Week 24

AP Precalculus

Homework bundle for this week

本周作业合集

exercises.lol

Contents 目录

4. Functions Involving Parameters, Vectors, and Matrices Starter Quiz · 课前小测

AP Precalculus Name: ______________ Score: ______

1. A **parametric function** describes a curve using...
   - ☐ two equations $x(t)$ and $y(t)$ driven by a parameter t
   - ☐ a single equation $y = f(x)$
   - ☐ only an angle
   - ☐ a matrix
2. When parametric functions model motion, the parameter t usually represents...
   - ☐ time
   - ☐ distance
   - ☐ the slope
   - ☐ the area
3. For a moving point, $\dfrac{dx}{dt}$ and $\dfrac{dy}{dt}$ are the...
   - ☐ horizontal and vertical velocity components
   - ☐ the total distance travelled
   - ☐ the position coordinates
   - ☐ the parameter values
4. Which **parametric equations** trace a circle of radius r centred at the origin?
   - ☐ $x = r\cos t,\ y = r\sin t$
   - ☐ $x = rt,\ y = rt$
   - ☐ $x = t,\ y = r$
   - ☐ $x = r + t,\ y = r - t$
5. An **implicitly defined** relationship is one where...
   - ☐ x and y appear mixed together, not solved for y
   - ☐ y is written alone as $y = f(x)$
   - ☐ there is no y at all
   - ☐ it uses a parameter t
6. For $\begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}$, the determinant is $3 \cdot 2 - 1 \cdot 1$. What is it?

   ____

7. A parametric curve can trace a shape that is not a function of x (like a full circle).
   - ☐ True ☐ False
8. A ball has $x = 3t$ and $y = 20t - 5t^2$. What is its height y at $t = 2$?

   ____
9. A point has $x = 3t$. What is its average horizontal rate of change $\Delta x/\Delta t$ over any interval?

   ____
10. A vector-valued function $\vec{r}(t) = \langle x(t), y(t)\rangle$ is closely related to a parametric function.
    - ☐ True ☐ False

1. **two equations $x(t)$ and $y(t)$ driven by a parameter t**

   A **parametric function** gives x and y each as a function of a **parameter** t: the pair $(x(t), y(t))$ traces the curve.

2. **time**

   In a **parametric function** modeling motion, t is time, and $(x(t), y(t))$ is the object's position then.

3. **horizontal and vertical velocity components**

   The **rate of change** of each coordinate is a velocity component: dx/dt across, dy/dt up.

4. $x = r\cos t,\ \ y = r\sin t$

   From the unit circle scaled by r: $x = r\cos t$ and $y = r\sin t$ trace the circle as t goes 0 to 2π.

5. **x and y appear mixed together, not solved for y**

   An **implicitly defined** curve, like $x^2 + y^2 = 25$, mixes x and y without isolating y.

6. **5**

   The **determinant** of $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is $ad - bc = 3 \cdot 2 - 1 \cdot 1 = 5$.

7. **True**

   Because both coordinates depend on t, the path can loop or double back — it need not pass the vertical-line test.

8. **20**

   $y = 20(2) - 5(2)^2 = 40 - 20 = 20$; and $x = 3(2) = 6$, so the ball is at $(6, 20)$.

9. **3**

   Since $x = 3t$ is linear, $\Delta x/\Delta t = 3$ everywhere — a constant horizontal velocity.

10. **True**

    They are two views of the same idea: the components $x(t), y(t)$ are exactly the parametric equations.

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4.1 Parametric Functions

Name: ____________ Class: ________ Date: __________ **Total: 9 marks**

KEY WORDS parametric function 参数函数 • parameter 参数

Objective

Build the skills to answer exam questions on **parametric functions**.

You must be able to:

- interpret a **parametric function** 参数函数 $x = f(t)$, $y = g(t)$
- eliminate the **parameter** 参数 to relate x and y

1 Worked examples

Study these first. Each one shows the method for a question type used later.

Parametric functions

Both coordinates depend on a **parameter** t: $x = f(t)$, $y = g(t)$. As t varies, (x, y) traces a curve. Eliminate t to get a direct relation between x and y.

Example

$x = t$, $y = t^2$: since $x = t$, substitute to get $y = x^2$.

2 Practice

2.1 State what a parameter is in a parametric function. [1]

2.2 For $x = t$, $y = 2t$, eliminate t. [2]

2.3 For $x = t + 1$, $y = t$, find the point at $t = 2$. [1]

3 Exam-style questions

3.1 In parametric form, x and y are both functions of [1]

- **A** each other
- **B** a parameter t
- **C** the slope
- **D** the origin

3.2 For $x = t$, $y = t^2$, eliminating t gives [1]

- **A** $y = x$
- **B** $y = x^2$
- **C** $x = y^2$
- **D** $y = 2x$

3.3 $x = 2t$, $y = t - 1$.

(a) Find the point at $t = 0$. [1]

(b) Find the point at $t = 3$. [1]

(c) Eliminate t to relate x and y. [1]

Solutions

2.1 an independent variable (t) that both x and y depend on.

2.2 $t = x$, so $y = 2x$.

2.3 $t = 2$: $x = 3$, $y = 2$, i.e. $(3, 2)$.

3.1 **B**.

3.2 **B**.

3.3 (a) $(0, -1)$. (b) $(6, 2)$. (c) $t = \frac{x}{2}$, so $y = \frac{x}{2} - 1$.

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4.2 Parametric Functions Modeling Planar Motion

Name: ____________ Class: ________ Date: __________ **Total: 9 marks**

KEY WORDS position 位置 • direction of motion 运动方向 • parametric function 参数函数

Objective

Build the skills to answer exam questions on **parametric functions modeling planar motion**.

You must be able to:

- read an object's **position 位置** $(x(t), y(t))$ at time t
- find the **direction of motion 运动方向** as t increases

1 Worked examples

Study these first. Each one shows the method for a question type used later.

Planar motion

Parametric equations model a moving object's position at time t: $(x(t), y(t))$. Increasing t traces the curve in the **direction of motion**.

Example

$x = t$, $y = t^2$: at $t = 0$ the object is at $(0, 0)$; as t grows it moves right and up.

2 Practice

2.1 State what $(x(t), y(t))$ represents in a motion problem. [1]

2.2 State how you find the direction of motion. [1]

2.3 For $x = t$, $y = 3t$, find the positions at $t = 0$ and $t = 1$. [2]

3 Exam-style questions

3.1 Parametric equations of motion give an object's [1]

- **A** speed only
- **B** position at each time
- **C** mass
- **D** energy

3.2 The direction of motion is found by [1]

- **A** increasing t
- **B** decreasing x
- **C** setting $t = 0$
- **D** the amplitude

3.3 An object has $x = t$, $y = 4 - t$.

(a) State the position at $t = 0$. [1]

(b) State the position at $t = 4$. [1]

(c) Describe the horizontal motion. [1]

Solutions

2.1 the object's position in the plane at time t.

2.2 trace the curve in the direction of increasing t.

2.3 $t = 0$: $(0, 0)$; $t = 1$: $(1, 3)$.

3.1 **B**.

3.2 **A**.

3.3 (a) $(0, 4)$. (b) $(4, 0)$. (c) it moves to the right, since $x = t$ increases with t.

4.3 Parametric Functions and Rates of Change

Name: ____________ Class: ________ Date: __________ **Total: 9 marks**

KEY WORDS parametric function 参数函数

Objective

Build the skills to answer exam questions on **parametric functions and rates of change**.

You must be able to:

- find the average rate of change of x and of y with respect to t
- find the slope $\dfrac{dy}{dx}$ of a parametric curve

1 Worked examples

Study these first. Each one shows the method for a question type used later.

Parametric rates of change

- rate of change of x: $\dfrac{\Delta x}{\Delta t}$; of y: $\dfrac{\Delta y}{\Delta t}$;
- the slope of the curve: $\dfrac{dy}{dx} = \dfrac{\Delta y/\Delta t}{\Delta x/\Delta t}$.

Example

$x = 2t$, $y = t^2$ over $[0, 1]$: $\Delta x = 2$, $\Delta y = 1$, slope $= \frac{1}{2}$.

2 Practice

2.1 State the average rate of change of x on $[t_1, t_2]$. [1]

2.2 State the slope $\dfrac{dy}{dx}$ in parametric terms. [1]

2.3 For $x = t$, $y = 2t$ over $[0, 3]$, find the average rates of change of x and y. [2]

3 Exam-style questions

3.1 The slope of a parametric curve is [1]

- **A** $\Delta x/\Delta y$
- **B** $\dfrac{\Delta y/\Delta t}{\Delta x/\Delta t}$
- **C** $\Delta t/\Delta y$
- **D** $\Delta x \cdot \Delta y$

3.2 The average rate of change of x over $[a, b]$ is [1]

- **A** $\dfrac{x(b) - x(a)}{b - a}$
- **B** $x(b) \cdot x(a)$
- **C** $b - a$
- **D** $x(b) + x(a)$

3.3 $x = 3t$, $y = t$.

(a) Find Δx over $[0, 2]$. [1]

(b) Find Δy over $[0, 2]$. [1]

(c) Find the slope $\dfrac{dy}{dx}$. [1]

Solutions

2.1 $\dfrac{x(t_2) - x(t_1)}{t_2 - t_1}$.

2.2 $\dfrac{dy}{dx} = \dfrac{\Delta y/\Delta t}{\Delta x/\Delta t}$.

2.3 x: $\frac{3-0}{3} = 1$; y: $\frac{6-0}{3} = 2$.

3.1 **B**.

3.2 **A**.

3.3 (a) $\Delta x = 6$. (b) $\Delta y = 2$. (c) $\dfrac{2}{6} = \frac{1}{3}$.

4.4 Parametrically Defined Circles and Lines

Name: ________ Class: ________ Date: ________ **Total: 9 marks**

KEY WORDS line 直线 • circle 圆

Objective

Build the skills to answer exam questions on **parametrically defined circles and lines**.

You must be able to:

- write parametric equations for a **circle** 圆 and a **line** 直线

1 Worked examples

Study these first. Each one shows the method for a question type used later.

Circles and lines

- **circle** of radius r (centred at the origin): $x = r\cos t$, $y = r\sin t$;
- **line** through (x_0, y_0) with direction (a, b): $x = x_0 + at$, $y = y_0 + bt$.

Example

$x = 3\cos t$, $y = 3\sin t$ is a circle of radius 3.

2 Practice

2.1 Write parametric equations for a circle of radius 5. [2]

2.2 State the radius of $x = 2\cos t$, $y = 2\sin t$. [1]

2.3 For the line $x = 1 + 2t$, $y = 3 + t$, state the point at $t = 0$. [1]

3 Exam-style questions

3.1 A circle of radius r is [1]

- **A** $x = rt$, $y = rt$
- **B** $x = r\cos t$, $y = r\sin t$
- **C** $x = t$, $y = r$
- **D** $x = r + t$, $y = r - t$

3.2 The line $x = x_0 + at$, $y = y_0 + bt$ passes through [1]

- **A** the origin
- **B** (x_0, y_0) at $t = 0$
- **C** (a, b)
- **D** (t, t)

3.3 $x = 4\cos t$, $y = 4\sin t$.

(a) State the shape. [1]

(b) State the radius. [1]

(c) State the point at $t = 0$. [1]

Solutions

2.1 $x = 5\cos t$, $y = 5\sin t$.

2.2 2.

2.3 $(1, 3)$.

3.1 **B**.

3.2 **B**.

3.3 (a) a circle. (b) 4. (c) $(4, 0)$.

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4.5 Implicitly Defined Functions

Name: ____________ Class: ________ Date: __________ **Total: 9 marks**

KEY WORDS implicit 隐式 • implicitly defined 隐式定义

Objective

Build the skills to answer exam questions on **implicitly defined functions**.

You must be able to:

- recognise an **implicitly defined** 隐式定义 relation such as $x^2 + y^2 = 25$
- solve it for y and decide whether it is a function

1 Worked examples

Study these first. Each one shows the method for a question type used later.

■ Implicit relations

An implicit equation relates x and y without solving for y — for example $x^2+y^2 = 25$ (a circle). It may fail the vertical-line test, so it need not be a function.

■ Example

$x^2 + y^2 = 25$ is a circle of radius 5; solving gives $y = \pm\sqrt{25 - x^2}$.

2 Practice

2.1 State what "implicitly defined" means. [1]

2.2 State the shape of $x^2 + y^2 = 9$. [1]

2.3 Solve $x^2 + y^2 = 25$ for y. [2]

3 Exam-style questions

3.1 An implicitly defined relation is one [1]

- **A** solved for y
- **B** written as an equation in x and y
- **C** always a function
- **D** always linear

3.2 The graph of $x^2 + y^2 = 16$ is [1]

- **A** a line
- **B** a circle of radius 4
- **C** a parabola
- **D** two points

3.3 $x^2 + y^2 = 25$.

(a) State the radius. [1]

(b) Solve for y. [1]

(c) State whether it is a function. [1]

Solutions

2.1 given by an equation in x and y that is not solved for y.

2.2 a circle of radius 3 centred at the origin.

2.3 $y = \pm\sqrt{25 - x^2}$.

3.1 B.

3.2 B.

3.3 (a) 5. (b) $y = \pm\sqrt{25 - x^2}$. (c) no (it fails the vertical-line test).

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4.6 Conic Sections

Name: ____________ Class: ________ Date: __________ **Total: 9 marks**

KEY WORDS conic section 圆锥曲线

Objective

Build the skills to answer exam questions on **conic sections**.

You must be able to:

- identify a **conic section** 圆锥曲线 (circle, ellipse, parabola, hyperbola) from its equation

1 Worked examples

Study these first. Each one shows the method for a question type used later.

Conic sections

Curves formed by slicing a cone. Standard equations:

- **circle**: $x^2 + y^2 = r^2$;
- **ellipse**: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$;
- **parabola**: $y = ax^2$;
- **hyperbola**: $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

Example

$\frac{x^2}{9} + \frac{y^2}{4} = 1$ is an ellipse (sum of squared terms = 1).

2 Practice

2.1 Name the four conic sections. [1]

2.2 State the standard equation of an ellipse. [1]

2.3 Identify the conic $\frac{x^2}{9} + \frac{y^2}{4} = 1$. [2]

3 Exam-style questions

3.1 The equation $\frac{x^2}{16} + \frac{y^2}{9} = 1$ is [1]

- **A** a circle
- **B** an ellipse
- **C** a parabola
- **D** a hyperbola

3.2 The equation $x^2 - y^2 = 1$ is [1]

- **A** a circle
- **B** an ellipse
- **C** a hyperbola
- **D** a line

3.3 Classify each conic.

(a) $x^2 + y^2 = 25$. [1]

(b) $y = x^2$. [1]

(c) $\frac{x^2}{4} - \frac{y^2}{9} = 1$. [1]

Solutions

2.1 circle, ellipse, parabola, hyperbola.

2.2 $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

2.3 an ellipse.

3.1 **B**.

3.2 **C**.

3.3 (a) circle. (b) parabola. (c) hyperbola.

4.7 Parametrization of Implicitly Defined Functions

Name: ______ Class: ______ Date: ______ **Total: 9 marks**

KEY WORDS parametric 参数 • implicit 隐式

Objective

Build the skills to answer exam questions on **parametrization of implicitly defined functions**.

You must be able to:

- convert an implicit curve to **parametric 参数** form
- use $\cos^2 t + \sin^2 t = 1$ to parametrize a circle

1 Worked examples

Study these first. Each one shows the method for a question type used later.

Parametrizing an implicit curve

Rewrite an implicit relation as $x = f(t)$, $y = g(t)$. The circle $x^2 + y^2 = r^2$ becomes

$$x = r\cos t, \quad y = r\sin t,$$

which works because $\cos^2 t + \sin^2 t = 1$.

Example

$x^2 + y^2 = 9 \;\Rightarrow\; x = 3\cos t,\ y = 3\sin t$.

2 Practice

2.1 Parametrize $x^2 + y^2 = 16$. [2]

2.2 State the identity used to parametrize a circle. [1]

2.3 Verify that $x = 2\cos t$, $y = 2\sin t$ satisfies $x^2 + y^2 = 4$. [1]

---

3 Exam-style questions

3.1 The circle $x^2 + y^2 = 25$ parametrizes as [1]

- **A** $x = 5t$, $y = 5t$
- **B** $x = 5\cos t$, $y = 5\sin t$
- **C** $x = t^2$, $y = t^2$
- **D** $x = 25t$, $y = 25t$

3.2 Parametrizing a circle uses the identity [1]

- **A** $\sin t + \cos t = 1$
- **B** $\cos^2 t + \sin^2 t = 1$
- **C** $\tan t = \sin t / \cos t$
- **D** $\sec t = 1/\cos t$

3.3 $x^2 + y^2 = 36$.

(a) State r. [1]

(b) Write $x(t)$. [1]

(c) Write $y(t)$. [1]

Solutions

2.1 $x = 4\cos t$, $y = 4\sin t$.

2.2 $\cos^2 t + \sin^2 t = 1$.

2.3 $x^2 + y^2 = 4\cos^2 t + 4\sin^2 t = 4(\cos^2 t + \sin^2 t) = 4$.

3.1 **B**.

3.2 **B**.

3.3 (a) 6. (b) $x = 6\cos t$. (c) $y = 6\sin t$.

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4.8 Vectors

Name: ____________ Class: ________ Date: __________ **Total: 9 marks**

KEY WORDS magnitude 大小 • vector 向量

Objective

Build the skills to answer exam questions on **vectors**.

You must be able to:

- find the **magnitude** 大小 of a **vector** 向量 $\langle a, b\rangle$
- add vectors and multiply by a scalar

1 Worked examples

Study these first. Each one shows the method for a question type used later.

Vectors

A vector $\langle a, b\rangle$ has **magnitude** $\sqrt{a^2+b^2}$ and a direction.

- add componentwise: $\langle a, b\rangle + \langle c, d\rangle = \langle a+c, b+d\rangle$;
- scalar multiple: $k\langle a, b\rangle = \langle ka, kb\rangle$.

Example

$\langle 3, 4\rangle$ has magnitude $\sqrt{9+16} = 5$.

2 Practice

2.1 State the magnitude of $\langle a, b\rangle$. [1]

2.2 Add $\langle 1, 2\rangle + \langle 3, 4\rangle$. [1]

2.3 Find the magnitude of $\langle 3, 4\rangle$. [2]

3 Exam-style questions

3.1 The magnitude of $\langle a, b\rangle$ is [1]

- **A** $a+b$
- **B** $\sqrt{a^2+b^2}$
- **C** a^2+b^2
- **D** a/b

3.2 $\langle 2, 1\rangle + \langle 3, 5\rangle$ equals [1]

- **A** $\langle 5, 6\rangle$
- **B** $\langle 6, 5\rangle$
- **C** $\langle 5, 5\rangle$
- **D** $\langle 6, 6\rangle$

3.3 $\vec{v} = \langle 6, 8\rangle$.

(a) Find the magnitude. [1]

(b) Find $2\vec{v}$. [1]

(c) Find $\vec{v} + \langle 1, 0\rangle$. [1]

Solutions

2.1 $\sqrt{a^2+b^2}$.

2.2 $\langle 4, 6\rangle$.

2.3 $\sqrt{3^2+4^2} = \sqrt{25} = 5$.

3.1 **B**.

3.2 **A**.

3.3 (a) $\sqrt{36+64} = 10$. (b) $\langle 12, 16\rangle$. (c) $\langle 7, 8\rangle$.

4.9 Vector-Valued Functions

Name: ____________ Class: ________ Date: __________ **Total: 9 marks**

KEY WORDS vector-valued function 向量值函数 • vector 向量 • parametric function 参数函数

Objective

Build the skills to answer exam questions on **vector-valued functions**.

You must be able to:

- interpret a **vector-valued function** 向量值函数 $\vec{r}(t) = \langle x(t), y(t)\rangle$
- evaluate it at a value of t

1 Worked examples

Study these first. Each one shows the method for a question type used later.

Vector-valued functions

$\vec{r}(t) = \langle x(t), y(t)\rangle$ gives a **position vector** at each t — the same information as parametric motion, written as a vector.

Example

$\vec{r}(t) = \langle t, t^2\rangle$: at $t = 2$, $\vec{r}(2) = \langle 2, 4\rangle$.

2 Practice

2.1 State what $\vec{r}(t) = \langle x(t), y(t)\rangle$ gives. [1]

2.2 For $\vec{r}(t) = \langle 2t, t\rangle$, find $\vec{r}(3)$. [2]

2.3 State the connection between vector-valued and parametric functions. [1]

3 Exam-style questions

3.1 A vector-valued function $\vec{r}(t) = \langle x(t), y(t) \rangle$ describes [1]

- **A** a single number
- **B** a position vector at each t
- **C** a slope
- **D** an area

3.2 For $\vec{r}(t) = \langle t, 2t \rangle$, $\vec{r}(1)$ equals [1]

- **A** $\langle 1, 2 \rangle$
- **B** $\langle 2, 1 \rangle$
- **C** $\langle 1, 1 \rangle$
- **D** $\langle 2, 2 \rangle$

3.3 $\vec{r}(t) = \langle t + 1, 3t \rangle$.

(a) Find $\vec{r}(0)$. [1]

(b) Find $\vec{r}(2)$. [1]

(c) State the x-component function. [1]

Solutions

2.1 a position vector for each value of t.

2.2 $\vec{r}(3) = \langle 6, 3 \rangle$.

2.3 they carry the same information — the components $x(t)$, $y(t)$ are the parametric equations.

3.1 **B**.

3.2 **A**.

3.3 (a) $\langle 1, 0 \rangle$. (b) $\langle 3, 6 \rangle$. (c) $x(t) = t + 1$.

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4.10 Matrices

Name: ____________ Class: ________ Date: __________ **Total: 9 marks**

KEY WORDS dimensions 维数 • matrix 矩阵 • determinant 行列式

Objective

Build the skills to answer exam questions on **matrices**.

You must be able to:

- state the **dimensions** 维数 of a **matrix** 矩阵
- add matrices and multiply a matrix by a scalar

1 Worked examples

Study these first. Each one shows the method for a question type used later.

Matrices

A matrix is a rectangular array of numbers with dimensions rows × columns. Add matrices of the **same size** entrywise; multiply by a scalar entrywise.

Example

$\begin{bmatrix}1 & 2\\3 & 4\end{bmatrix}$ is 2×2. Adding $\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$ gives $\begin{bmatrix}2 & 2\\3 & 5\end{bmatrix}$.

2 Practice

2.1 State the dimensions of a matrix with 2 rows and 3 columns. [1]

2.2 State when two matrices can be added. [1]

2.3 Add $\begin{bmatrix}1 & 2\\3 & 4\end{bmatrix}+\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}$. [2]

3 Exam-style questions

3.1 A matrix with 3 rows and 2 columns is [1]

- **A** 2×3
- **B** 3×2
- **C** 6×1
- **D** 2×2

3.2 Two matrices can be added only if they [1]

- **A** are square
- **B** have the same dimensions
- **C** are diagonal
- **D** have determinant 0

3.3 $A=\begin{bmatrix}2 & 0\\1 & 3\end{bmatrix}$.

(a) State the dimensions. [1]

(b) Find $2A$. [1]

(c) Find $A+\begin{bmatrix}1 & 1\\1 & 1\end{bmatrix}$. [1]

Solutions

2.1 2×3.

2.2 when they have the same dimensions.

2.3 $\begin{bmatrix} 1 & 3 \\ 4 & 4 \end{bmatrix}$.

3.1 **B**.

3.2 **B**.

3.3 (a) 2×2. (b) $\begin{bmatrix} 4 & 0 \\ 2 & 6 \end{bmatrix}$. (c) $\begin{bmatrix} 3 & 1 \\ 2 & 4 \end{bmatrix}$.

4.11 The Inverse and Determinant of a Matrix

Name: ____________ Class: ________ Date: __________ **Total: 9 marks**

KEY WORDS determinant 行列式 • inverse 逆矩阵 • matrix 矩阵

Objective

Build the skills to answer exam questions on **the inverse and determinant of a matrix**.

You must be able to:

- compute the **determinant** 行列式 of a 2×2 matrix
- decide whether a matrix has an **inverse** 逆矩阵

1 Worked examples

Study these first. Each one shows the method for a question type used later.

Determinant and inverse

For $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, the determinant is $\det(A) = ad - bc$.

A has an inverse if and only if $\det(A) \neq 0$, and then

$$A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

Example

$A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$: $\det = 2(1) - 1(1) = 1$, so $A^{-1} = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$.

2 Practice

2.1 State the determinant of $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$. [1]

2.2 State when a matrix has an inverse. [1]

2.3 Find the determinant of $\begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix}$. [2]

3 Exam-style questions

3.1 $\det \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ equals [1]

- **A** $ad + bc$
- **B** $ad - bc$
- **C** $ab - cd$
- **D** $a + d$

3.2 A 2×2 matrix has an inverse if and only if [1]

- **A** it is square
- **B** $\det \neq 0$
- **C** it is diagonal
- **D** all entries are positive

3.3 $A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$.

(a) Find $\det(A)$. [1]

(b) State whether A is invertible. [1]

(c) Find A^{-1}. [1]

Solutions

2.1 $ad - bc$.

2.2 when its determinant is non-zero.

2.3 $3(4) - 2(1) = 12 - 2 = 10$.

3.1 **B**.

3.2 **B**.

3.3 (a) $2(3) - 0 = 6$. (b) yes, since $\det = 6 \neq 0$. (c) $\begin{bmatrix} 1/2 & 0 \\ 0 & 1/3 \end{bmatrix}$.

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4.12 Linear Transformations and Matrices

Name: ____________ Class: ________ Date: __________ **Total: 9 marks**

KEY WORDS linear transformation 线性变换 • matrix-vector multiplication 矩阵向量乘法 • matrix 矩阵 • vector 向量

Objective

Build the skills to answer exam questions on **linear transformations and matrices**.

You must be able to:

- apply a matrix to a vector by **matrix-vector multiplication** 矩阵向量乘法
- describe the geometric effect (scaling, identity)

1 Worked examples

Study these first. Each one shows the method for a question type used later.

■ **Linear transformations**

A 2×2 matrix acts on a vector:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax + by \\ cx + dy \end{bmatrix}.$$

This transforms the plane — scaling, rotation, or reflection.

■ **Example**

$\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 6 \end{bmatrix}$ — a scaling by 2.

2 Practice

2.1 State how a matrix acts on a vector. [1]

2.2 Compute $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ 2 \end{bmatrix}$. [1]

2.3 Compute $\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. [2]

3 Exam-style questions

3.1 The identity matrix $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ maps a vector to [1]

- **A** the zero vector
- **B** itself
- **C** its reverse
- **D** twice itself

3.2 $\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$ applied to a vector gives a [1]

- **A** rotation
- **B** scaling by 3
- **C** reflection
- **D** shift

3.3 $A = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$.

(a) Compute $A \begin{bmatrix} 1 \\ 4 \end{bmatrix}$. [1]

(b) Compute $A \begin{bmatrix} 3 \\ 0 \end{bmatrix}$. [1]

(c) Describe the effect. [1]

Solutions

2.1 by matrix-vector multiplication, giving $\begin{bmatrix} ax+by \\ cx+dy \end{bmatrix}$.

2.2 $\begin{bmatrix} 5 \\ 2 \end{bmatrix}$.

2.3 $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$.

3.1 **B**.

3.2 **B**.

3.3 (a) $\begin{bmatrix} 2 \\ 4 \end{bmatrix}$. (b) $\begin{bmatrix} 6 \\ 0 \end{bmatrix}$. (c) it stretches the x-component by 2 and leaves y unchanged.

4.13 Matrices as Functions

Name: ____________ Class: ________ Date: __________ **Total: 9 marks**

KEY WORDS function 函数 • matrix 矩阵 • vector 向量

Objective

Build the skills to answer exam questions on **matrices as functions**.

You must be able to:

- view a matrix transformation $T(\vec{v}) = A\vec{v}$ as a **function** 函数 on vectors
- relate composition of transformations to the matrix product

1 Worked examples

Study these first. Each one shows the method for a question type used later.

Matrices as functions

A matrix transformation $T(\vec{v}) = A\vec{v}$ is a function taking an input vector to an output vector. Applying two transformations in sequence corresponds to **multiplying** their matrices: applying A then B is $B(A\vec{v}) = (BA)\vec{v}$.

Example

$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ swaps components: $T\left(\begin{bmatrix} 2 \\ 5 \end{bmatrix}\right) = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$.

2 Practice

2.1 State what the input and output of a matrix transformation are. [1]

2.2 State what composing two matrix transformations corresponds to. [1]

2.3 If $T(\vec{v}) = A\vec{v}$ and $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, find $T\left(\begin{bmatrix} 2 \\ 5 \end{bmatrix}\right)$. [2]

3 Exam-style questions

3.1 A matrix transformation $T(\vec{v}) = A\vec{v}$ takes [1]

- **A** a number to a number
- **B** a vector to a vector
- **C** a matrix to a scalar
- **D** nothing

3.2 Applying transformation A then B equals the single matrix [1]

- **A** $A + B$
- **B** BA
- **C** $A - B$
- **D** AB only if diagonal

3.3 $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

(a) Find $A\begin{bmatrix} 2 \\ 5 \end{bmatrix}$. [1]

(b) Find $A\begin{bmatrix} 7 \\ 0 \end{bmatrix}$. [1]

(c) Describe the effect. [1]

Solutions

2.1 the input is a vector and the output is a vector.

2.2 multiplying their matrices.

2.3 $\begin{bmatrix} 5 \\ 2 \end{bmatrix}$.

3.1 **B**.

3.2 **B**.

3.3 (a) $\begin{bmatrix} 5 \\ 2 \end{bmatrix}$. (b) $\begin{bmatrix} 0 \\ 7 \end{bmatrix}$. (c) it swaps the two components (a reflection across $y = x$).

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4.14 Matrices Modeling Contexts

Name: ____________ Class: ________ Date: __________ **Total: 8 marks**

KEY WORDS transition matrix 转移矩阵 • matrix 矩阵 • vector 向量

Objective

Build the skills to answer exam questions on **matrices modeling contexts**.

You must be able to:

- use a **transition matrix 转移矩阵** to step a state vector forward

1 Worked examples

Study these first. Each one shows the method for a question type used later.

■ **Transition matrices**

A transition matrix models a process that moves between states each step:

$$\text{next state} = (\text{transition matrix}) \times (\text{current state}).$$

Repeated multiplication steps the model forward in time.

■ **Example**

State $\begin{bmatrix} \text{city} \\ \text{suburb} \end{bmatrix}$: each year a transition matrix redistributes the population; multiply to get next year's distribution.

2 Practice

2.1 State what a transition matrix models. [1]

2.2 State how to get the next state from the current state. [1]

2.3 If the current state is $\begin{bmatrix} 10 \\ 20 \end{bmatrix}$ and the transition is the identity matrix, state the next state. [1]

3 Exam-style questions

3.1 A transition matrix models [1]

- **A** a single number
- **B** movement between states over steps
- **C** a static picture
- **D** an angle

3.2 The next state equals [1]

- **A** current state minus the matrix
- **B** matrix times current state
- **C** matrix plus current state
- **D** the current state alone

3.3 Current state $\begin{bmatrix} 100 \\ 0 \end{bmatrix}$; transition $\begin{bmatrix} 0.5 & 0 \\ 0.5 & 1 \end{bmatrix}$ (half of state 1 moves to state 2 each step).

(a) Compute the next state. [1]

(b) State the new value of state 1. [1]

(c) State the new value of state 2. [1]

Solutions

2.1 a process that moves between states each step.

2.2 multiply the transition matrix by the current state vector.

2.3 $\begin{bmatrix} 10 \\ 20 \end{bmatrix}$ (unchanged).

3.1 **B**.

3.2 **B**.

3.3 (a) $\begin{bmatrix} 0.5(100)+0 \\ 0.5(100)+1(0) \end{bmatrix} = \begin{bmatrix} 50 \\ 50 \end{bmatrix}$. (b) 50. (c) 50.