centers, vertices, axes, and asymptotes.

Worked example. Identify $4x^2 + 9y^2 - 16x + 18y - 11 = 0$.

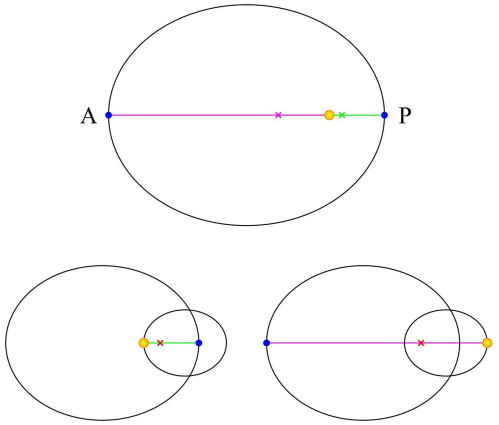
Complete the square in each variable:

$$4(x^2-4x) + 9(y^2+2y) = 11 \;\Rightarrow\; 4(x-2)^2 - 16 + 9(y+1)^2 - 9 = 11 \;\Rightarrow\; 4(x-2)^2 + 9(y+1)^2 = 36.$$

Divide by 36: $\dfrac{(x-2)^2}{9} + \dfrac{(y+1)^2}{4} = 1$. This is an **ellipse** 椭圆 with center $(2, -1)$, reaching 3 units left and right (to $(-1, -1)$ and $(5, -1)$) and 2 units up and down (to $(2, 1)$ and $(2, -3)$).

Worked example. The **hyperbola** 双曲线 $\dfrac{x^2}{16} - \dfrac{y^2}{9} = 1$ has vertices $(\pm 4, 0)$ and asymptotes $y = \pm\frac{3}{4}x$. The minus sign makes it a hyperbola, and the positive x^2 term means it opens left and right.

Common mistake. Forgetting to subtract the constants that completing the square adds (-16 and -9 above), or dividing only one term by 36.



Elliptical orbits: conic sections describe closed paths with two foci – a key parametric model

Parametrizing Implicit Curves

Many implicit curves can be **parametrized** – rewritten as $\big(x(t), y(t)\big)$ – which makes them easier to graph and to treat as motion. The circle above is the basic example; ellipses use $x = h + A\cos t,\ y = k + B\sin t$.

For the ellipse above, $x = 2 + 3\cos t,\ y = -1 + 2\sin t,\ 0 \le t < 2\pi$.

Vectors

A **vector** 向量 has both **magnitude** 大小 (length) and **direction**. Write it by components $\langle a, b\rangle$. Add vectors component-by-component; scale by multiplying each component; the magnitude is $\sqrt{a^2 + b^2}$. Vectors model displacements, velocities, and forces.

> **Worked example.** Let $\vec{u} = \langle 3, -4\rangle$ and $\vec{v} = \langle -1, 2\rangle$. Then $\vec{u} + 2\vec{v} = \langle 3-2,\ -4+4\rangle = \langle 1, 0\rangle$, and $|\vec{u}| = \sqrt{9 + 16} = 5$. The **unit vector** in the direction of $\vec{u}$ divides by its magnitude: $\frac{1}{5}\vec{u} = \langle \frac{3}{5}, -\frac{4}{5}\rangle$.

The **dot product** 点积 of $\vec{u} = \langle u_1, u_2\rangle$ and $\vec{v} = \langle v_1, v_2\rangle$ is the number $\vec{u}\cdot\vec{v} = u_1v_1 + u_2v_2$. It also equals $|\vec{u}||\vec{v}|\cos\theta$, where θ is the angle between the vectors. For two nonzero vectors, a dot product of zero means they are **perpendicular**.

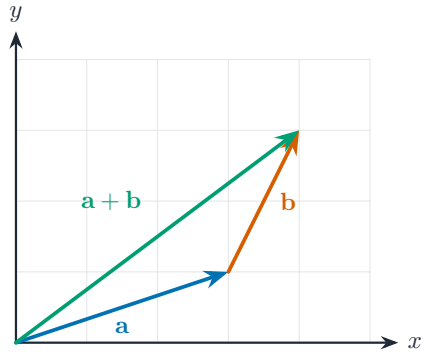
> **Worked example.** Find the angle between $\vec{u} = \langle 2, 1\rangle$ and $\vec{v} = \langle 1, 3\rangle$.

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$$\vec{u}\cdot\vec{v} = 2(1) + 1(3) = 5, \qquad |\vec{u}| = \sqrt{5}, \qquad |\vec{v}| = \sqrt{10}$$

$$\cos\theta = \frac{5}{\sqrt{5}\,\sqrt{10}} = \frac{5}{\sqrt{50}} = \frac{1}{\sqrt{2}} \quad \Rightarrow \quad \theta = \frac{\pi}{4}$$

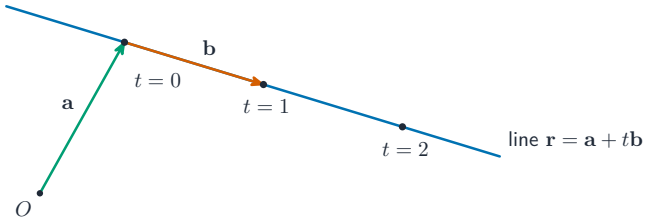
For $\langle 3, 4\rangle$ and $\langle 4, -3\rangle$, the dot product is $12 - 12 = 0$: they are perpendicular.



Adding vectors by the triangle law

Vector-Valued Functions

A **vector-valued function** 向量值函数 outputs a vector for each input, e.g. $\vec{r}(t) = \langle x(t), y(t)\rangle$ – the same information as a parametric function, packaged as a moving position vector.



A vector-valued line: start at a, then slide by t lots of the direction b

Matrices

A **matrix** 矩阵 is a rectangular array of numbers. Add matrices of the same size entry-by-entry; multiply a matrix by a compatible one by combining rows with columns. Matrices store and transform data compactly.

Worked example.

$$\begin{bmatrix}1 & 2\\ 0 & 1\end{bmatrix}\begin{bmatrix}3 & 0\\ 1 & 2\end{bmatrix}=\begin{bmatrix}1\cdot 3+2\cdot 1 & 1\cdot 0+2\cdot 2\\ 0\cdot 3+1\cdot 1 & 0\cdot 0+1\cdot 2\end{bmatrix}=\begin{bmatrix}5 & 4\\ 1 & 2\end{bmatrix}.$$

In the other order the product is $\begin{bmatrix}3 & 6\\ 1 & 4\end{bmatrix}$, which is different. Matrix multiplication is not **commutative**: order matters.

The Inverse and Determinant of a Matrix

The **determinant** 行列式 of a 2×2 matrix $\begin{bmatrix}a & b\\ c & d\end{bmatrix}$ is $ad-bc$. A matrix has an **inverse** 逆矩阵 exactly when its determinant is nonzero; the inverse undoes the matrix, and it solves matrix equations (like a reciprocal for numbers).

Worked example. For $A=\begin{bmatrix}3 & 1\\ 2 & 4\end{bmatrix}$, the determinant is $ad-bc=(3)(4)-(1)(2)=10$. Since it is nonzero, A is invertible, and $A^{-1}=\frac{1}{10}\begin{bmatrix}4 & -1\\ -2 & 3\end{bmatrix}$ (swap the diagonal, negate the off-diagonal, divide by the determinant).

Worked example. Solve $3x+y=7$, $2x+4y=8$ with A^{-1}. In matrix form $A\begin{bmatrix}x\\ y\end{bmatrix}=\begin{bmatrix}7\\ 8\end{bmatrix}$, so

$$\begin{bmatrix}x\\ y\end{bmatrix}=A^{-1}\begin{bmatrix}7\\ 8\end{bmatrix}=\frac{1}{10}\begin{bmatrix}4\cdot 7-1\cdot 8\\ -2\cdot 7+3\cdot 8\end{bmatrix}=\frac{1}{10}\begin{bmatrix}20\\ 10\end{bmatrix}=\begin{bmatrix}2\\ 1\end{bmatrix}.$$

Check: $3(2)+1=7$ and $2(2)+4(1)=8$.

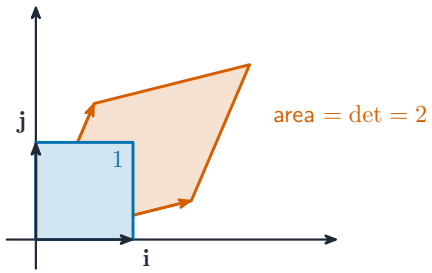
Linear Transformations and Matrices

A 2×2 matrix acts as a **linear transformation** 线性变换 of the plane – rotating, reflecting, stretching, or shearing points – by multiplying each position vector. The absolute value of the determinant measures how the transformation scales area; its sign tells whether orientation flips.

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Worked example. The matrix $\begin{bmatrix}2 & 0\\ 0 & 3\end{bmatrix}$ stretches the plane by 2 horizontally and 3 vertically, sending the vector $\begin{bmatrix}1\\ 1\end{bmatrix}$ to $\begin{bmatrix}2\\ 3\end{bmatrix}$. Its determinant $2\times 3=6$ means every area is multiplied by 6, so the unit square becomes a 2×3 rectangle.

Worked example. The rotation by angle θ counterclockwise about the origin is $\begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}$. For $\theta=90°$ it is $\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}$, which sends $\begin{bmatrix}3\\ 1\end{bmatrix}$ to $\begin{bmatrix}-1\\ 3\end{bmatrix}$. Its determinant is $0\cdot 0-(-1)(1)=1$: a rotation keeps every area the same.



A 2x2 matrix maps the unit square to a parallelogram; the absolute determinant is the area scale

Matrices as Functions

Because a matrix maps input vectors to output vectors, it **is** a function on the plane. Composing transformations corresponds to multiplying their matrices, and the inverse matrix reverses the mapping.

Worked example: rotate, then reflect. Let $Q=\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}$ rotate by 90° counterclockwise and $R=\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}$ reflect across the x-axis. "Rotate first, then reflect" is $R(Q\vec{x})=(RQ)\vec{x}$, so the matrix on the right acts first:

$$RQ=\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}=\begin{bmatrix}0 & -1\\ -1 & 0\end{bmatrix}$$

Check with $\langle 1,0\rangle$: rotating gives $\langle 0,1\rangle$, reflecting gives $\langle 0,-1\rangle$, and $RQ\langle 1,0\rangle=\langle 0,-1\rangle$. ✓ The determinant of RQ is $0-1=-1$: areas are unchanged but orientation is reversed.

Matrices Modeling Contexts

Matrices model systems that step forward in stages – for example, populations moving between states each year. Repeated multiplication by a **transition matrix** advances the model one step at a time, so matrix powers predict long-run behavior.

> **Worked example.** Each year 90% of a town's residents who use bus A keep using it and 10% switch to bus B; 20% of bus B users switch to A and 80% stay. Write the model with each column holding where one group goes:

$$T = \begin{bmatrix} 0.9 & 0.2 \\ 0.1 & 0.8 \end{bmatrix}, \qquad T \begin{bmatrix} 600 \\ 400 \end{bmatrix} = \begin{bmatrix} 0.9(600) + 0.2(400) \\ 0.1(600) + 0.8(400) \end{bmatrix} = \begin{bmatrix} 620 \\ 380 \end{bmatrix}.$$

After one year, 620 use A and 380 use B. The numbers stop changing when the flows balance, $0.1A = 0.2B$, so $A = 2B$. Out of 1000 residents, that is about 667 on A and 333 on B in the long run.

Exam tips

- A **parametric** function gives $x(t)$ and $y(t)$ separately – track the direction of motion as t increases.
- A **vector** has magnitude and direction; add component-by-component and find magnitude with $\sqrt{a^2 + b^2}$.
- The **determinant** of a 2×2 matrix is $ad - bc$ (mind the minus sign); its absolute value scales area under the transformation.
- A 2×2 matrix transforms the plane (rotate, reflect, stretch); an inverse exists only when the determinant is non-zero.
- Note this unit is not tested on the AP Precalculus Exam (Units 1–3 only), but it underpins later courses.

Common mistakes.

- Giving a whole curve after eliminating the parameter; the t-interval limits the path and sets its direction.
- Writing the determinant as $ad + bc$, or the inverse without swapping a and d and negating b and c.
- Multiplying transformation matrices in the wrong order: the matrix applied first goes on the right.
- Treating the dot product as a vector. It is a single number.

Practice.

1. A particle has $x(t) = t + 1$ and $y(t) = t^2 - 2$ for $-2 \le t \le 1$. (a) Find its start and end points. (b) Eliminate the parameter and state the x-interval. (c) Find the average rate of change of y with respect to t on $-2 \le t \le 0$, and say what it means for the motion.
2. Parametrize the circle with centre $(-1, 4)$ and radius 3, traced **clockwise** from $(2, 4)$.
3. Identify the conic $9x^2 + 4y^2 - 36 = 0$ and give its vertices.
4. For $\vec{u} = \langle 1, 2 \rangle$ and $\vec{v} = \langle 3, -1 \rangle$, find $\vec{u} + \vec{v}$, $|\vec{v}|$, $\vec{u} \cdot \vec{v}$ and the angle between them.
5. Let $A = \begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix}$. Find $\det A$ and A^{-1}, then use A^{-1} to solve $2x + y = 4$, $5x + 3y = 11$.
6. Using the bus transition matrix T above, find the numbers using buses A and B after **two** years, starting from 600 and 400.

Answers.

1. (a) Start $(-1, 2)$ at $t = -2$; end $(2, -1)$ at $t = 1$. (b) $t = x - 1$, so $y = (x-1)^2 - 2$ for $-1 \le x \le 2$. (c) $\dfrac{y(0) - y(-2)}{0 - (-2)} = \dfrac{-2 - 2}{2} = -2$: on average the particle moves down 2 units per unit of time.
2. $x = -1 + 3\cos t$, $y = 4 - 3\sin t$ for $0 \le t < 2\pi$ (starts at $(2, 4)$; the minus sign reverses the direction).
3. $\dfrac{x^2}{4} + \dfrac{y^2}{9} = 1$: an ellipse centred at the origin, vertices $(0, \pm 3)$ and co-vertices $(\pm 2, 0)$.
4. $\langle 4, 1 \rangle$; $\sqrt{10}$; $3 - 2 = 1$; $\cos\theta = \dfrac{1}{\sqrt{5}\sqrt{10}} = \dfrac{1}{\sqrt{50}} \approx 0.141$, so $\theta \approx 1.43$ radians (about $82°$).
5. $\det A = 6 - 5 = 1$; $A^{-1} = \begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix}$; $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3(4) - 11 \\ -5(4) + 2(11) \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.
6. After one year $(620, 380)$. Then $T \begin{bmatrix} 620 \\ 380 \end{bmatrix} = \begin{bmatrix} 558 + 76 \\ 62 + 304 \end{bmatrix} = \begin{bmatrix} 634 \\ 366 \end{bmatrix}$.

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