Course Companion

AP Precalculus

Every topic handout in one volume

全部专题讲义合集

exercises.lol

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Polynomial and Rational Functions

AP Precalculus

Change in Tandem

A **function** 函数 is a rule that maps each input to **exactly one** output. The set of allowed inputs is the **domain** 定义域; the set of outputs is the **range** 值域. The input variable is the **independent variable** 自变量 and the output variable is the **dependent variable** 因变量. A function rule can be shown graphically, numerically, analytically (a formula), or verbally.

As the input changes, the output changes "in tandem" – together. Over an interval, a function is:

- **increasing** 递增 if, whenever $a < b$, then $f(a) < f(b)$ (bigger input, bigger output);
- **decreasing** 递减 if, whenever $a < b$, then $f(a) > f(b)$ (bigger input, smaller output).

A graph of a function shows all its input–output pairs, so you can read this behavior straight off the picture.

Rates of Change

The **average rate of change** 平均变化率 of a function over an interval is the change in output divided by the change in input – the single constant rate that would give the same total change. Over $[a, b]$:

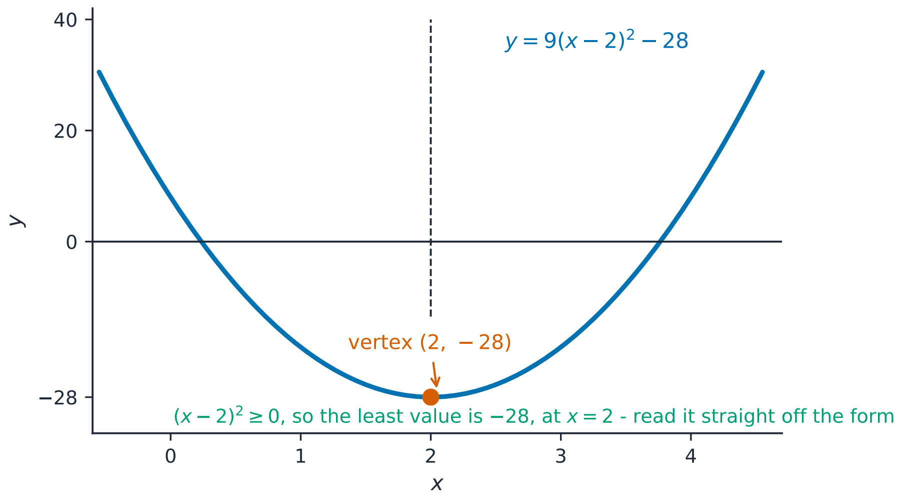
$$\text{avg rate} = \frac{f(b) - f(a)}{b - a}.$$

The **rate of change at a point** measures how fast the output changes right at that input. You **approximate** it with average rates over small intervals around the point. Comparing two points, the one with the larger average rate (over small enough intervals) is changing faster. A **positive** rate means both quantities move the same way; a **negative** rate means they move in opposite ways.

Worked example. For $f(x) = x^2$, the average rate of change over $[1, 4]$ is $\dfrac{f(4) - f(1)}{4 - 1} = \dfrac{16 - 1}{3} = 5$. Over $[1, 2]$ it is $\dfrac{4 - 1}{1} = 3$ – the rate itself changes, which is exactly why f is not linear.

Rates of Change in Linear and Quadratic Functions

The average rate of change over $[a, b]$ is the slope of the **secant line** 割线 from $(a, f(a))$ to $(b, f(b))$.



Completing the square gives the vertex of a parabola

- For a **linear** 线性 function, the average rate of change over any interval is **constant** – so the rate at which the rate changes is zero.
- For a **quadratic** 二次 function, the average rates of change over equal-length intervals themselves form a **linear** pattern – so those rates change at a **constant** rate.

This "rate of the rate" idea distinguishes function types: a constant second difference signals a quadratic.

A hanging bridge cable takes the shape of a parabola-like curve — a quadratic you can see in steel

Polynomial Functions and Rates of Change

A smooth bridge curve: polynomial and rational graphs model real shapes with asymptotes and turns

A nonconstant **polynomial** 多项式 has the form

$$p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0, \quad a_n \neq 0.$$

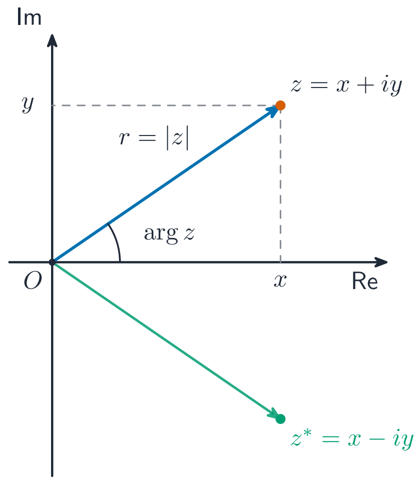
Its **degree** 次数 is n, its **leading term** 首项 is $a_n x^n$, and its **leading coefficient** 首项系数 is a_n. A constant is a polynomial of degree 0.

Key features:

- Where a polynomial switches between increasing and decreasing, it has a **local (relative) extremum** 局部极值. The greatest local maximum is the **global (absolute) maximum** 全局最大值 *only when the polynomial has one* — but most polynomials are unbounded (every odd-degree polynomial runs off to $\pm\infty$), so they have no global maximum or minimum at all.
- Between any two distinct real zeros there is at least one local maximum or minimum.
- An **even-degree** polynomial has either a global maximum or a global minimum.
- A **point of inflection** 拐点 is where the graph changes concavity – from **concave up** 上凹 to **concave down** 下凹 or the reverse – i.e. where the rate of change switches between increasing and decreasing.
- A function has **symmetry** when it is **even** 偶函数 – $f(-x) = f(x)$, graph symmetric across the y-axis (like x^2, x^4) – or **odd** 奇函数 – $f(-x) = -f(x)$, graph symmetric by rotation about the origin (like x^3, x^5). Test by substituting $-x$ and simplifying; later, $\cos\theta$ is even and $\sin\theta$ is odd.

Polynomial Functions and Complex Zeros

A **zero** 零点 (or **root** 根) of p is a number a with $p(a) = 0$. For a real a, $(x - a)$ is a factor of p exactly when a is a zero. If the factor $(x - a)$ is repeated n times, that zero has **multiplicity** 重数 n. Counting multiplicities, a degree-n polynomial has exactly n **complex** 复数 zeros.

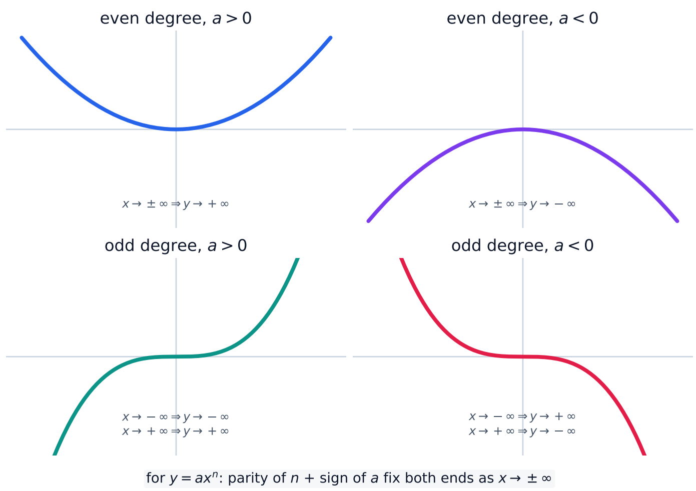


A complex number on an Argand diagram: modulus is the distance, argument the angle

- A real zero a gives an x**-intercept** at $(a, 0)$; real zeros are the endpoints of the intervals where $p(x) \geq 0$ or ≤ 0.
- Non-real zeros come in **conjugate** 共轭 pairs: if $a + bi$ is a zero, so is $a - bi$.
- At a real zero of **even** multiplicity the graph is **tangent** to the x-axis (it touches but does not cross); at odd multiplicity it crosses.
- The degree equals the least n for which the nth **successive differences** of equally-spaced outputs become constant.

Polynomial Functions and End Behavior

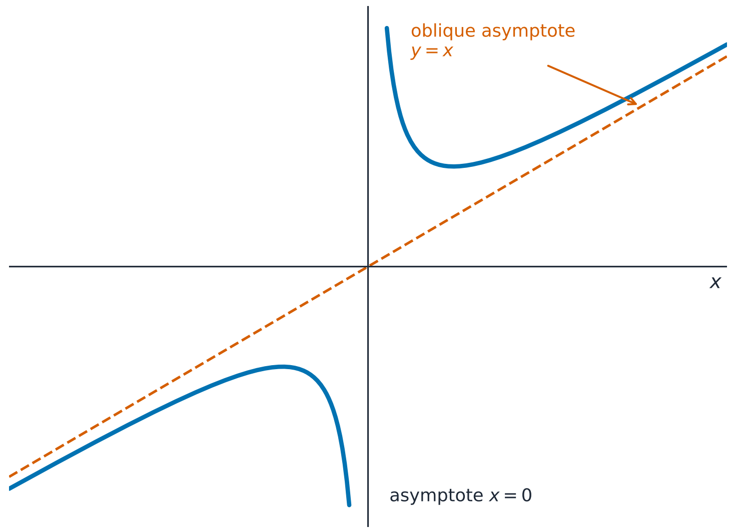
End behavior 末端行为 describes where a function heads as the input grows without bound. For a nonconstant polynomial, as $x \to \pm\infty$ the output goes to $+\infty$ or $-\infty$, written e.g. $\lim_{x\to\infty} p(x) = \infty$. Which way depends entirely on the **leading term** $a_n x^n$, because for large $|x|$ it **dominates** all lower-degree terms: the sign of a_n and whether n is even or odd fix both ends.



Every polynomial's ends are decided by its degree's parity and the sign of its leading coefficient

Rational Functions and End Behavior

A **rational function 有理函数** is a quotient of two polynomials, $r(x) = \dfrac{\text{numerator}}{\text{denominator}}$. Its end behavior is governed by the **quotient of the leading terms**:



When the numerator has higher degree, the curve approaches a slanting asymptote

- **Numerator degree > denominator degree:** the quotient is a nonconstant polynomial, and r follows that polynomial's end behavior. If that quotient is linear, the graph has a **slant asymptote 斜渐近线**.
- **Equal degrees:** the quotient is a constant, giving a **horizontal asymptote 水平渐近线** $y =$ the ratio of leading coefficients.
- **Numerator degree < denominator degree:** the quotient tends to 0, so the horizontal asymptote is $y = 0$.

At a horizontal asymptote $y = b$, the outputs get and stay arbitrarily close to b: $\lim_{x\to\pm\infty} r(x) = b$.

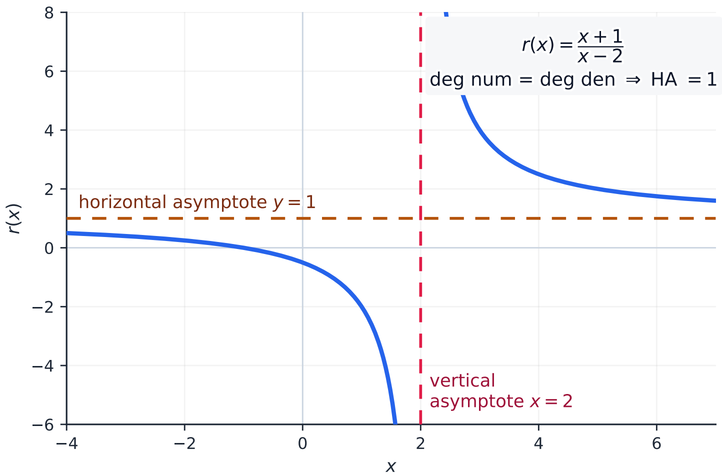
Rational Functions and Zeros

The real **zeros** of a rational function are the real zeros of its **numerator** that are still in the domain. These zeros, together with the zeros of the denominator, split the number line into the intervals you test when solving inequalities $r(x) \geq 0$ or $r(x) \leq 0$.

Rational Functions and Vertical Asymptotes

A **vertical asymptote** 垂直渐近线 occurs at $x = a$ when a is a zero of the **denominator** but not cancelled by the numerator – more precisely, when its multiplicity in the denominator exceeds its multiplicity in the numerator. Near it, the denominator is nearly zero, so r shoots to $\pm\infty$: $\lim_{x\to a^\pm} r(x) = \pm\infty$. Check each side, since the two sides can go opposite ways.

Worked example. Describe $r(x) = \dfrac{2x^2+3}{x^2-1}$. The numerator and denominator have **equal degree**, so the horizontal asymptote is $y = \dfrac{2}{1} = 2$. The denominator $x^2 - 1$ is zero at $x = \pm 1$ and neither cancels, so there are **vertical asymptotes** at $x = 1$ and $x = -1$.



A rational function approaching a vertical asymptote and a horizontal asymptote

Rational Functions and Holes

A **hole** 空洞 (removable point) occurs at $x = c$ when a factor cancels – the multiplicity of the zero c in the numerator is at least its multiplicity in the denominator. The graph is missing a single point. Its height is the limit of the simplified function: if inputs near c give outputs near L, the hole is at (c, L), and $\lim_{x\to c} r(x) = L$.

Equivalent Representations of Polynomial and Rational Expressions

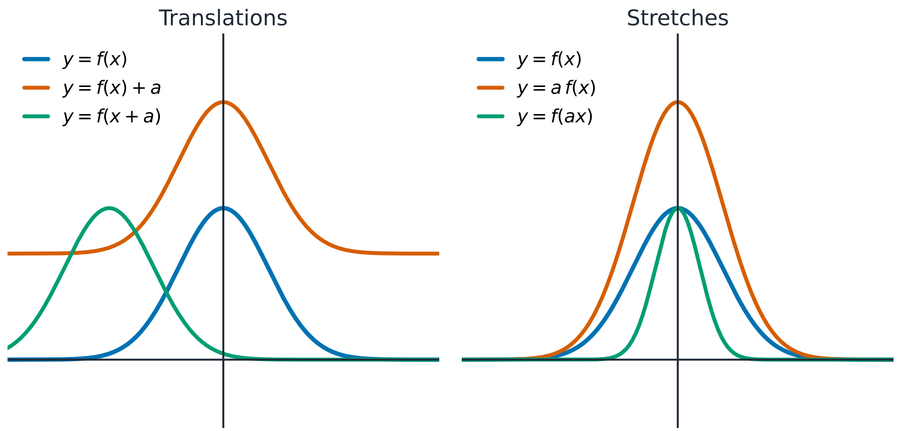
The same expression, written differently, reveals different features:

- **Factored form** shows real zeros, and hence x-intercepts, holes, vertical asymptotes, and domain.
- **Standard form** (expanded) shows the degree and leading term, and hence end behavior.

Polynomial long division 多项式长除法 rewrites $f(x) = g(x)\,q(x) + r(x)$, where q is the **quotient** 商 and r the **remainder** 余数 (of smaller degree than g). The quotient gives the equation of a **slant asymptote** when the numerator's degree is one more than the denominator's.

Transformations of Functions

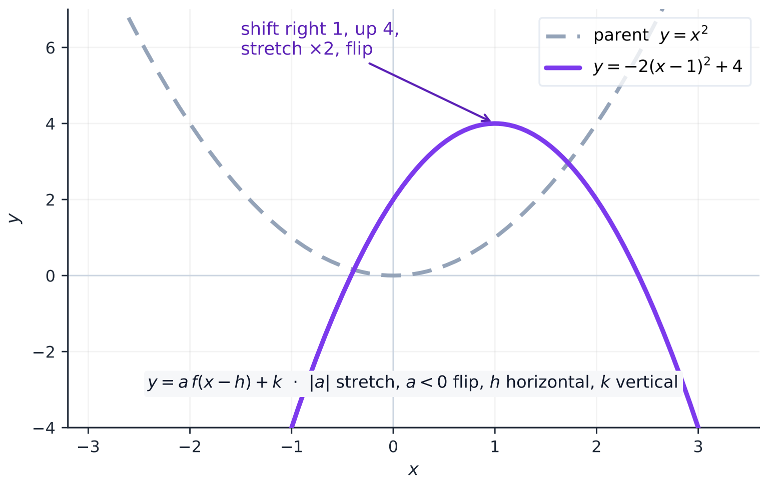
A **transformation** 变换 builds a new function from an old one f:



Adding to the output or input shifts the curve; a multiplier stretches it

- **Translations** 平移 (shifts): $f(x) + k$ moves up/down; $f(x - h)$ moves right/left.
- **Dilations** 伸缩 (stretches): $a\,f(x)$ stretches vertically; $f(bx)$ stretches horizontally.
- **Reflections** 反射: $-f(x)$ flips over the x-axis; $f(-x)$ flips over the y-axis.

Combine additive shifts and multiplicative stretches to model shifted, scaled versions of a known shape.



Transforming the parent parabola: shift, stretch, and reflect

Function Model Selection and Assumption Articulation

Choosing a **model** 模型 means matching a function type to how a quantity changes. A linear model fits a constant rate of change; a quadratic fits a constant second difference; a polynomial fits data with several turns. In geometric situations the number of dimensions is a strong hint: quantities about **area** (two dimensions) are typically **quadratic**, and quantities about **volume** (three dimensions) are typically **cubic** – for example a box's volume as a function of a cut length. State the **assumptions** 假设 your model relies on (for example, that the pattern continues), because a model is only valid where those assumptions hold.

Function Model Construction and Application

To **construct** a model, use given features – points, intercepts, zeros with multiplicities, end behavior – to write the function, then use it to answer questions in context. Always check the answer against the **domain** that makes sense for the situation, and interpret outputs with their real-world units.

A **piecewise-defined function** 分段函数 uses different rules over non-overlapping intervals of the domain. To evaluate it, pick the branch whose interval contains the input. For example,

$$f(x) = \begin{cases} x^2, & x < 0 \\ 2x, & x \geq 0 \end{cases}$$

gives $f(-3) = 9$ from the first branch but $f(3) = 6$ from the second – useful when behaviour changes at a threshold (a tiered price, a speed limit that changes).

Exam tips

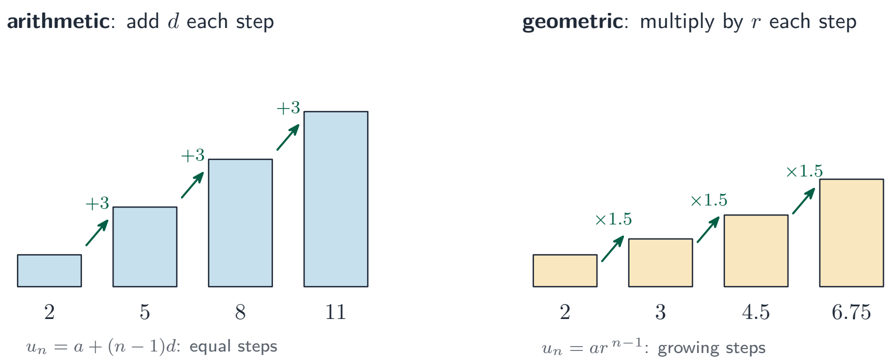
- Read a function through its **rate of change**: average rate = secant slope; a constant rate means linear, a linear-changing rate means quadratic.
- Factor a polynomial to find **zeros** (x-intercepts) and their multiplicity; even multiplicity touches, odd crosses the axis.
- End behaviour is set by the **leading term**; a rational function's asymptotes come from the degrees of top and bottom.
- Build models from key features (points, zeros, end behaviour) and state your assumptions.
- Describe transformations precisely: shift $f(x-h)+k$, stretch $af(bx)$, reflect $-f(x)$ / $f(-x)$.

Exponential and Logarithmic Functions

AP Precalculus

Change in Arithmetic and Geometric Sequences

A **sequence** 数列 is a function from the whole numbers to the real numbers, so its graph is **discrete points**, not a curve.



An arithmetic sequence climbs in equal steps; a geometric one multiplies by a ratio

- An **arithmetic sequence** 等差数列 has a **common difference** 公差 d (a constant rate of change): $a_n = a_0 + dn$, or from a known term, $a_n = a_k + d(n-k)$.
- A **geometric sequence** 等比数列 has a **common ratio** 公比 r (a constant proportional change): $g_n = g_0\, r^{\,n}$, or $g_n = g_k\, r^{\,(n-k)}$.

An increasing arithmetic sequence grows by the **same amount** each step; an increasing geometric sequence grows by a **larger amount** each step.

Worked example. An arithmetic sequence with $a_0 = 3$ and $d = 5$ has $a_4 = 3+5(4) = 23$. A geometric sequence with $g_0 = 2$ and $r = 3$ has $g_4 = 2 \cdot 3^4 = 162$ – addition versus multiplication makes the geometric one pull far ahead.

Change in Linear and Exponential Functions

A sprinter: exponential and logarithmic models capture growth and decay rates in context

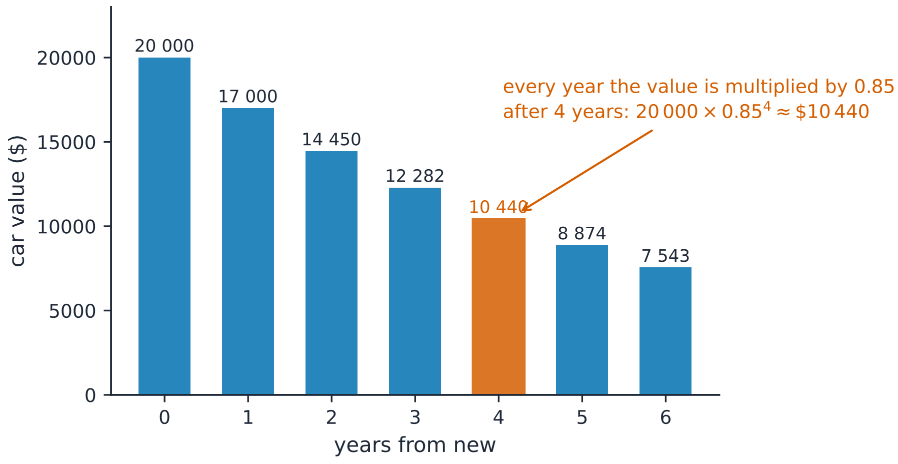
Sequences have continuous cousins:

- **Linear functions** $f(x) = b + mx$ mirror arithmetic sequences – an initial value plus repeated addition of the slope m. Point form: $f(x) = y_i + m(x - x_i)$.
- **Exponential functions** $f(x) = ab^x$ mirror geometric sequences – an initial value times repeated multiplication by the base b. Point form: $f(x) = y_i\, r^{\,(x-x_i)}$.

The difference: linear = repeated **addition**, exponential = repeated **multiplication**. (A sequence and its function may have different domains.)

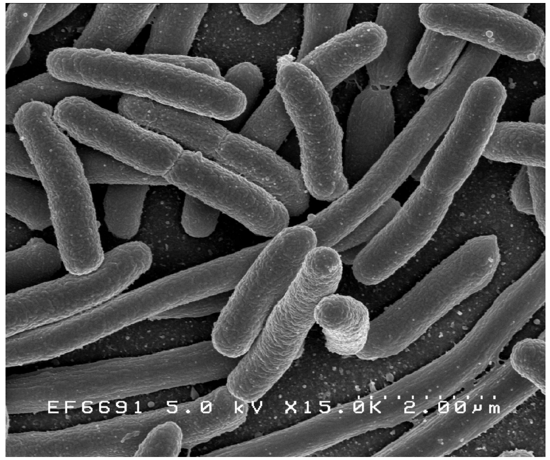
Exponential Functions

The general **exponential function** 指数函数 is $f(x) = ab^x$ with **initial value** 初始值 $a \neq 0$ and **base** 底数 $b > 0,\ b \neq 1$. Domain: all real numbers.



Exponential decay: a fixed percentage is lost each period

- $a > 0,\ b > 1$ gives **exponential growth** 指数增长; $a > 0,\ 0 < b < 1$ gives **exponential decay** 指数衰减.
- Outputs are **proportional** over equal-length input intervals. So an exponential is always increasing or always decreasing, and always concave up or always concave down – it has **no extrema** (except on a closed interval) and **no points of inflection**.



Bacteria under a microscope: exponential models describe populations that grow by a constant factor each time step

Exponential Function Manipulation

The exponent rules reshape exponential expressions and connect to graph transformations:

- **Product:** $b^m b^n = b^{m+n}$. A horizontal shift b^{x+k} equals a vertical stretch ab^x with $a = b^k$.
- **Power:** $(b^m)^n = b^{mn}$. A horizontal stretch b^{cx} equals a base change $(b^c)^x$.
- **Negative exponent:** $b^{-n} = \dfrac{1}{b^n}$.
- **Unit-fraction exponent:** $b^{1/k} = \sqrt[k]{b}$ (the kth **root** 方根).

Exponential Function Context and Data Modeling

Exponentials model quantities that grow by a **constant proportion** over equal intervals (repeated multiplication).

- Build a model from a ratio and an initial value, or from **two points** (solve the system for a and b).
- Sometimes a constant must be **added** to the data to reveal the proportional pattern.
- Use **exponential regression** 指数回归 on technology to fit a data set.
- The **natural base** 自然底数 $e \approx 2.718$ is the standard base for real-world models.

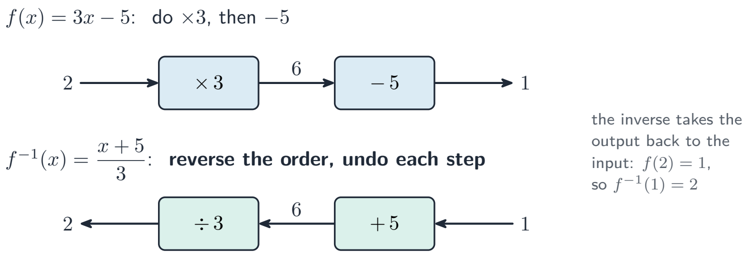
Competing Function Model Validation

When a rate of change shifts only slightly, linear, quadratic, and exponential models may all seem to fit. Choose using context and fit quality:

- A model is **appropriate** if its **residual plot** 残差图 shows **no pattern** (a **residual** 残差 is actual minus predicted).
- The **error** is the gap between predicted and actual; context decides whether an over- or under-estimate is safer.

Composition of Functions

The **composite function** 复合函数 $(f \circ g)(x) = f(g(x))$ feeds g's output into f. Its domain is the inputs of g whose outputs lie in f's domain.



A function as a machine; its inverse runs the machine backwards

- Composition is **not commutative**: $f(g(x))$ and $g(f(x))$ usually differ.
- To build $f(g(x))$ analytically, **substitute** $g(x)$ for every x in f.
- The **identity function** 恒等函数 $f(x) = x$ leaves any function unchanged under composition.
- A function can also be **decomposed** into simpler pieces – useful for seeing an additive shift as composing with $x + k$, or a dilation as composing with kx.

Matryoshka dolls: function composition nests layers — evaluate the inside first

Inverse Functions

A function is **invertible** 可逆 on a domain where each output comes from a **unique** input (you may restrict the domain to force this). The **inverse function** 反函数 f^{-1} reverses the mapping: if $f(a) = b$ then $f^{-1}(b) = a$.

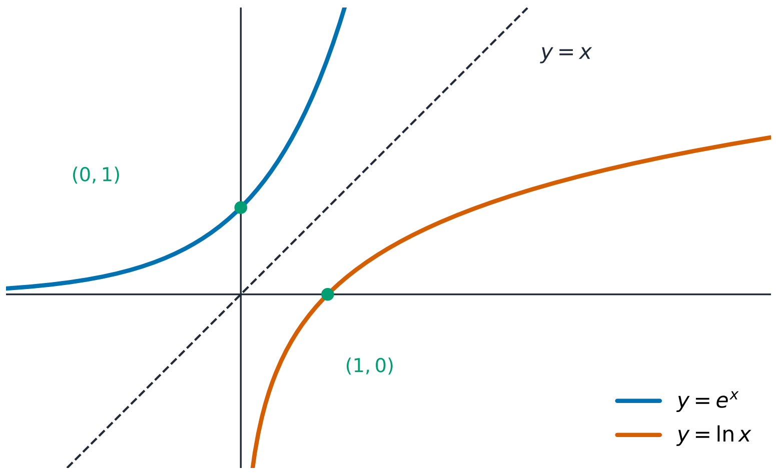
- $f\big(f^{-1}(x)\big) = f^{-1}\big(f(x)\big) = x$ (composition gives the identity).
- Domain and range **swap**: reverse each (a, b) to (b, a) in a table; reflect the graph over the line $y = x$.
- To find a formula: swap x and y in $y = f(x)$, then solve for y. Context may further limit where the inverse applies.

Logarithmic Expressions

The **logarithm** 对数 answers "what exponent?": $\log_b c = a$ means exactly $b^a = c$ (with $b > 0,\ b \neq 1$). An unwritten base means the **common logarithm** 常用对数 (base 10). On a **logarithmic scale**, each unit is a **multiplicative** step of the base $(\ldots, 10^0, 10^1, 10^2, \ldots)$.

Inverses of Exponential Functions

The logarithm is the **inverse of the exponential**: $y = b^x$ and $y = \log_b x$ undo each other, so their graphs are reflections over $y = x$. Hence $\log_b(b^x) = x$ and $b^{\log_b x} = x$. The **natural logarithm** 自然对数 $\ln x = \log_e x$ is the inverse of e^x.



e^x and ln x are reflections of each other in the line $y = x$

Logarithmic Functions

The **logarithmic function** 对数函数 $f(x) = \log_b x$ has domain $x > 0$ and range all reals. It is always increasing (for $b > 1$) or always decreasing (for $0 < b < 1$), always concave one way, with a **vertical asymptote** at $x = 0$ – the mirror image of the exponential's horizontal asymptote. It grows very slowly for large x.

Logarithmic Function Manipulation

The log properties reverse the exponent rules:

$$\log_b(xy) = \log_b x + \log_b y, \quad \log_b \frac{x}{y} = \log_b x - \log_b y, \quad \log_b(x^n) = n \log_b x.$$

The **change-of-base** 换底 formula lets you compute any log with technology: $\log_b x = \dfrac{\log x}{\log b} = \dfrac{\ln x}{\ln b}$.

Exponential and Logarithmic Equations and Inequalities

To solve, use the fact that exponentials and logs are inverses:

- Isolate the exponential, then take a log of both sides (bring the exponent down with the power property).
- Isolate the log, then exponentiate both sides.
- Always check for **extraneous solutions** – the argument of a log must stay **positive**.

Worked example. Solve $2 \cdot 3^x = 54$. Divide by 2: $3^x = 27 = 3^3$, so $x = 3$. When the sides are not tidy powers, take logs instead: $5^x = 20$ gives $x = \dfrac{\ln 20}{\ln 5} \approx 1.86$.

Logarithmic Function Context and Data Modeling

Logarithms model quantities that change over huge multiplicative ranges (sound, acid strength, earthquakes). Build a log model from data, and use **logarithmic regression** for a data set. Because a log compresses large values, it turns proportional growth into a straight-line pattern – the idea behind semi-log plots.

Semi-log Plots

A **semi-log plot** 半对数图 puts the output on a **logarithmic** axis and the input on a normal axis. On these axes, an **exponential** function $y = ab^x$ becomes a **straight line**, because $\log y = \log a + (\log b)\,x$ is linear in x. So: if data look linear on a semi-log plot, an exponential model fits; the line's slope gives $\log b$ and its intercept gives $\log a$.

Exam tips

- Distinguish **arithmetic** (add a common difference) from **geometric** (multiply a common ratio) sequences and their linear/exponential function cousins.
- Exponential = repeated **multiplication**, so it eventually outgrows any linear or polynomial model.
- The **logarithm is the inverse** of the exponential ($\log_b c = a \Leftrightarrow b^a = c$); use it to solve $b^x = k$.
- Apply the log laws (product, quotient, power) and the change-of-base formula; the argument of a log must be **positive**.
- On a **semi-log plot** an exponential model becomes a straight line.

Trigonometric and Polar Functions

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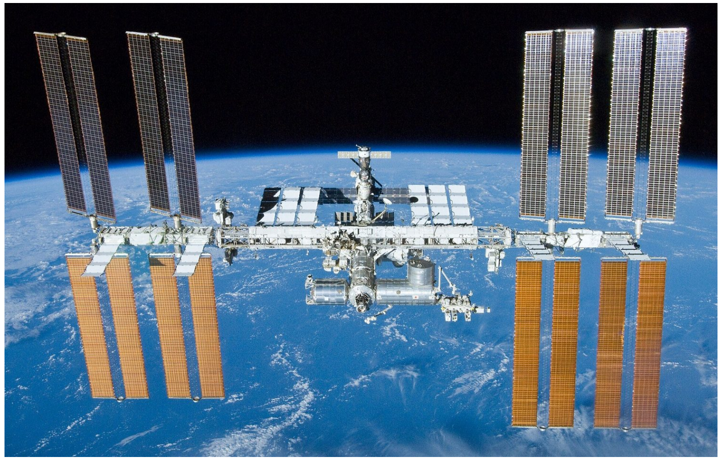
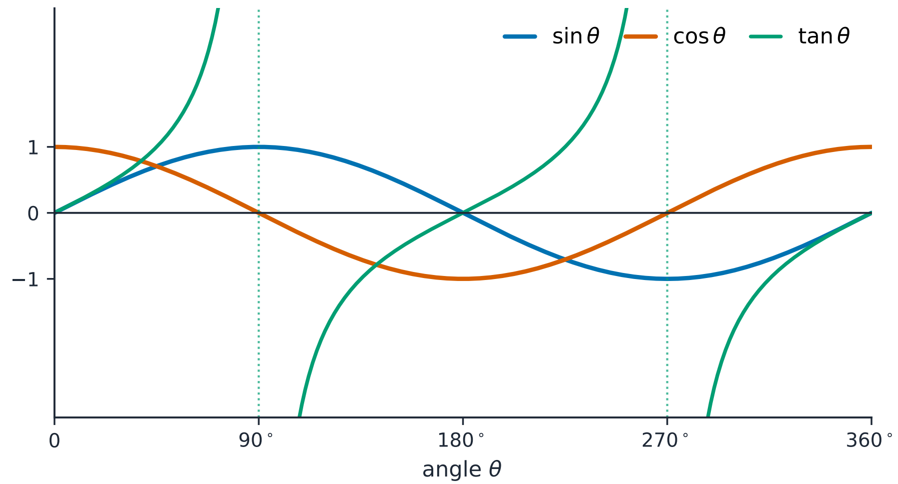
Periodic Phenomena



The ISS in orbit: trigonometric and polar functions describe periodic and circular motion

A relationship is **periodic** 周期性 if its output pattern repeats at regular input steps. The **period** 周期 is the smallest positive k with $f(x+k) = f(x)$ for all x. You can build the whole graph by copying a single **cycle** 周期段, and estimate the period by finding how far apart the pattern repeats. Within each cycle a periodic function still has intervals of increase/decrease and maxima/minima.



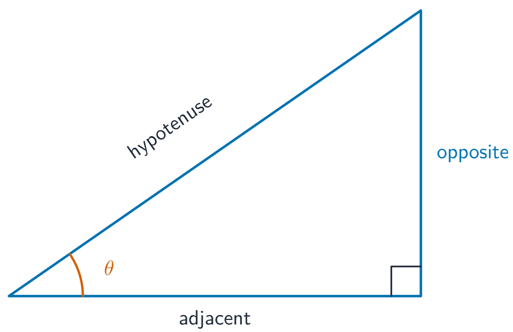
sin and cos wave between -1 and 1; tan breaks at 90 and 270 degrees

As a Ferris wheel turns steadily, your height rises and falls over and over — a real sine wave in time

Sine, Cosine, and Tangent

An angle is in **standard position** 标准位置 when its vertex is at the origin and its initial ray lies on the positive x-axis. Its **radian** 弧度 measure is the arc length on a unit circle. For the point P where the terminal ray meets a circle of radius r:

$$\cos\theta = \frac{x}{r}, \qquad \sin\theta = \frac{y}{r}, \qquad \tan\theta = \frac{y}{x} \text{ (the slope of the terminal ray).}$$



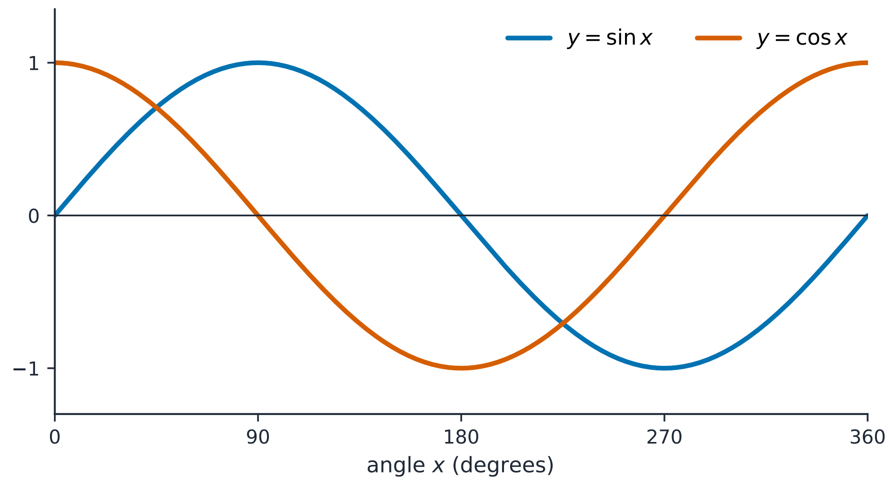
Naming the sides of a right triangle from the angle

Sine and Cosine Function Values

On a circle of radius r, $x = r\cos\theta$ and $y = r\sin\theta$. Exact values at the special angles $(0, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2}, \ldots)$ come from **isosceles right** and **equilateral triangle** 等边三角形 geometry, with signs set by the **quadrant** 象限 of the angle.

Sine and Cosine Function Graphs

On the **unit circle** 单位圆 $(r = 1)$, $\sin\theta$ is the y-coordinate and $\cos\theta$ is the x-coordinate. As θ increases, both **oscillate** 振荡 smoothly between -1 and 1 with period 2π. Sine starts at 0 (rising); cosine starts at 1. They are the same wave shifted by $\frac{\pi}{2}$.



y = sin x and y = cos x are smooth waves between -1 and 1

Sinusoidal Functions

A **sinusoidal function** 正弦型函数 is any transformation of sine (or cosine). Its key features:

- **Period** and **frequency** 频率 are reciprocals; $\sin\theta$ has period 2π.
- **Amplitude** 振幅 = half the distance between the maximum and minimum output.
- **Midline** 中线 $y = d$ = the average of the maximum and minimum (the horizontal center line).

The graph alternates concave up and concave down each half-cycle.

Worked example. For $f(\theta) = 3\sin(2\theta) + 1$: the amplitude is 3, the period is $\dfrac{2\pi}{2} = \pi$, and the midline is $y = 1$. So the maximum output is $1 + 3 = 4$ and the minimum is $1 - 3 = -2$.

Sinusoidal Function Transformations

The general sinusoid is

$$f(\theta) = a\sin\big(b(\theta+c)\big)+d \quad \text{or} \quad a\cos\big(b(\theta+c)\big)+d,$$

where $|a|$ is the **amplitude**, $\dfrac{2\pi}{|b|}$ is the **period**, $-c$ is the horizontal **phase shift** 相移 (a $+c$ inside the bracket shifts the graph **left** by c), and d is the **vertical shift** (midline). A negative a reflects the wave.

Sinusoidal Function Context and Data Modeling

To model periodic data (tides, daylight, temperature): read the **period** from where the pattern repeats, get the **amplitude** and **midline** from the max and min, and set the phase shift so a peak lands at the right input. Technology fits a sinusoidal regression. Such a model is only trustworthy over its **contextual domain**.

The Tangent Function

$\tan\theta = \dfrac{\sin\theta}{\cos\theta}$ is the **slope** of the terminal ray. Its slope values repeat every half-revolution, so its period is π. It has **vertical asymptotes** where $\cos\theta = 0$ (at $\theta = \frac{\pi}{2} + k\pi$), and it is always increasing between consecutive asymptotes, switching concavity at each zero.

Inverse Trigonometric Functions

The **inverse trigonometric functions** 反三角函数 – **arcsine** 反正弦, **arccosine**, **arctangent** – reverse the trig functions, so they take a ratio and return an angle. Because sine, cosine, and tangent repeat, their domains must be **restricted** (e.g. sine to $[-\frac{\pi}{2}, \frac{\pi}{2}]$) before an inverse can exist.

Trigonometric Equations and Inequalities

Use inverse trig functions to solve trig equations. Because the functions are **periodic**, there are usually **infinitely many** solutions – write the general solution by adding multiples of the period, then keep those in the required (often contextual) domain.

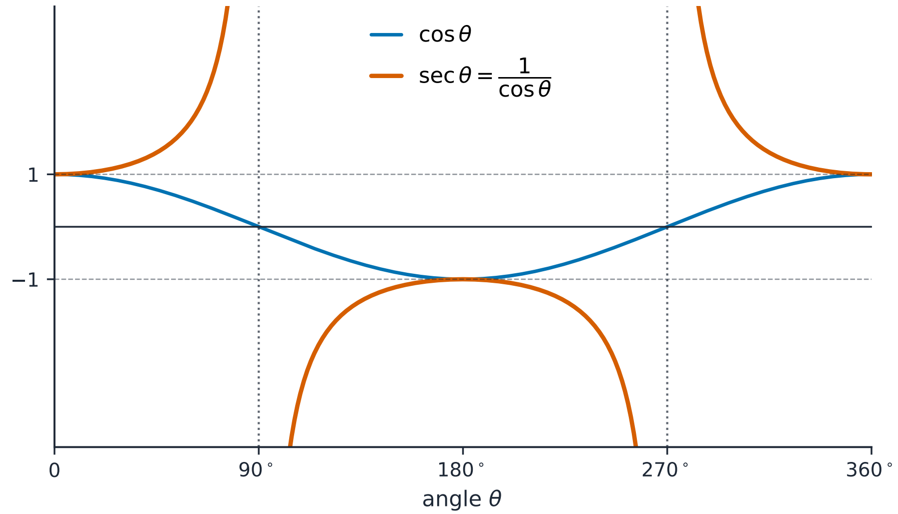
Worked example. Solve $2\sin\theta = 1$ on $[0, 2\pi)$. Then $\sin\theta = \frac{1}{2}$, and on the unit circle sine is positive in the first and second quadrants, so $\theta = \dfrac{\pi}{6}$ or $\dfrac{5\pi}{6}$. Over all reals you would add $2\pi k$ to each.

The Secant, Cosecant, and Cotangent Functions

These are the **reciprocals** 倒数 of the main three:

$$\sec\theta = \frac{1}{\cos\theta}, \qquad \csc\theta = \frac{1}{\sin\theta}, \qquad \cot\theta = \frac{1}{\tan\theta}.$$

Secant and cosecant have vertical asymptotes where cosine and sine are zero; cotangent has them where tangent is zero (where sine is zero).



sec is the reciprocal of cos, with asymptotes where cos is zero

Equivalent Representations of Trigonometric Functions

The **Pythagorean identity** 毕达哥拉斯恒等式 comes from $x^2 + y^2 = 1$ on the unit circle:

$$\sin^2\theta + \cos^2\theta = 1,$$

which rearranges into forms with sec and tan. The **sum identities** 和角公式:

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta, \qquad \cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta,$$

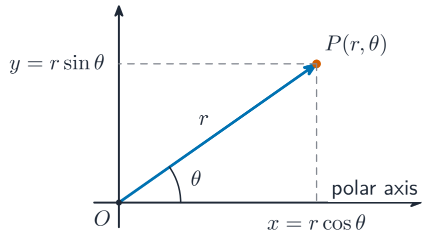
also give the difference and **double-angle** 二倍角 identities (set $\beta = \alpha$).

Trigonometry and Polar Coordinates

Polar coordinates 极坐标 locate a point by (r, θ) – distance r from the origin at angle θ. Convert with

$$x = r\cos\theta, \quad y = r\sin\theta, \qquad r = \sqrt{x^2 + y^2}, \quad \tan\theta = \frac{y}{x}.$$

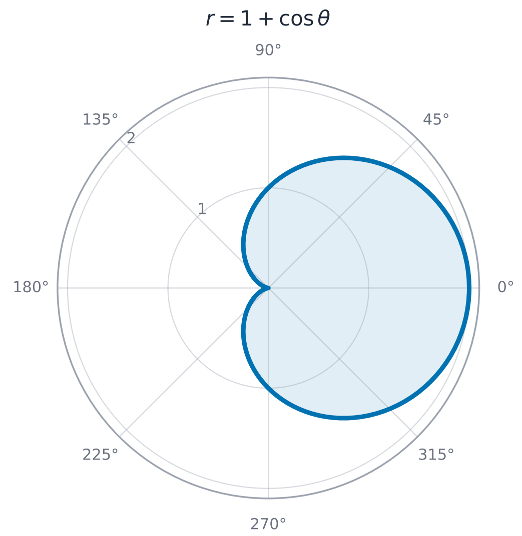
A **complex number** 复数 is the point $(x, y) = x + yi$ and can likewise be written with r and θ.



Polar coordinates give a point by its distance r and angle theta

Polar Function Graphs

The graph of $r = f(\theta)$ is all points (r, θ) as θ sweeps around. Restricting the domain of θ draws only part of the curve. As θ increases, r grows or shrinks, tracing spirals, circles, roses, and limaçons.



A cardioid r = 1 + cos theta, sketched directly in polar form

Rates of Change in Polar Functions

As θ increases, the **distance from the origin** r changes:

- r positive and increasing (or negative and decreasing) $\Rightarrow$ moving **away** from the origin;
- r positive and decreasing (or negative and increasing) $\Rightarrow$ moving **toward** the origin.

Where r switches between increasing and decreasing, the distance reaches a relative extreme. The **average rate of change** of r with respect to θ, $\dfrac{\Delta r}{\Delta\theta}$, estimates how fast the curve moves in or out over an interval.

Exam tips

- Use radians and the **unit circle**: $\cos\theta$ and $\sin\theta$ are the coordinates, always between -1 and 1.
- For a sinusoid $a\sin(b(\theta+c))+d$: $|a|$ is the amplitude, $\frac{2\pi}{|b|}$ the period, d the midline, and the phase shift is $-c$ (a $+c$ shifts left).
- Model periodic data by reading period, amplitude, and midline from the max and min.
- **Inverse trig** functions need a restricted domain and return an angle; trig equations have infinitely many solutions (add the period).
- Convert polar↔rectangular with $x = r\cos\theta$, $y = r\sin\theta$.

Functions Involving Parameters, Vectors, and Matrices

AP Precalculus

Parametric Functions

A roller coaster: vectors and parametric motion describe path and velocity together

Unit 4 is part of the course but is not assessed on the AP Precalculus Exam (which covers only Units 1–3); it builds tools used in later courses.

A **parametric function** 参数函数 describes a curve by giving both coordinates as functions of a third variable, the **parameter** 参数 t: $\big(x(t),\, y(t)\big)$. As t runs through its domain, the point traces a path – and, unlike $y = f(x)$, the path may loop or cross itself.

Parametric Functions and Planar Motion

Reading t as **time**, a parametric function models motion in the plane. $x(t)$ and $y(t)$ give the horizontal and vertical position; the point's **direction** of travel is the order in which the curve is drawn as t increases (mark it with arrows).

Parametric Functions and Rates of Change

Over an interval of t, the average rate of change of x is $\frac{\Delta x}{\Delta t}$ and of y is $\frac{\Delta y}{\Delta t}$. Their signs tell you which way the point moves (right/left, up/down); together they describe the motion's speed and direction along the path.

Parametrically Defined Circles and Lines

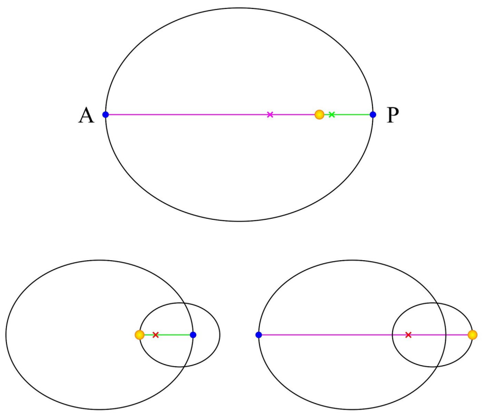
A **circle** of radius R centered at (h, k) is $x = h + R\cos t,\ y = k + R\sin t$. A **line** through (x_0, y_0) with direction (a, b) is $x = x_0 + at,\ y = y_0 + bt$. Adjusting the coefficients changes the start point, speed, and direction of tracing.

Implicitly Defined Functions

An **implicit** 隐式 equation relates x and y without solving for either, such as $x^2+y^2 = 25$. Its graph may fail the vertical-line test (not a function), so it is often split into pieces or described parametrically.

Conic Sections

Conic sections 圆锥曲线 – circles, ellipses, parabolas, and hyperbolas – are the curves from slicing a cone, each given by a quadratic equation in x and y. Their standard forms reveal centers, vertices, axes, and asymptotes.



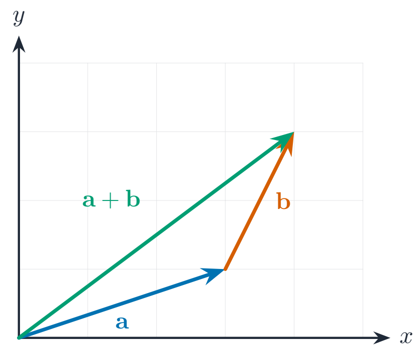
Elliptical orbits: conic sections describe closed paths with two foci — a key parametric model

Parametrizing Implicit Curves

Many implicit curves can be **parametrized** – rewritten as $\big(x(t), y(t)\big)$ – which makes them easier to graph and to treat as motion. The circle above is the basic example; ellipses use $x = h + A\cos t,\ y = k + B\sin t$.

Vectors

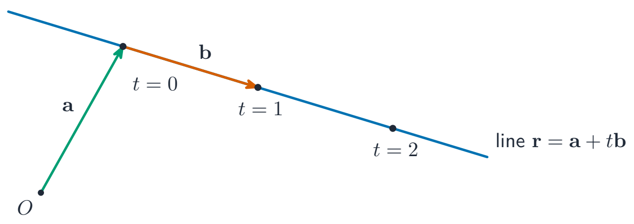
A **vector** 向量 has both **magnitude** 大小 (length) and **direction**. Write it by components $\langle a, b\rangle$. Add vectors component-by-component; scale by multiplying each component; the magnitude is $\sqrt{a^2+b^2}$. Vectors model displacements, velocities, and forces.



Adding vectors by the triangle law

Vector-Valued Functions

A **vector-valued function** 向量值函数 outputs a vector for each input, e.g. $\vec{r}(t) = \langle x(t), y(t)\rangle$ – the same information as a parametric function, packaged as a moving position vector.



A vector-valued line: start at a, then slide by t lots of the direction b

Matrices

A **matrix** 矩阵 is a rectangular array of numbers. Add matrices of the same size entry-by-entry; multiply a matrix by a compatible one by combining rows with columns. Matrices store and transform data compactly.

The Inverse and Determinant of a Matrix

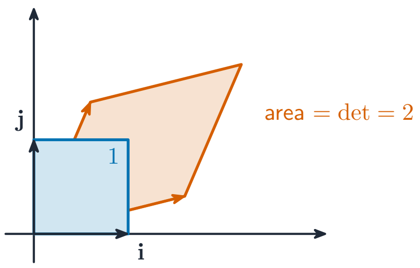
The **determinant** 行列式 of a 2×2 matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is $ad-bc$. A matrix has an **inverse** 逆矩阵 exactly when its determinant is nonzero; the inverse undoes the matrix, and it solves matrix equations (like a reciprocal for numbers).

Worked example. For $A = \begin{bmatrix} 3 & 1 \\ 2 & 4 \end{bmatrix}$, the determinant is $ad-bc = (3)(4)-(1)(2) = 10$. Since it is nonzero, A is invertible, and $A^{-1} = \frac{1}{10}\begin{bmatrix} 4 & -1 \\ -2 & 3 \end{bmatrix}$ (swap the diagonal, negate the off-diagonal, divide by the determinant).

Linear Transformations and Matrices

A 2×2 matrix acts as a **linear transformation** 线性变换 of the plane – rotating, reflecting, stretching, or shearing points – by multiplying each position vector. The determinant measures how the transformation scales area (and its sign tells whether orientation flips).

Worked example. The matrix $\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$ stretches the plane by 2 horizontally and 3 vertically, sending the vector $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ to $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$. Its determinant $2 \times 3 = 6$ means every area is multiplied by 6, so the unit square becomes a 2×3 rectangle.



A 2x2 matrix maps the unit square to a parallelogram; the determinant is the area scale

Matrices as Functions

Because a matrix maps input vectors to output vectors, it **is** a function on the plane. Composing transformations corresponds to multiplying their matrices, and the inverse matrix reverses the mapping.

Matrices Modeling Contexts

Matrices model systems that step forward in stages – for example, populations moving between states each year. Repeated multiplication by a **transition matrix** advances the model one step at a time, so matrix powers predict long-run behavior.

Exam tips

- A **parametric** function gives $x(t)$ and $y(t)$ separately — track the direction of motion as t increases.
- A **vector** has magnitude and direction; add component-by-component and find magnitude with $\sqrt{a^2 + b^2}$.
- The **determinant** of a 2×2 matrix is $ad - bc$ (mind the minus sign); it scales area under the transformation.
- A 2×2 matrix transforms the plane (rotate, reflect, stretch); an inverse exists only when the determinant is non-zero.
- Note this unit is not tested on the AP Precalculus Exam (Units 1–3 only), but it underpins later courses.