Functions Involving Parameters, Vectors, and Matrices

AP Precalculus

Parametric Functions

Unit 4 is part of the course but is not assessed on the AP Exam's free-response section; it builds tools used in later courses.

A **parametric function** 参数函数 describes a curve by giving both coordinates as functions of a third variable, the **parameter** 参数 t: $\big(x(t),\, y(t)\big)$. As t runs through its domain, the point traces a path –and, unlike $y = f(x)$, the path may loop or cross itself.

Parametric Functions and Planar Motion

Reading t as **time**, a parametric function models motion in the plane. $x(t)$ and $y(t)$ give the horizontal and vertical position; the point's **direction** of travel is the order in which the curve is drawn as t increases (mark it with arrows).

Parametric Functions and Rates of Change

Over an interval of t, the average rate of change of x is $\dfrac{\Delta x}{\Delta t}$ and of y is $\dfrac{\Delta y}{\Delta t}$. Their signs tell you which way the point moves (right/left, up/down); together they describe the motion's speed and direction along the path.

Parametrically Defined Circles and Lines

A **circle** of radius R centered at (h, k) is $x = h + R\cos t,\ y = k + R\sin t$. A **line** through (x_0, y_0) with direction (a, b) is $x = x_0 + at,\ y = y_0 + bt$. Adjusting the coefficients changes the start point, speed, and direction of tracing.

Implicitly Defined Functions

An **implicit** 隐式 equation relates x and y without solving for either, such as $x^2+y^2 = 25$. Its graph may fail the vertical-line test (not a function), so it is often split into pieces or described parametrically.

Conic Sections

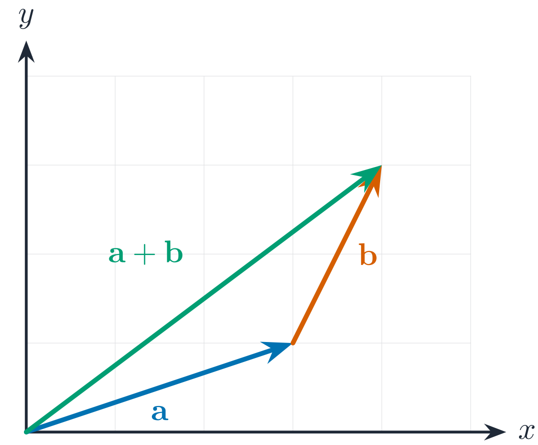
Conic sections 圆锥曲线–circles, ellipses, parabolas, and hyperbolas –are the curves from slicing a cone, each given by a quadratic equation in x and y. Their standard forms reveal centers, vertices, axes, and asymptotes.

Parametrizing Implicit Curves

Many implicit curves can be **parametrized** –rewritten as $\big(x(t), y(t)\big)$ –which makes them easier to graph and to treat as motion. The circle above is the basic example; ellipses use $x = h + A\cos t,\ y = k + B\sin t$.

Vectors

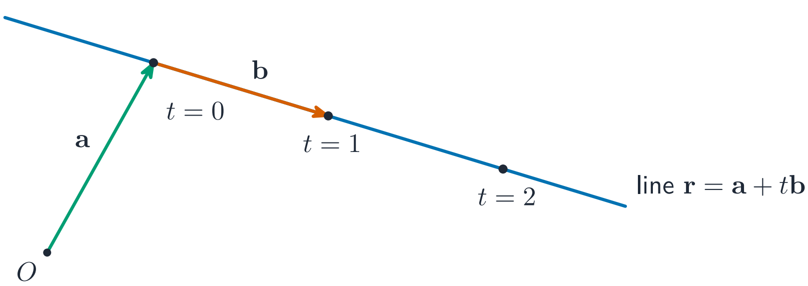
A **vector** 向量 has both **magnitude** 大小 (length) and **direction**. Write it by components $\langle a, b\rangle$. Add vectors component-by-component; scale by multiplying each component; the magnitude is $\sqrt{a^2+b^2}$. Vectors model displacements, velocities, and forces.



Adding vectors by the triangle law

Vector-Valued Functions

A **vector-valued function** 向量值函数 outputs a vector for each input, e.g. $\vec{r}(t) = \langle x(t), y(t)\rangle$ –the same information as a parametric function, packaged as a moving position vector.



A vector-valued line: start at a, then slide by t lots of the direction b

Matrices

A **matrix** 矩阵 is a rectangular array of numbers. Add matrices of the same size entry-by-entry; multiply a matrix by a compatible one by combining rows with columns. Matrices store and transform data compactly.

The Inverse and Determinant of a Matrix

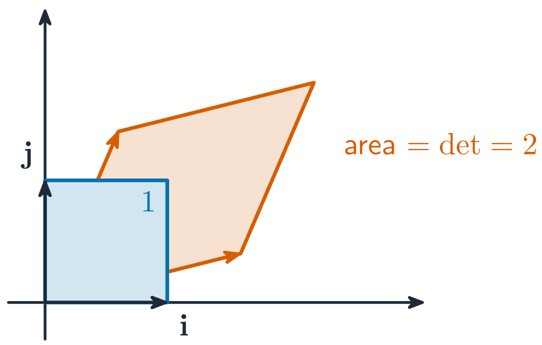
The **determinant** 行列式 of a 2×2 matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is $ad - bc$. A matrix has an **inverse** 逆矩阵 exactly when its determinant is nonzero; the inverse undoes the matrix, and it solves matrix equations (like a reciprocal for numbers).

Worked example. For $A = \begin{bmatrix} 3 & 1 \\ 2 & 4 \end{bmatrix}$, the determinant is $ad - bc = (3)(4) - (1)(2) = 10$. Since it is nonzero, A is invertible, and $A^{-1} = \frac{1}{10}\begin{bmatrix} 4 & -1 \\ -2 & 3 \end{bmatrix}$ (swap the diagonal, negate the off-diagonal, divide by the determinant).

Linear Transformations and Matrices

A 2×2 matrix acts as a **linear transformation** 线性变换 of the plane –rotating, reflecting, stretching, or shearing points –by multiplying each position vector. The determinant measures how the transformation scales area (and its sign tells whether orientation flips).

Worked example. The matrix $\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$ stretches the plane by 2 horizontally and 3 vertically, sending the vector $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ to $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$. Its determinant $2 \times 3 = 6$ means every area is multiplied by 6, so the unit square becomes a 2×3 rectangle.



A 2x2 matrix maps the unit square to a parallelogram; the determinant is the area scale

Matrices as Functions

Because a matrix maps input vectors to output vectors, it **is** a function on the plane. Composing transformations corresponds to multiplying their matrices, and the inverse matrix reverses the mapping.

Matrices Modeling Contexts

Matrices model systems that step forward in stages –for example, populations moving between states each year. Repeated multiplication by a **transition matrix** advances the model one step at a time, so matrix powers predict long-run behavior.

Exam tips

- A **parametric** function gives $x(t)$ and $y(t)$ separately —track the direction of motion as t increases.
- A **vector** has magnitude and direction; add component-by-component and find magnitude with $\sqrt{a^2 + b^2}$.
- The **determinant** of a 2×2 matrix is $ad - bc$ (mind the minus sign); it scales area under the transformation.
- A 2×2 matrix transforms the plane (rotate, reflect, stretch); an inverse exists only when the determinant is non-zero.
- Note this unit is not on the AP exam's free-response, but it underpins later courses.