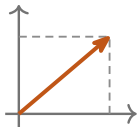


Parametric, Vectors & Matrices

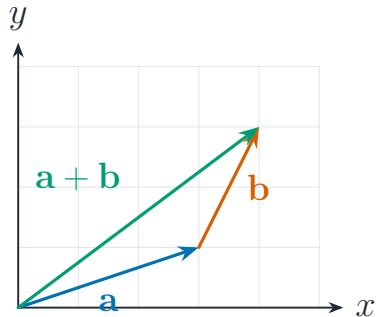
AP Precalculus ■ Unit 4

AP Precalculus



What we will cover

- 1 Parametric functions
- 2 Planar motion & rates
- 3 Circles lines & conics
- 4 Implicit curves
- 5 Vectors
- 6 Matrices & determinants
- 7 Linear transformations



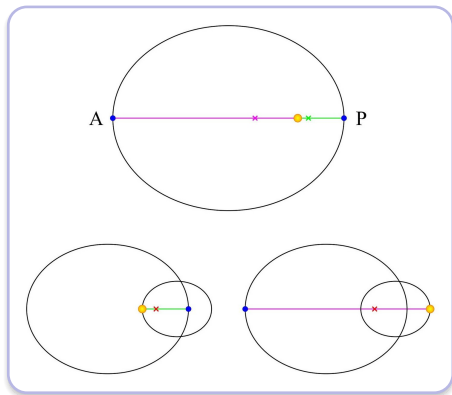


Parametric functions

A **parametric function** gives x and y each as functions of a **parameter** t :

$$x = x(t), \quad y = y(t).$$

The curve traced **need not be a function of x** – it may loop or cross itself.



Elliptical orbit

Parametrically Defined Circles and Lines

A **circle** of radius R centered at (h, k) is $x = h + R \cos t$, $y = k + R \sin t$.

A **line** through (x_0, y_0) with direction (a, b) is $x = x_0 + at$, $y = y_0 + bt$.

Parametrising an implicit curve

what it means

rewriting a curve as $(x(t), y(t))$ – a position at each value of t

why bother

it is easier to graph, and it turns the curve into **motion** along a path

the circle

$x = r \cos t$, $y = r \sin t$ traces $x^2 + y^2 = r^2$ once as t runs 0 to 2π

the ellipse

$x = a \cos t$, $y = b \sin t$ – the same idea with different radii

the extra information

a parametrisation also fixes the **direction** and **speed** of travel, which the implicit equation does not

One curve has many parametrisations. Choosing one adds information – where you start, which way you go, how fast – so say what yours does.

Planar motion and rates of change

As t advances, $(x(t), y(t))$ traces the path of a moving point – so t carries **direction** and **timing**, which a plain curve cannot show.

Read each rate **separately**: $x(t)$ increasing moves right, $y(t)$ decreasing moves down.



Vector coaster

Implicit curves and conic sections

An **implicitly defined** curve is an equation in x and y – not solved for y . The **conic sections** :

- circle $x^2 + y^2 = r^2$
- ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$
- hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$
- parabola $y = ax^2$

To **parametrize** an implicit curve, find $x(t)$ and $y(t)$ that satisfy the equation for every t – for a circle, $\cos^2 t + \sin^2 t = 1$ does it **automatically**.

Implicit equations and the four conics

implicit

an equation relating x and y **without solving** for either – $x^2 + y^2 = 25$

the consequence

its graph may **fail the vertical-line test**, so it need not be a function at all

conic sections

circles, ellipses, parabolas and hyperbolas – the curves you get by slicing a cone

each is a quadratic

one general second-degree equation in x and y covers all four

telling them apart

the signs and coefficients of x^2 and y^2 : equal – circle; same sign, unequal – ellipse; one missing – parabola; opposite signs – hyperbola

Complete the square in both variables to get the standard form, and the centre, radius or axes read straight off it.

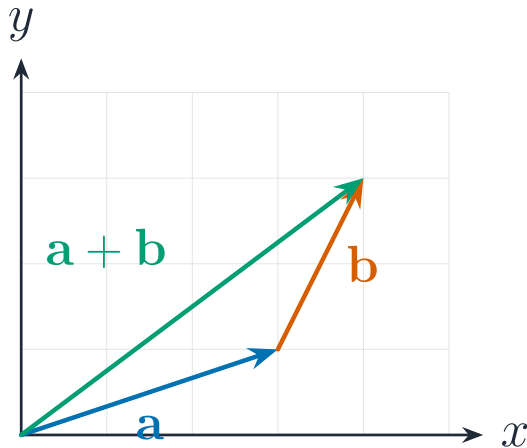


Vectors

A **vector** has **magnitude** and **direction**, written $\langle a, b \rangle$:

$$|\vec{v}| = \sqrt{a^2 + b^2}, \quad \theta = \arctan \frac{b}{a}.$$

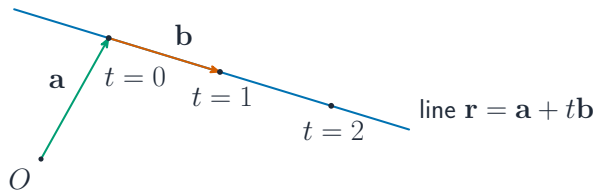
Add **component-wise**; a **scalar** multiple $k\vec{v}$ scales the length (and flips it if $k < 0$).



Vector-valued functions

A vector-valued function

$\vec{p}(t) = \langle x(t), y(t) \rangle$ is a **position vector** for each t – the same idea as a parametric function, written as one object.



A line: start at $\vec{a}$, then slide t lots of the direction $\vec{b}$ – $\vec{p}(t) = \vec{a} + t\vec{b}$.



Matrices, determinant and inverse

For $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, the **determinant** is

$$\det A = ad - bc.$$

A is **invertible** exactly when $\det A \neq 0$. The inverse **undoes** the transformation.

$$A = \begin{bmatrix} 3 & 1 \\ 2 & 4 \end{bmatrix}$$

$\det A = (3)(4) - (1)(2) = 10 \neq 0$, so A is invertible:

$$A^{-1} = \frac{1}{10} \begin{bmatrix} 4 & -1 \\ -2 & 3 \end{bmatrix}.$$

Swap the diagonal, negate the off-diagonal, divide by det.

Matrix arithmetic

a matrix

a rectangular array of numbers

addition

entry by entry, so the two must be the **same size**

multiplication

combine **rows of the first with columns of the second**; A 's column count must equal B 's row count

not commutative

$AB \neq BA$ in general – and often only one of the two is even defined

the identity

I leaves any compatible matrix unchanged: $AI = IA = A$

the inverse

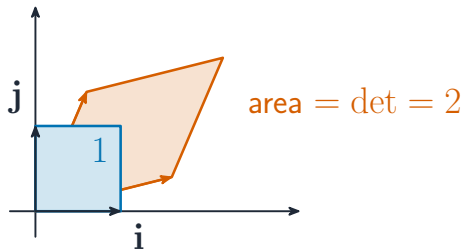
A^{-1} exists only when $\det A \neq 0$, and then $AA^{-1} = I$

As a transformation, $\det A$ is the **area scale factor**. A zero determinant collapses the plane onto a line, which is exactly why the inverse cannot exist.

Matrices as linear transformations

$\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$ stretches the plane

It sends $\begin{bmatrix} 1 \\ 1 \end{bmatrix} \mapsto \begin{bmatrix} 2 \\ 3 \end{bmatrix}$: $2\times$ horizontally, $3\times$ vertically. Its determinant $2 \times 3 = 6$ means every area is multiplied by 6 – the unit square becomes a 2×3 rectangle.

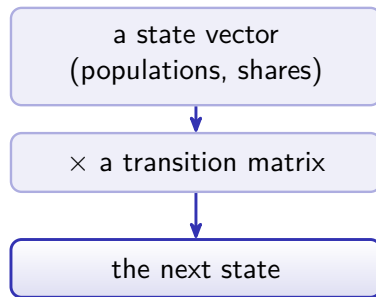


The determinant is an **area scale factor** – a negative one flips orientation.

Matrices as functions, and as models

A matrix **is** a function on vectors:
composition of transformations is matrix
multiplication, and the **identity matrix**
leaves every vector unchanged.

Order matters: AB and BA usually differ
– just like function composition.



A matrix is a function, and it models a system in stages

the claim

a matrix maps input vectors to output vectors, so it *is* a function on the plane

composition

applying one transformation then another is **matrix multiplication** – and the order matters

the inverse

A^{-1} undoes A , and exists exactly when $\det A \neq 0$

modelling in stages

matrices model systems that **step forward in stages** – populations moving between states

a transition matrix

each entry is the proportion moving from one state to another in one step

 n steps ahead

multiply by A n times: $\mathbf{v}_n = A^n \mathbf{v}_0$

Because composition is multiplication, “do this transformation n times” becomes “raise the matrix to the n th power” – which is what makes the stage model tractable.

Exam tips

- A **parametric** function gives $x(t)$ and $y(t)$ separately – track the direction of motion as t increases.
- A **vector** has magnitude and direction; add component-by-component and find magnitude with $\sqrt{a^2 + b^2}$.
- The **determinant** of a 2×2 matrix is $ad - bc$ (mind the minus sign); it scales area under the transformation.
- A 2×2 matrix transforms the plane (rotate, reflect, stretch); an inverse exists only when the determinant is non-zero.
- Note this unit is not tested on the AP Precalculus Exam (Units 1–3 only), but it underpins later courses.



Unit 4 – key takeaways

1

Parametric

t adds direction and timing a plain curve cannot show

2

Conics

implicit curves; parametrize with $\cos^2 t + \sin^2 t = 1$

3

Vectors

magnitude and direction; add component-wise

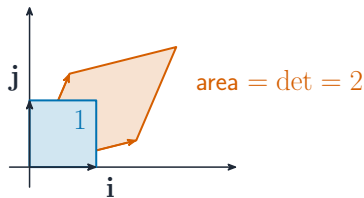
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Matrices

$\det = ad - bc$ is the area scale factor; $\neq 0$ means invertible

Check yourself

- 1 Parametrize a circle of radius r centred at (h, k) .
- 2 Find $\det \begin{bmatrix} 3 & 1 \\ 2 & 4 \end{bmatrix}$.
- 3 What does a determinant of 6 do to areas?
- 4 Give the magnitude of $\langle a, b \rangle$.



Answers 1. $x = h + r \cos t$, $y = k + r \sin t$ 2. 10 3. multiplies every area by 6
4. $\sqrt{a^2 + b^2}$