Course Companion

AP Precalculus

Every topic handout in one volume

全部专题讲义合集

exercises.lol

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AP Precalculus

Change in Tandem

A **function** 函数 is a rule that maps each input to **exactly one** output. The set of allowed inputs is the **domain** 定义域; the set of outputs is the **range** 值域. The input variable is the **independent variable** 自变量 and the output variable is the **dependent variable** 因变量. A function rule can be shown graphically, numerically, analytically (a formula), or verbally.

As the input changes, the output changes "in tandem" –together. Over an interval, a function is:

- **increasing** 递增 if, whenever $a < b$, then $f(a) < f(b)$ (bigger input, bigger output);
- **decreasing** 递减 if, whenever $a < b$, then $f(a) > f(b)$ (bigger input, smaller output).

A graph of a function shows all its input–output pairs, so you can read this behavior straight off the picture.

Rates of Change

The **average rate of change** 平均变化率 of a function over an interval is the change in output divided by the change in input –the single constant rate that would give the same total change. Over $[a, b]$:

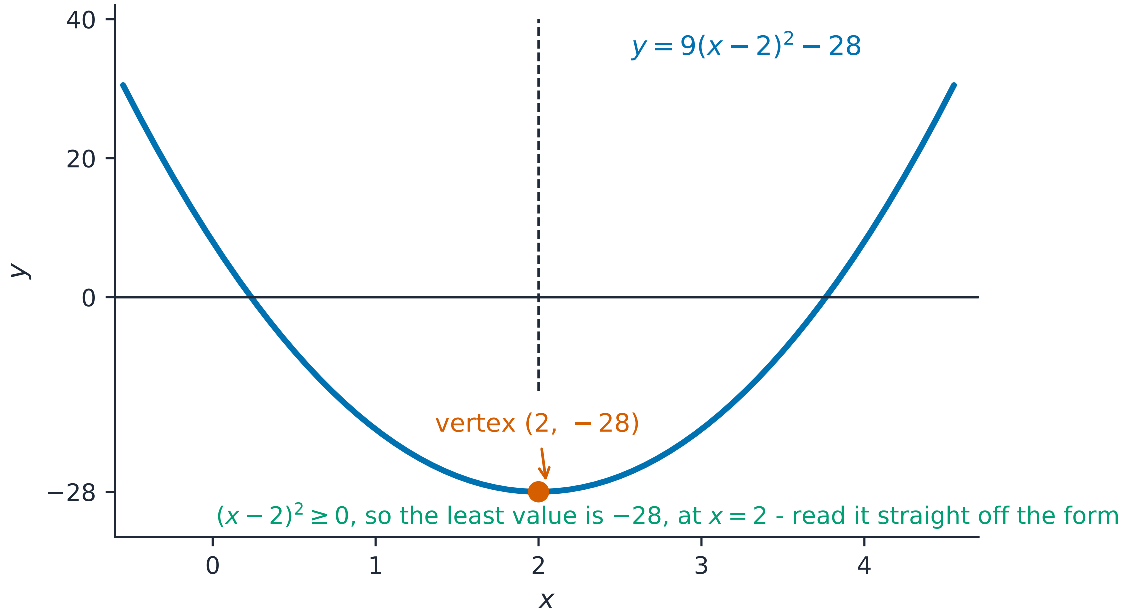
$$\text{avg rate} = \frac{f(b) - f(a)}{b - a}.$$

The **rate of change at a point** measures how fast the output changes right at that input. You **approximate** it with average rates over small intervals around the point. Comparing two points, the one with the larger average rate (over small enough intervals) is changing faster. A **positive** rate means both quantities move the same way; a **negative** rate means they move in opposite ways.

Worked example. For $f(x) = x^2$, the average rate of change over $[1, 4]$ is $\dfrac{f(4) - f(1)}{4 - 1} = \dfrac{16 - 1}{3} = 5$. Over $[1, 2]$ it is $\dfrac{4 - 1}{1} = 3$ –the rate itself changes, which is exactly why f is not linear.

Rates of Change in Linear and Quadratic Functions

The average rate of change over $[a, b]$ is the slope of the **secant line** 割线 from $(a, f(a))$ to $(b, f(b))$.



Completing the square gives the vertex of a parabola

- For a **linear** 线性 function, the average rate of change over any interval is **constant** –so the rate at which the rate changes is zero.
- For a **quadratic** 二次 function, the average rates of change over equal-length intervals themselves form a **linear** pattern –so those rates change at a **constant** rate.

This ”rate of the rate” idea distinguishes function types: a constant second difference signals a quadratic.

A hanging bridge cable takes the shape of a parabola-like curve —a quadratic you can see in steel

Polynomial Functions and Rates of Change

A smooth bridge curve: polynomial and rational graphs model real shapes with asymptotes and turns

A nonconstant **polynomial 多项式** has the form

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0, \quad a_n \neq 0.$$

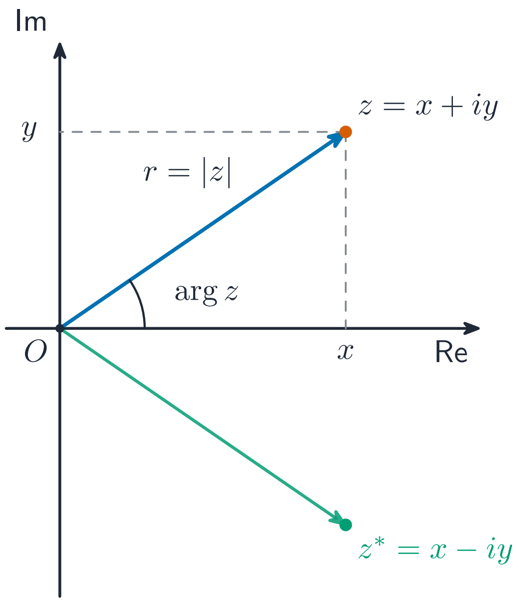
Its **degree** 次数 is n, its **leading term** 首项 is $a_n x^n$, and its **leading coefficient** 首项系数 is a_n. A constant is a polynomial of degree 0.

Key features:

- Where a polynomial switches between increasing and decreasing, it has a **local (relative) extremum** 局部极值. The greatest local maximum is the **global (absolute) maximum** 全局最大值 *only when the polynomial has one* —but most polynomials are unbounded (every odd-degree polynomial runs off to $\pm\infty$), so they have no global maximum or minimum at all.
- Between any two distinct real zeros there is at least one local maximum or minimum.
- An **even-degree** polynomial has either a global maximum or a global minimum.
- A **point of inflection** 拐点 is where the graph changes concavity –from **concave up** 上凹 to **concave down** 下凹 or the reverse –i.e. where the rate of change switches between increasing and decreasing.
- A function has **symmetry** when it is **even** 偶函数–$f(-x) = f(x)$, graph symmetric across the y-axis (like x^2, x^4) –or **odd** 奇函数–$f(-x) = -f(x)$, graph symmetric by rotation about the origin (like x^3, x^5). Test by substituting $-x$ and simplifying; later, $\cos\theta$ is even and $\sin\theta$ is odd.

Polynomial Functions and Complex Zeros

A **zero** 零点 (or **root** 根) of p is a number a with $p(a) = 0$. For a real a, $(x - a)$ is a factor of p exactly when a is a zero. If the factor $(x - a)$ is repeated n times, that zero has **multiplicity** 重数 n. Counting multiplicities, a degree-n polynomial has exactly n **complex** 复数 zeros.

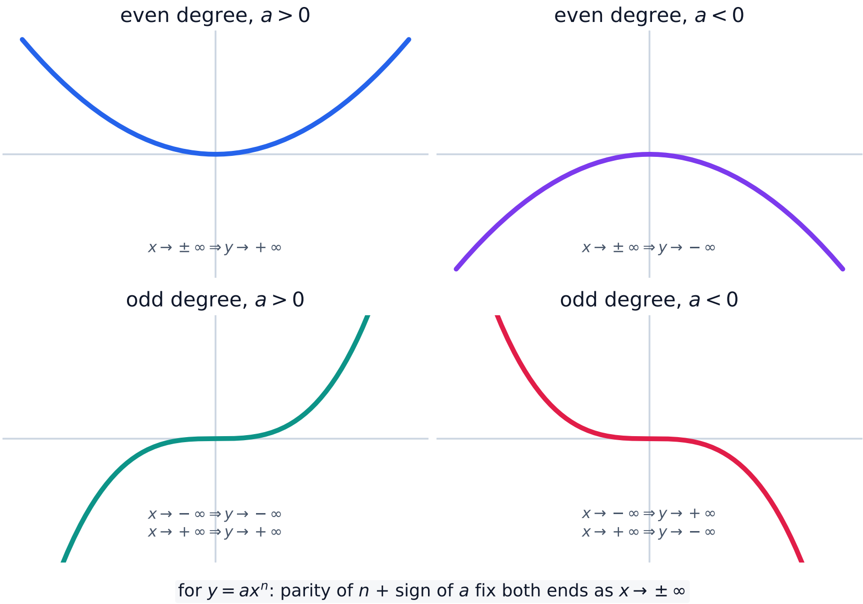


A complex number on an Argand diagram: modulus is the distance, argument the angle

- A real zero a gives an x-**intercept** at $(a, 0)$; real zeros are the endpoints of the intervals where $p(x) \geq 0$ or ≤ 0.
- Non-real zeros come in **conjugate** 共轭 pairs: if $a + bi$ is a zero, so is $a - bi$.
- At a real zero of **even** multiplicity the graph is **tangent** to the x-axis (it touches but does not cross); at odd multiplicity it crosses.
- The degree equals the least n for which the nth **successive differences** of equally-spaced outputs become constant.

Polynomial Functions and End Behavior

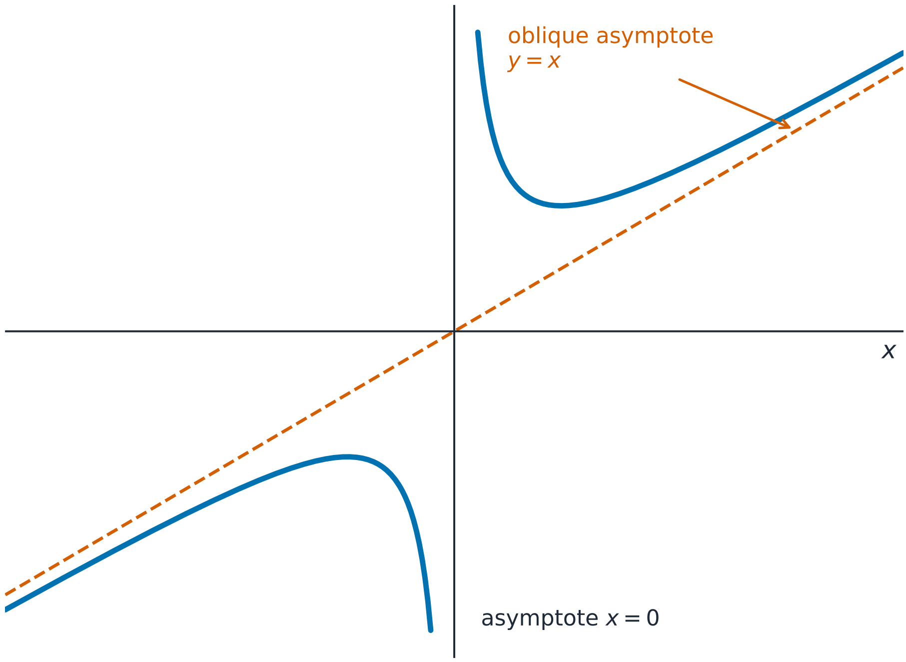
End behavior 末端行为 describes where a function heads as the input grows without bound. For a nonconstant polynomial, as $x \to \pm\infty$ the output goes to $+\infty$ or $-\infty$, written e.g. $\lim_{x\to\infty} p(x) = \infty$. Which way depends entirely on the **leading term** $a_n x^n$, because for large $|x|$ it **dominates** all lower-degree terms: the sign of a_n and whether n is even or odd fix both ends.



Every polynomial's ends are decided by its degree's parity and the sign of its leading coefficient

Rational Functions and End Behavior

A **rational function** 有理函数 is a quotient of two polynomials, $r(x) = \dfrac{\text{numerator}}{\text{denominator}}$. Its end behavior is governed by the **quotient of the leading terms**:



When the numerator has higher degree, the curve approaches a slanting asymptote

- **Numerator degree > denominator degree:** the quotient is a nonconstant polynomial, and r follows that polynomial's end behavior. If that quotient is linear, the graph has a **slant asymptote** 斜渐近线.
- **Equal degrees:** the quotient is a constant, giving a **horizontal asymptote** 水平渐近线 $y =$ the ratio of leading coefficients.
- **Numerator degree < denominator degree:** the quotient tends to 0, so the horizontal asymptote is $y = 0$.

At a horizontal asymptote $y = b$, the outputs get and stay arbitrarily close to b: $\lim_{x\to\pm\infty} r(x) = b$.

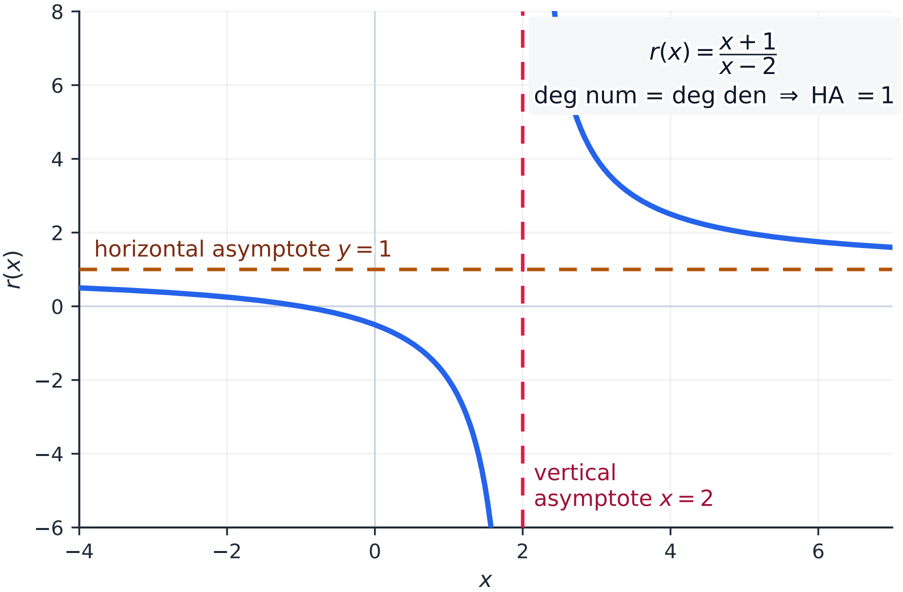
Rational Functions and Zeros

The real **zeros** of a rational function are the real zeros of its **numerator** that are still in the domain. These zeros, together with the zeros of the denominator, split the number line into the intervals you test when solving inequalities $r(x) \geq 0$ or $r(x) \leq 0$.

Rational Functions and Vertical Asymptotes

A **vertical asymptote** 垂直渐近线 occurs at $x = a$ when a is a zero of the **denominator** but not cancelled by the numerator –more precisely, when its multiplicity in the denominator exceeds its multiplicity in the numerator. Near it, the denominator is nearly zero, so r shoots to $\pm\infty$: $\lim_{x\to a^\pm} r(x) = \pm\infty$. Check each side, since the two sides can go opposite ways.

Worked example. Describe $r(x) = \dfrac{2x^2+3}{x^2-1}$. The numerator and denominator have **equal degree**, so the horizontal asymptote is $y = \dfrac{2}{1} = 2$. The denominator $x^2 - 1$ is zero at $x = \pm 1$ and neither cancels, so there are **vertical asymptotes** at $x = 1$ and $x = -1$.



A rational function approaching a vertical asymptote and a horizontal asymptote

Rational Functions and Holes

A **hole** 空洞 (removable point) occurs at $x = c$ when a factor cancels –the multiplicity of the zero c in the numerator is at least its multiplicity in the denominator. The graph is missing a single point. Its height is the limit of the simplified function: if inputs near c give outputs near L, the hole is at (c, L), and $\lim_{x\to c} r(x) = L$.

Equivalent Representations of Polynomial and Rational Expressions

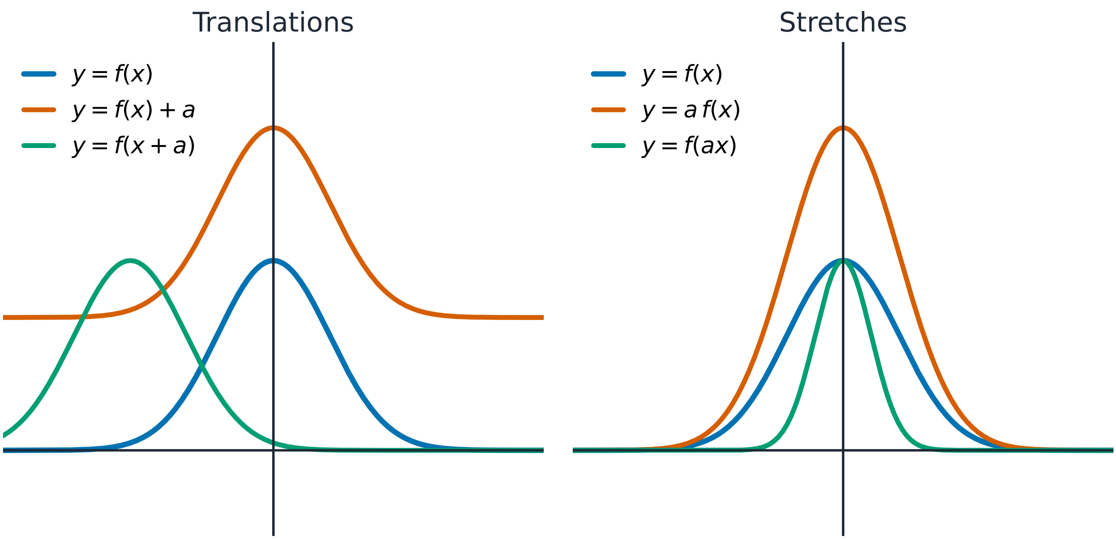
The same expression, written differently, reveals different features:

- **Factored form** shows real zeros, and hence x-intercepts, holes, vertical asymptotes, and domain.
- **Standard form** (expanded) shows the degree and leading term, and hence end behavior.

Polynomial long division 多项式长除法 rewrites $f(x) = g(x)\,q(x)+r(x)$, where q is the **quotient** 商 and r the **remainder** 余数 (of smaller degree than g). The quotient gives the equation of a **slant asymptote** when the numerator's degree is one more than the denominator's.

Transformations of Functions

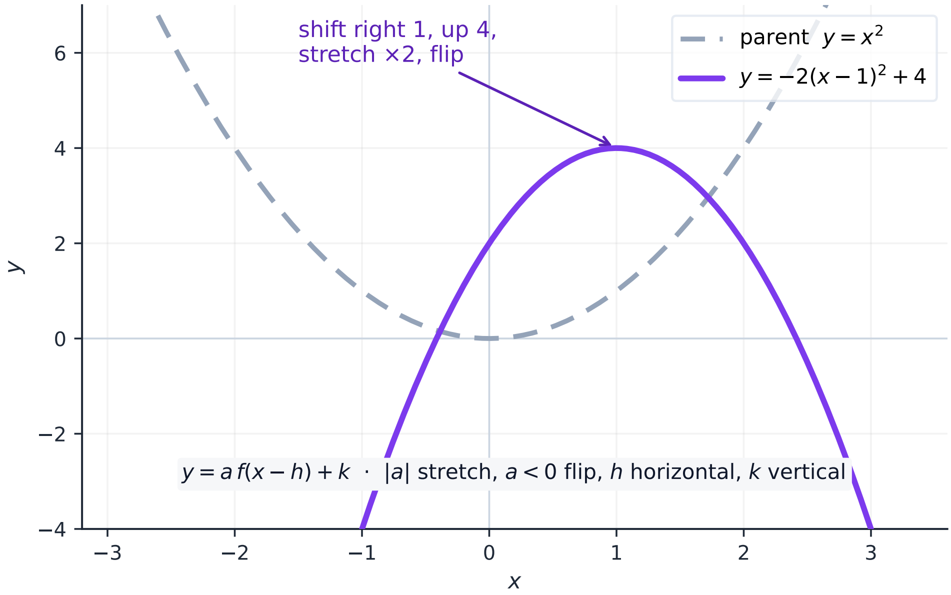
A **transformation** 变换 builds a new function from an old one f:



Adding to the output or input shifts the curve; a multiplier stretches it

- **Translations** 平移 (shifts): $f(x) + k$ moves up/down; $f(x - h)$ moves right/left.
- **Dilations** 伸缩 (stretches): $a\,f(x)$ stretches vertically; $f(bx)$ stretches horizontally.
- **Reflections** 反射: $-f(x)$ flips over the x-axis; $f(-x)$ flips over the y-axis.

Combine additive shifts and multiplicative stretches to model shifted, scaled versions of a known shape.



Transforming the parent parabola: shift, stretch, and reflect

Function Model Selection and Assumption Articulation

Choosing a **model** 模型 means matching a function type to how a quantity changes. A linear model fits a constant rate of change; a quadratic fits a constant second difference; a polynomial fits data with several turns. In geometric situations the number of dimensions is a strong hint: quantities about **area** (two dimensions) are typically **quadratic**, and quantities about **volume** (three dimensions) are typically **cubic** – for example a box's volume as a function of a cut length. State the **assumptions** 假设 your model relies on (for example, that the pattern continues), because a model is only valid where those assumptions hold.

Function Model Construction and Application

To **construct** a model, use given features –points, intercepts, zeros with multiplicities, end behavior –to write the function, then use it to answer questions in context. Always check the answer against the **domain** that makes sense for the situation, and interpret outputs with their real-world units.

A **piecewise-defined function** 分段函数 uses different rules over non-overlapping intervals of the domain. To evaluate it, pick the branch whose interval contains the input. For example,

$$f(x) = \begin{cases} x^2, & x < 0 \\ 2x, & x \geq 0 \end{cases}$$

gives $f(-3) = 9$ from the first branch but $f(3) = 6$ from the second –useful when behaviour changes at a threshold (a tiered price, a speed limit that changes).

Exam tips

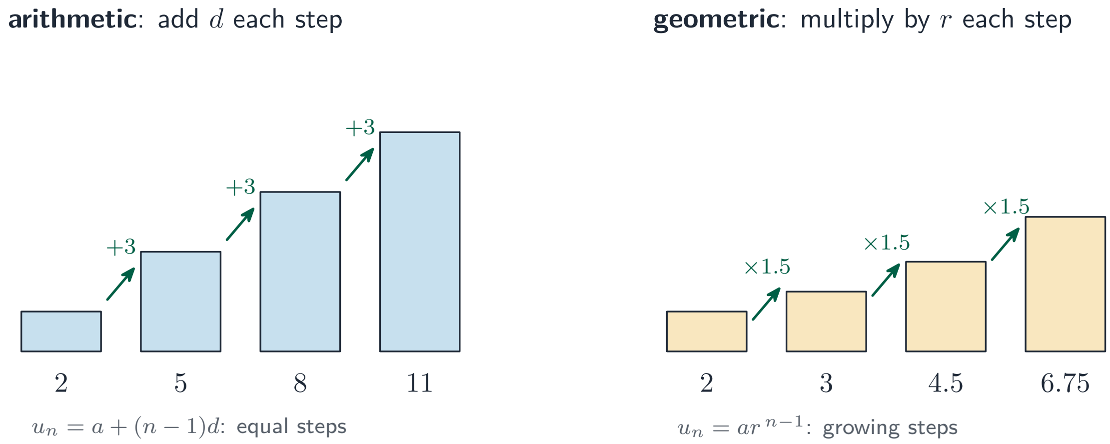
- Read a function through its **rate of change**: average rate = secant slope; a constant rate means linear, a linear-changing rate means quadratic.
- Factor a polynomial to find **zeros** (x-intercepts) and their multiplicity; even multiplicity touches, odd crosses the axis.
- End behaviour is set by the **leading term**; a rational function's asymptotes come from the degrees of top and bottom.
- Build models from key features (points, zeros, end behaviour) and state your assumptions.
- Describe transformations precisely: shift $f(x-h)+k$, stretch $af(bx)$, reflect $-f(x)$ / $f(-x)$.

Exponential and Logarithmic Functions

AP Precalculus

Change in Arithmetic and Geometric Sequences

A **sequence** 数列 is a function from the whole numbers to the real numbers, so its graph is **discrete points**, not a curve.



An arithmetic sequence climbs in equal steps; a geometric one multiplies by a ratio

- An **arithmetic sequence** 等差数列 has a **common difference** 公差 d (a constant rate of change): $a_n = a_0 + dn$, or from a known term, $a_n = a_k + d(n-k)$.
- A **geometric sequence** 等比数列 has a **common ratio** 公比 r (a constant proportional change): $g_n = g_0\, r^{\,n}$, or $g_n = g_k\, r^{\,(n-k)}$.

An increasing arithmetic sequence grows by the **same amount** each step; an increasing geometric sequence grows by a **larger amount** each step.

Worked example. An arithmetic sequence with $a_0 = 3$ and $d = 5$ has $a_4 = 3+5(4) = 23$. A geometric sequence with $g_0 = 2$ and $r = 3$ has $g_4 = 2 \cdot 3^4 = 162$ –addition versus multiplication makes the geometric one pull far ahead.

Change in Linear and Exponential Functions

A sprinter: exponential and logarithmic models capture growth and decay rates in context

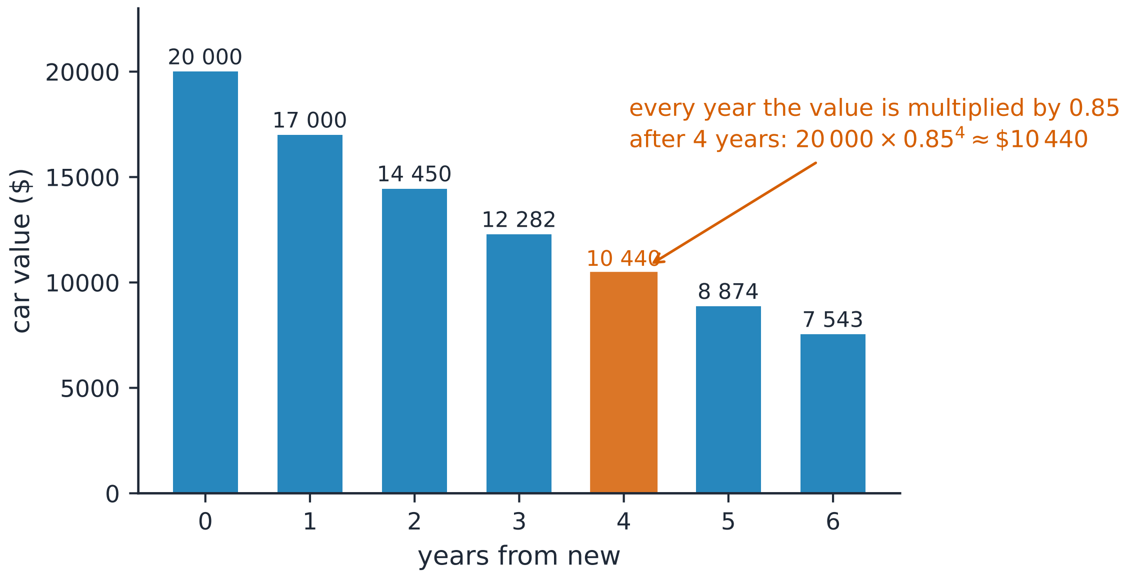
Sequences have continuous cousins:

- **Linear functions** $f(x) = b + mx$ mirror arithmetic sequences –an initial value plus repeated addition of the slope m. Point form: $f(x) = y_i + m(x - x_i)$.
- **Exponential functions** $f(x) = ab^x$ mirror geometric sequences –an initial value times repeated multiplication by the base b. Point form: $f(x) = y_i\, r^{(x-x_i)}$.

The difference: linear = repeated **addition**, exponential = repeated **multiplication**. (A sequence and its function may have different domains.)

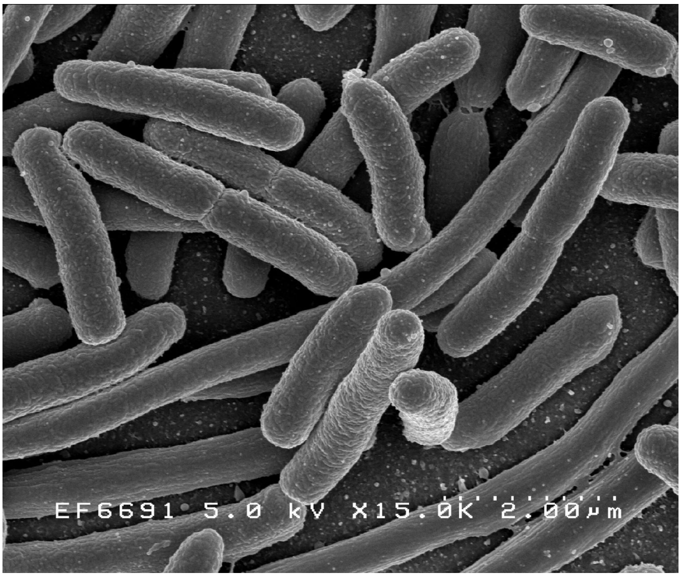
Exponential Functions

The general **exponential function** 指数函数 is $f(x) = ab^x$ with **initial value** 初始值 $a \neq 0$ and **base** 底数 $b > 0,\ b \neq 1$. Domain: all real numbers.



Exponential decay: a fixed percentage is lost each period

- $a > 0,\ b > 1$ gives **exponential growth** 指数增长; $a > 0,\ 0 < b < 1$ gives **exponential decay** 指数衰减.
- Outputs are **proportional** over equal-length input intervals. So an exponential is always increasing or always decreasing, and always concave up or always concave down –it has **no extrema** (except on a closed interval) and **no points of inflection**.



Bacteria under a microscope: exponential models describe populations that grow by a constant factor each time step

Exponential Function Manipulation

The exponent rules reshape exponential expressions and connect to graph transformations:

- **Product:** $b^m b^n = b^{m+n}$. A horizontal shift b^{x+k} equals a vertical stretch ab^x with $a = b^k$.
- **Power:** $(b^m)^n = b^{mn}$. A horizontal stretch b^{cx} equals a base change $(b^c)^x$.
- **Negative exponent:** $b^{-n} = \dfrac{1}{b^n}$.
- **Unit-fraction exponent:** $b^{1/k} = \sqrt[k]{b}$ (the kth **root** 方根).

Exponential Function Context and Data Modeling

Exponentials model quantities that grow by a **constant proportion** over equal intervals (repeated multiplication).

- Build a model from a ratio and an initial value, or from **two points** (solve the system for a and b).
- Sometimes a constant must be **added** to the data to reveal the proportional pattern.
- Use **exponential regression** 指数回归 on technology to fit a data set.
- The **natural base** 自然底数 $e \approx 2.718$ is the standard base for real-world models.

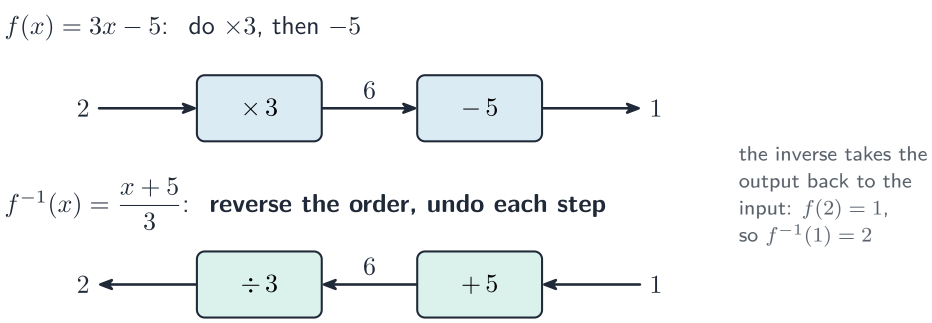
Competing Function Model Validation

When a rate of change shifts only slightly, linear, quadratic, and exponential models may all seem to fit. Choose using context and fit quality:

- A model is **appropriate** if its **residual plot** 残差图 shows **no pattern** (a **residual** 残差 is actual minus predicted).
- The **error** is the gap between predicted and actual; context decides whether an over- or under-estimate is safer.

Composition of Functions

The **composite function** 复合函数 $(f \circ g)(x) = f(g(x))$ feeds g's output into f. Its domain is the inputs of g whose outputs lie in f's domain.



A function as a machine; its inverse runs the machine backwards

- Composition is **not commutative**: $f(g(x))$ and $g(f(x))$ usually differ.
- To build $f(g(x))$ analytically, **substitute** $g(x)$ for every x in f.
- The **identity function** 恒等函数 $f(x) = x$ leaves any function unchanged under composition.
- A function can also be **decomposed** into simpler pieces –useful for seeing an additive shift as composing with $x + k$, or a dilation as composing with kx.

Matryoshka dolls: function composition nests layers —evaluate the inside first

Inverse Functions

A function is **invertible** 可逆 on a domain where each output comes from a **unique** input (you may restrict the domain to force this). The **inverse function** 反函数 f^{-1} reverses the mapping: if $f(a) = b$ then $f^{-1}(b) = a$.

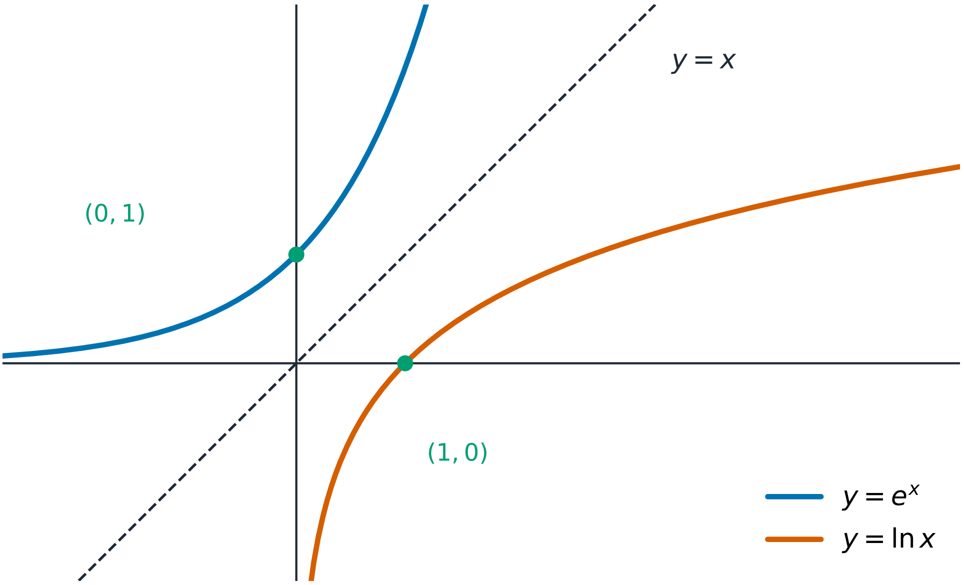
- $f\big(f^{-1}(x)\big) = f^{-1}\big(f(x)\big) = x$ (composition gives the identity).
- Domain and range **swap**: reverse each (a, b) to (b, a) in a table; reflect the graph over the line $y = x$.
- To find a formula: swap x and y in $y = f(x)$, then solve for y. Context may further limit where the inverse applies.

Logarithmic Expressions

The **logarithm** 对数 answers "what exponent?": $\log_b c = a$ means exactly $b^a = c$ (with $b > 0,\ b \neq 1$). An unwritten base means the **common logarithm** 常用对数 (base 10). On a **logarithmic scale**, each unit is a **multiplicative** step of the base ($\cdots, 10^0, 10^1, 10^2, \cdots$).

Inverses of Exponential Functions

The logarithm is the **inverse of the exponential**: $y = b^x$ and $y = \log_b x$ undo each other, so their graphs are reflections over $y = x$. Hence $\log_b(b^x) = x$ and $b^{\log_b x} = x$. The **natural logarithm** 自然对数 $\ln x = \log_e x$ is the inverse of e^x.



e^x and ln x are reflections of each other in the line $y = x$

Logarithmic Functions

The **logarithmic function** 对数函数 $f(x) = \log_b x$ has domain $x > 0$ and range all reals. It is always increasing (for $b > 1$) or always decreasing (for $0 < b < 1$), always concave one way, with a **vertical asymptote** at $x = 0$ –the mirror image of the exponential's horizontal asymptote. It grows very slowly for large x.

Logarithmic Function Manipulation

The log properties reverse the exponent rules:

$$\log_b(xy) = \log_b x + \log_b y, \quad \log_b \frac{x}{y} = \log_b x - \log_b y, \quad \log_b(x^n) = n \log_b x.$$

The **change-of-base** 换底 formula lets you compute any log with technology: $\log_b x = \dfrac{\log x}{\log b} = \dfrac{\ln x}{\ln b}$.

Exponential and Logarithmic Equations and Inequalities

To solve, use the fact that exponentials and logs are inverses:

- Isolate the exponential, then take a log of both sides (bring the exponent down with the power property).
- Isolate the log, then exponentiate both sides.
- Always check for **extraneous solutions** –the argument of a log must stay **positive**.

Worked example. Solve $2 \cdot 3^x = 54$. Divide by 2: $3^x = 27 = 3^3$, so $x = 3$. When the sides are not tidy powers, take logs instead: $5^x = 20$ gives $x = \dfrac{\ln 20}{\ln 5} \approx 1.86$.

Logarithmic Function Context and Data Modeling

Logarithms model quantities that change over huge multiplicative ranges (sound, acid strength, earthquakes). Build a log model from data, and use **logarithmic regression** for a data set. Because a log compresses large values, it turns proportional growth into a straight-line pattern –the idea behind semi-log plots.

Semi-log Plots

A **semi-log plot** 半对数图 puts the output on a **logarithmic** axis and the input on a normal axis. On these axes, an **exponential** function $y = ab^x$ becomes a **straight line**, because $\log y = \log a + (\log b)\, x$ is linear in x. So: if data look linear on a semi-log plot, an exponential model fits; the line's slope gives $\log b$ and its intercept gives $\log a$.

Exam tips

- Distinguish **arithmetic** (add a common difference) from **geometric** (multiply a common ratio) sequences and their linear/exponential function cousins.
- Exponential = repeated **multiplication**, so it eventually outgrows any linear or polynomial model.
- The **logarithm is the inverse** of the exponential ($\log_b c = a \Leftrightarrow b^a = c$); use it to solve $b^x = k$.
- Apply the log laws (product, quotient, power) and the change-of-base formula; the argument of a log must be **positive**.
- On a **semi-log plot** an exponential model becomes a straight line.

Trigonometric and Polar Functions

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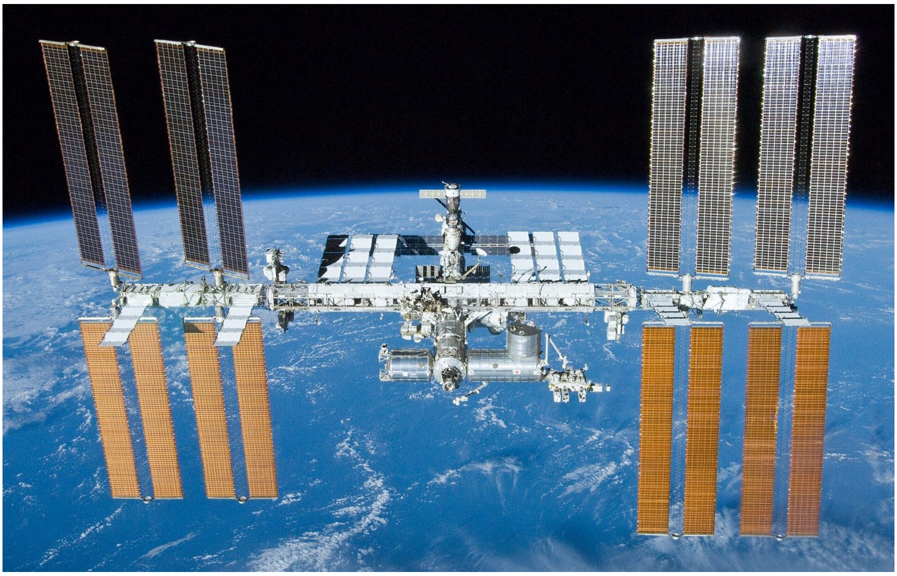
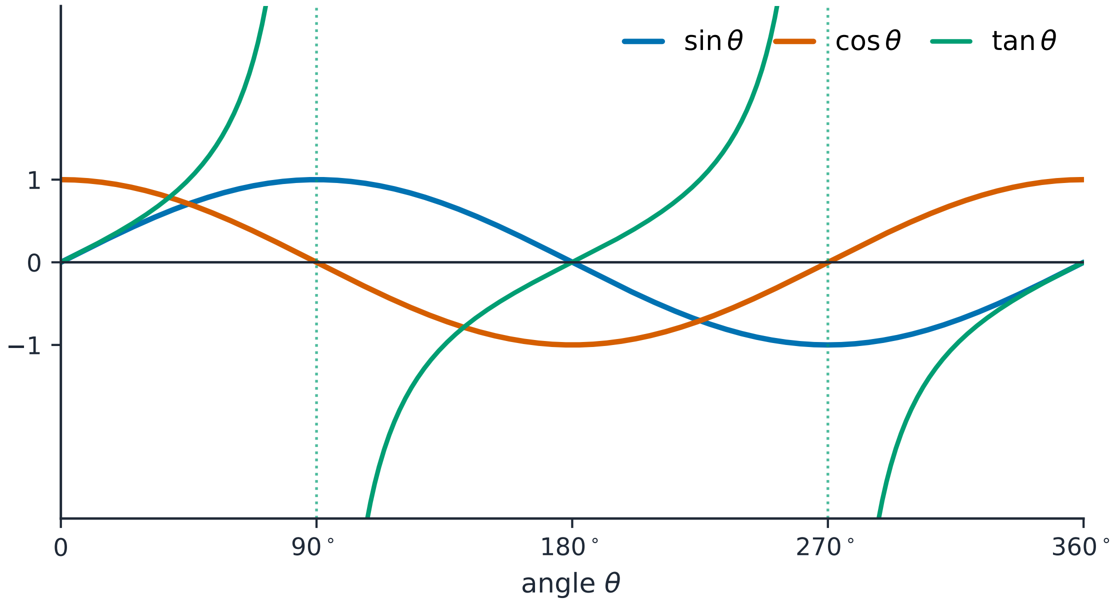
Periodic Phenomena



The ISS in orbit: trigonometric and polar functions describe periodic and circular motion

A relationship is **periodic** 周期性 if its output pattern repeats at regular input steps. The **period** 周期 is the smallest positive k with $f(x+k) = f(x)$ for all x. You can build the whole graph by copying a single **cycle** 周期段, and estimate the period by finding how far apart the pattern repeats. Within each cycle a periodic function still has intervals of increase/decrease and maxima/minima.



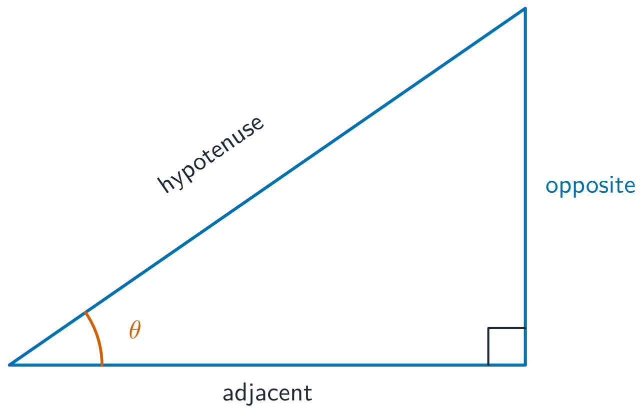
sin and cos wave between -1 and 1; tan breaks at 90 and 270 degrees

As a Ferris wheel turns steadily, your height rises and falls over and over —a real sine wave in time

Sine, Cosine, and Tangent

An angle is in **standard position** 标准位置 when its vertex is at the origin and its initial ray lies on the positive x-axis. Its **radian** 弧度 measure is the arc length on a unit circle. For the point P where the terminal ray meets a circle of radius r:

$$\cos\theta = \frac{x}{r}, \qquad \sin\theta = \frac{y}{r}, \qquad \tan\theta = \frac{y}{x} \text{ (the slope of the terminal ray).}$$



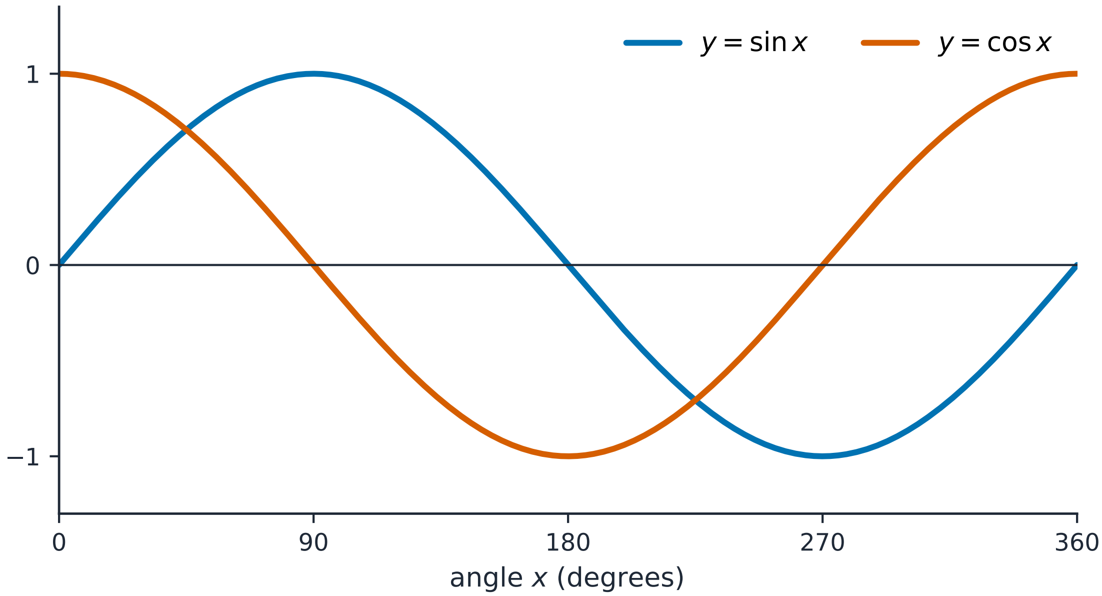
Naming the sides of a right triangle from the angle

Sine and Cosine Function Values

On a circle of radius r, $x = r\cos\theta$ and $y = r\sin\theta$. Exact values at the special angles $(0, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2}, \dots)$ come from **isosceles right** and **equilateral triangle** 等边三角形 geometry, with signs set by the **quadrant** 象限 of the angle.

Sine and Cosine Function Graphs

On the **unit circle** 单位圆 ($r = 1$), $\sin\theta$ is the y-coordinate and $\cos\theta$ is the x-coordinate. As θ increases, both **oscillate** 振荡 smoothly between -1 and 1 with period 2π. Sine starts at 0 (rising); cosine starts at 1. They are the same wave shifted by $\frac{\pi}{2}$.



y = sin x and y = cos x are smooth waves between -1 and 1

Sinusoidal Functions

A **sinusoidal function** 正弦型函数 is any transformation of sine (or cosine). Its key features:

- **Period** and **frequency** 频率 are reciprocals; $\sin\theta$ has period 2π.
- **Amplitude** 振幅 = half the distance between the maximum and minimum output.
- **Midline** 中线 $y = d$ = the average of the maximum and minimum (the horizontal center line).

The graph alternates concave up and concave down each half-cycle.

Worked example. For $f(\theta) = 3\sin(2\theta) + 1$: the amplitude is 3, the period is $\dfrac{2\pi}{2} = \pi$, and the midline is $y = 1$. So the maximum output is $1 + 3 = 4$ and the minimum is $1 - 3 = -2$.

Sinusoidal Function Transformations

The general sinusoid is

$$f(\theta) = a\sin\big(b(\theta + c)\big) + d \quad \text{or} \quad a\cos\big(b(\theta + c)\big) + d,$$

where $|a|$ is the **amplitude**, $\dfrac{2\pi}{|b|}$ is the **period**, $-c$ is the horizontal **phase shift** 相移 (a $+c$ inside the bracket shifts the graph **left** by c), and d is the **vertical shift** (midline). A negative a reflects the wave.

Sinusoidal Function Context and Data Modeling

To model periodic data (tides, daylight, temperature): read the **period** from where the pattern repeats, get the **amplitude** and **midline** from the max and min, and set the phase shift so a peak lands at the right input. Technology fits a sinusoidal regression. Such a model is only trustworthy over its **contextual domain**.

The Tangent Function

$\tan\theta = \dfrac{\sin\theta}{\cos\theta}$ is the **slope** of the terminal ray. Its slope values repeat every half-revolution, so its period is π. It has **vertical asymptotes** where $\cos\theta = 0$ (at $\theta = \frac{\pi}{2} + k\pi$), and it is always increasing between consecutive asymptotes, switching concavity at each zero.

Inverse Trigonometric Functions

The **inverse trigonometric functions** 反三角函数–**arcsine** 反正弦, **arccosine**, **arctangent** –reverse the trig functions, so they take a ratio and return an angle. Because sine, cosine, and tangent repeat, their domains must be **restricted** (e.g. sine to $[-\frac{\pi}{2}, \frac{\pi}{2}]$) before an inverse can exist.

Trigonometric Equations and Inequalities

Use inverse trig functions to solve trig equations. Because the functions are **periodic**, there are usually **infinitely many** solutions –write the general solution by adding multiples of the period, then keep those in the required (often contextual) domain.

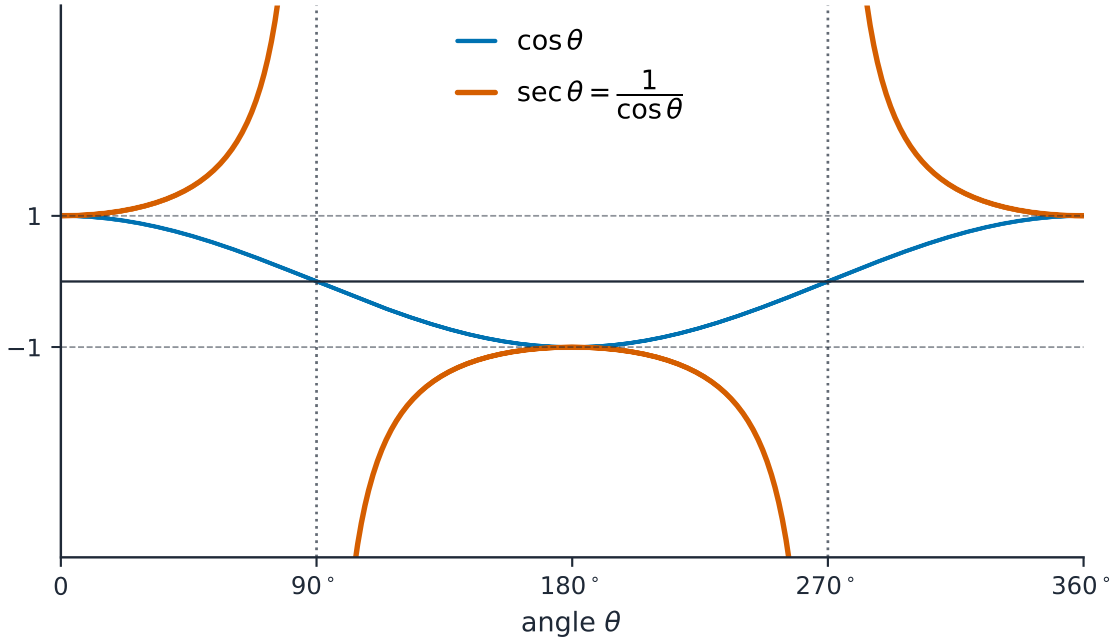
Worked example. Solve $2\sin\theta = 1$ on $[0, 2\pi)$. Then $\sin\theta = \frac{1}{2}$, and on the unit circle sine is positive in the first and second quadrants, so $\theta = \dfrac{\pi}{6}$ or $\dfrac{5\pi}{6}$. Over all reals you would add $2\pi k$ to each.

The Secant, Cosecant, and Cotangent Functions

These are the **reciprocals** 倒数 of the main three:

$$\sec\theta = \frac{1}{\cos\theta}, \qquad \csc\theta = \frac{1}{\sin\theta}, \qquad \cot\theta = \frac{1}{\tan\theta}.$$

Secant and cosecant have vertical asymptotes where cosine and sine are zero; cotangent has them where tangent is zero (where sine is zero).



sec is the reciprocal of cos, with asymptotes where cos is zero

Equivalent Representations of Trigonometric Functions

The **Pythagorean identity** 毕达哥拉斯恒等式 comes from $x^2 + y^2 = 1$ on the unit circle:

$$\sin^2\theta + \cos^2\theta = 1,$$

which rearranges into forms with sec and tan. The **sum identities** 和角公式:

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta, \qquad \cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta,$$

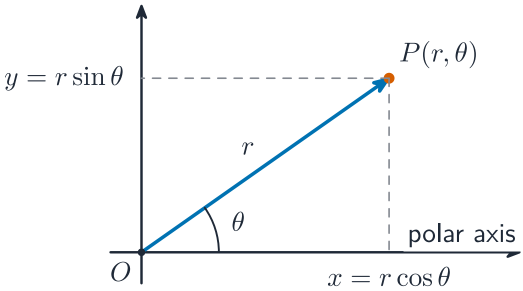
also give the difference and **double-angle** 二倍角 identities (set $\beta = \alpha$).

Trigonometry and Polar Coordinates

Polar coordinates 极坐标 locate a point by (r, θ) –distance r from the origin at angle θ. Convert with

$$x = r\cos\theta, \quad y = r\sin\theta, \qquad r = \sqrt{x^2 + y^2}, \quad \tan\theta = \frac{y}{x}.$$

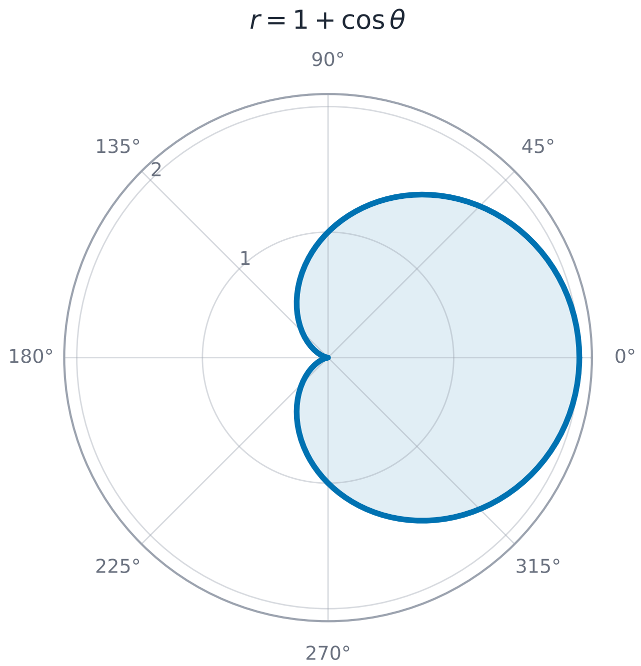
A **complex number** 复数 is the point $(x, y) = x + yi$ and can likewise be written with r and θ.



Polar coordinates give a point by its distance r and angle theta

Polar Function Graphs

The graph of $r = f(\theta)$ is all points (r, θ) as θ sweeps around. Restricting the domain of θ draws only part of the curve. As θ increases, r grows or shrinks, tracing spirals, circles, roses, and limaçons.



A cardioid r = 1 + cos theta, sketched directly in polar form

Rates of Change in Polar Functions

As θ increases, the **distance from the origin** r changes:

- r positive and increasing (or negative and decreasing) $\Rightarrow$ moving **away** from the origin;
- r positive and decreasing (or negative and increasing) $\Rightarrow$ moving **toward** the origin.

Where r switches between increasing and decreasing, the distance reaches a relative extreme. The **average rate of change** of r with respect to θ, $\dfrac{\Delta r}{\Delta \theta}$, estimates how fast the curve moves in or out over an interval.

Exam tips

- Use radians and the **unit circle**: $\cos\theta$ and $\sin\theta$ are the coordinates, always between -1 and 1.
- For a sinusoid $a\sin(b(\theta+c))+d$: $|a|$ is the amplitude, $\frac{2\pi}{|b|}$ the period, d the midline, and the phase shift is $-c$ (a $+c$ shifts left).
- Model periodic data by reading period, amplitude, and midline from the max and min.
- **Inverse trig** functions need a restricted domain and return an angle; trig equations have infinitely many solutions (add the period).
- Convert polar rectangular with $x = r\cos\theta$, $y = r\sin\theta$.

Functions Involving Parameters, Vectors, and Matrices

AP Precalculus

Parametric Functions

A roller coaster: vectors and parametric motion describe path and velocity together

Unit 4 is part of the course but is not assessed on the AP Precalculus Exam (which covers only Units 1–3); it builds tools used in later courses.

A **parametric function** 参数函数 describes a curve by giving both coordinates as functions of a third variable, the **parameter** 参数 t: $\big(x(t),\, y(t)\big)$. As t runs through its domain, the point traces a path –and, unlike $y = f(x)$, the path may loop or cross itself.

Parametric Functions and Planar Motion

Reading t as **time**, a parametric function models motion in the plane. $x(t)$ and $y(t)$ give the horizontal and vertical position; the point's **direction** of travel is the order in which the curve is drawn as t increases (mark it with arrows).

Parametric Functions and Rates of Change

Over an interval of t, the average rate of change of x is $\dfrac{\Delta x}{\Delta t}$ and of y is $\dfrac{\Delta y}{\Delta t}$. Their signs tell you which way the point moves (right/left, up/down); together they describe the motion's speed and direction along the path.

Parametrically Defined Circles and Lines

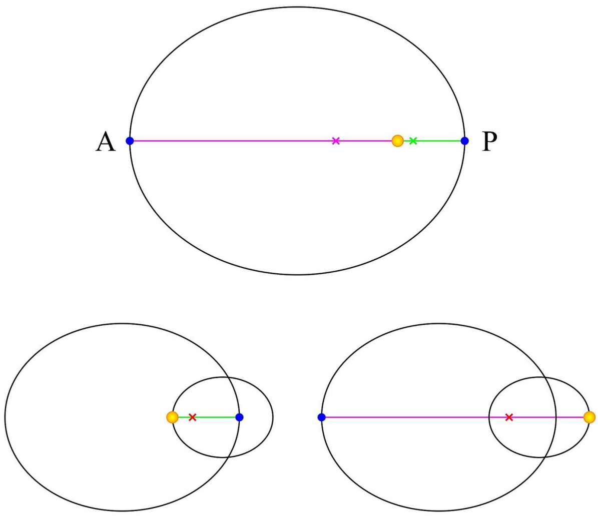
A **circle** of radius R centered at (h,k) is $x = h + R\cos t,\ y = k + R\sin t$. A **line** through (x_0, y_0) with direction (a,b) is $x = x_0 + at,\ y = y_0 + bt$. Adjusting the coefficients changes the start point, speed, and direction of tracing.

Implicitly Defined Functions

An **implicit** 隐式 equation relates x and y without solving for either, such as $x^2+y^2 = 25$. Its graph may fail the vertical-line test (not a function), so it is often split into pieces or described parametrically.

Conic Sections

Conic sections 圆锥曲线–circles, ellipses, parabolas, and hyperbolas –are the curves from slicing a cone, each given by a quadratic equation in x and y. Their standard forms reveal centers, vertices, axes, and asymptotes.



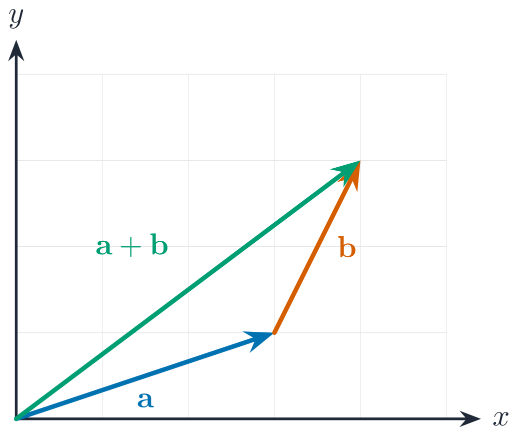
Elliptical orbits: conic sections describe closed paths with two foci —a key parametric model

Parametrizing Implicit Curves

Many implicit curves can be **parametrized** –rewritten as $\big(x(t), y(t)\big)$ –which makes them easier to graph and to treat as motion. The circle above is the basic example; ellipses use $x = h + A\cos t,\ y = k + B\sin t$.

Vectors

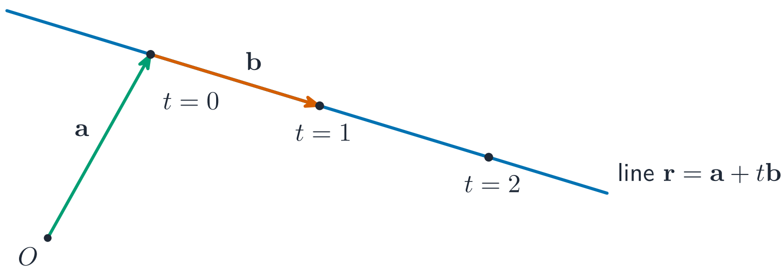
A **vector** 向量 has both **magnitude** 大小 (length) and **direction**. Write it by components $\langle a, b \rangle$. Add vectors component-by-component; scale by multiplying each component; the magnitude is $\sqrt{a^2 + b^2}$. Vectors model displacements, velocities, and forces.



Adding vectors by the triangle law

Vector-Valued Functions

A **vector-valued function** 向量值函数 outputs a vector for each input, e.g. $\vec{r}(t) = \langle x(t), y(t) \rangle$ –the same information as a parametric function, packaged as a moving position vector.



A vector-valued line: start at a, then slide by t lots of the direction b

Matrices

A **matrix** 矩阵 is a rectangular array of numbers. Add matrices of the same size entry-by-entry; multiply a matrix by a compatible one by combining rows with columns. Matrices store and transform data compactly.

The Inverse and Determinant of a Matrix

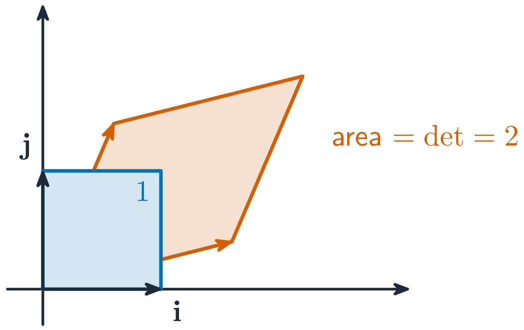
The **determinant** 行列式 of a 2×2 matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is $ad - bc$. A matrix has an **inverse** 逆矩阵 exactly when its determinant is nonzero; the inverse undoes the matrix, and it solves matrix equations (like a reciprocal for numbers).

Worked example. For $A = \begin{bmatrix} 3 & 1 \\ 2 & 4 \end{bmatrix}$, the determinant is $ad - bc = (3)(4) - (1)(2) = 10$. Since it is nonzero, A is invertible, and $A^{-1} = \dfrac{1}{10} \begin{bmatrix} 4 & -1 \\ -2 & 3 \end{bmatrix}$ (swap the diagonal, negate the off-diagonal, divide by the determinant).

Linear Transformations and Matrices

A 2×2 matrix acts as a **linear transformation** 线性变换 of the plane –rotating, reflecting, stretching, or shearing points –by multiplying each position vector. The determinant measures how the transformation scales area (and its sign tells whether orientation flips).

Worked example. The matrix $\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$ stretches the plane by 2 horizontally and 3 vertically, sending the vector $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ to $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$. Its determinant $2 \times 3 = 6$ means every area is multiplied by 6, so the unit square becomes a 2×3 rectangle.



A 2x2 matrix maps the unit square to a parallelogram; the determinant is the area scale

Matrices as Functions

Because a matrix maps input vectors to output vectors, it **is** a function on the plane. Composing transformations corresponds to multiplying their matrices, and the inverse matrix reverses the mapping.

Matrices Modeling Contexts

Matrices model systems that step forward in stages –for example, populations moving between states each year. Repeated multiplication by a **transition matrix** advances the model one step at a time, so matrix powers predict long-run behavior.

Exam tips

- A **parametric** function gives $x(t)$ and $y(t)$ separately —track the direction of motion as t increases.
- A **vector** has magnitude and direction; add component-by-component and find magnitude with $\sqrt{a^2 + b^2}$.
- The **determinant** of a 2×2 matrix is $ad - bc$ (mind the minus sign); it scales area under the transformation.
- A 2×2 matrix transforms the plane (rotate, reflect, stretch); an inverse exists only when the determinant is non-zero.
- Note this unit is not tested on the AP Precalculus Exam (Units 1–3 only), but it underpins later courses.