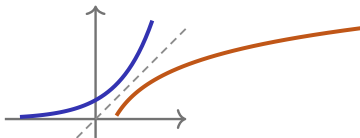


Exponential & Logarithmic Functions

AP Precalculus ■ Unit 2

AP Precalculus



What we will cover

- 1 Arithmetic & geometric sequences
- 2 Linear vs exponential change
- 3 Exponential functions
- 4 Composition & inverses
- 5 Logarithms
- 6 Solving equations
- 7 Modelling & semi-log plots



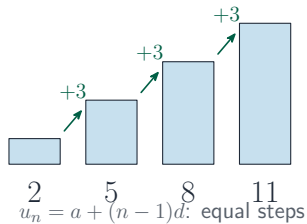
Arithmetic and geometric sequences

A **sequence** maps whole numbers to reals – so its graph is **discrete points**, not a curve.

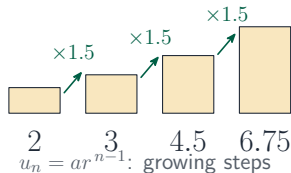
- **arithmetic : common difference** d , $a_n = a_0 + dn$
- **geometric : common ratio** r , $g_n = g_0 r^n$

Arithmetic and geometric sequences, in detail

arithmetic: add d each step



geometric: multiply by r each step



Arithmetic grows by the **same amount** each step; geometric by a **larger amount** each step.

A **sequence** maps whole numbers to reals – so its graph is **discrete points**, not a curve.

Worked example – addition versus multiplication

Compare the fourth terms

Arithmetic, $a_0 = 3$, $d = 5$:

$$a_4 = 3 + 5(4) = 23$$

Geometric, $g_0 = 2$, $r = 3$:

$$g_4 = 2 \cdot 3^4 = 162$$

Multiplication makes the geometric one pull far ahead.

The continuous cousins: linear $f(x) = b + mx$ mirrors arithmetic (repeated addition); exponential $f(x) = ab^x$ mirrors geometric (repeated multiplication).

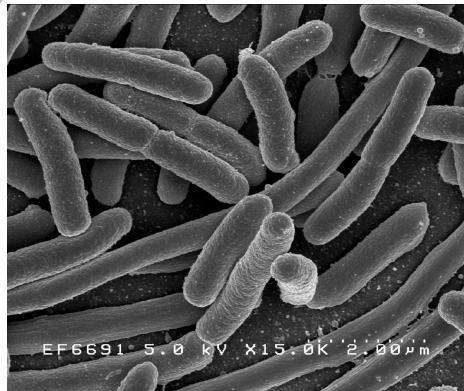


Exponential functions

$$f(x) = ab^x, \quad a \neq 0, \quad b > 0, \quad b \neq 1$$

initial value a , **base** b ; domain: all reals.

- $b > 1$: **growth**
- $0 < b < 1$: **decay**



Bacteria exponential

What makes a function exponential

the defining property

outputs are **proportional** over equal-length input intervals
– a constant *ratio*, where a linear function has a constant difference

the consequence

an exponential is always increasing or always decreasing, and always concave up or always concave down

exponential growth

$b > 1$: the value multiplies each step

exponential decay

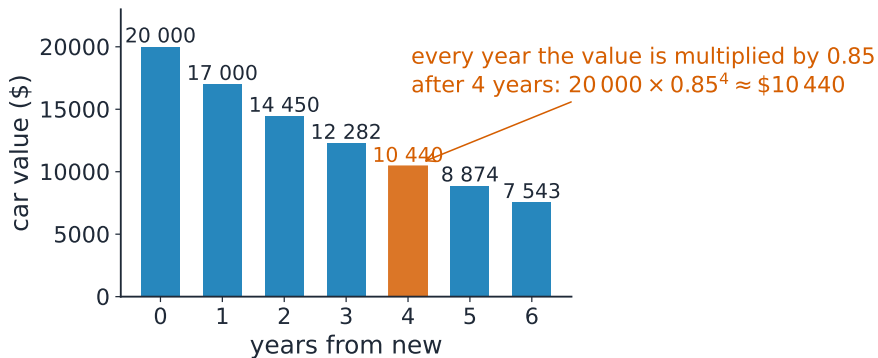
$0 < b < 1$: it multiplies by less than one, approaching zero without reaching it

in data

sometimes a **constant must be added** to the data before the proportional pattern shows – a shifted asymptote hides it

Test data by **dividing** consecutive outputs, not subtracting. A constant ratio is exponential; a constant difference is linear.

Exponential decay



Exponential decay: a fixed percentage is lost each period

Exponent rules are graph transformations

Product $b^m b^n = b^{m+n}$

a horizontal **shift** b^{x+k}
is a vertical **stretch**

Power $(b^m)^n = b^{mn}$

a horizontal **stretch** b^{cx}
is a **base change** $(b^c)^x$

Negative $b^{-n} = \frac{1}{b^n}$

Unit fraction $b^{1/k} = \sqrt[k]{b}$ (the k th **root**)

Exponent rules are graph transformations, in detail



Composition matryoshka

Modelling, and choosing between models

Exponentials model growth by a **constant proportion** over equal intervals.

- build from a ratio + an initial value, or from **two points**
- **exponential regression** fits a data set
- the **natural base** $e \approx 2.718$ is the standard



Exponential sprint

Choosing between competing models

the problem

when a rate of change shifts only slightly, linear, quadratic and exponential can all seem to fit

residual

actual minus predicted – the gap the model leaves at each point

residual plot

a model is **appropriate** when its residual plot shows **no pattern**

a pattern means

the model is missing structure the data has – a curve in the residuals means you fitted a line to a curve

the error

the gap between predicted and actual; **context** decides whether over- or under-estimating is safer

A high R^2 with a patterned residual plot is still the wrong model. The residual plot is the check that catches what the summary number hides.



Composition of functions

The **composite** $(f \circ g)(x) = f(g(x))$ feeds g 's output into f .

Composition is **not commutative**: $f(g(x))$ and $g(f(x))$ usually differ.

To build it, **substitute** $g(x)$ for every x in f . The **identity** $f(x) = x$ changes nothing.

Composition of functions, in detail

$f(x) = 3x - 5$: do $\times 3$, then -5



$f^{-1}(x) = \frac{x + 5}{3}$: **reverse the order, undo each step**



the inverse takes the
output back to the
input: $f(2) = 1$,
so $f^{-1}(1) = 2$

The **composite** $(f \circ g)(x) = f(g(x))$ feeds g 's output into f .

Composing, decomposing, and inverting

composite function

$(f \circ g)(x) = f(g(x))$ – inner function first, and the order matters

the domain

x must be allowed in g , **and** $g(x)$ must be allowed in f

decomposing

splitting a function into simpler pieces – an additive shift is composing with $x + k$, a dilation is composing with bx

finding an inverse

swap x and y in $y = f(x)$, then **solve for y**

the check

$f(f^{-1}(x)) = x$: composing a function with its inverse gives the **identity function**

the restriction

context, or a failed horizontal-line test, may limit **where** the inverse applies

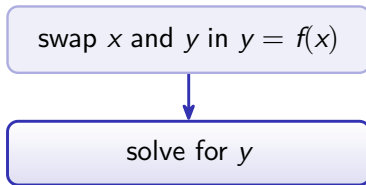
Decomposition is why transformations and composition are the same topic: every shift and stretch is a composition with a simple function.

Inverse functions

f is **invertible** where each output comes from a **unique** input. Then
 $f(a) = b \Rightarrow f^{-1}(b) = a$, and

$$f(f^{-1}(x)) = f^{-1}(f(x)) = x.$$

Domain and range **swap**; the graph reflects over $y = x$.



You may **restrict the domain** to force invertibility.



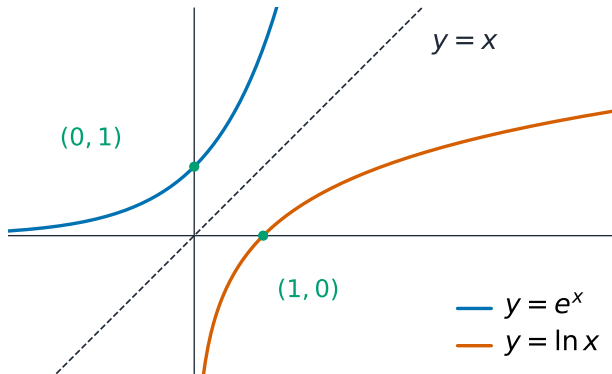
The logarithm answers “what exponent?”

$$\log_b c = a \iff b^a = c$$

The **logarithm** is the **inverse of the exponential**, so

$$\log_b(b^x) = x, \quad b^{\log_b x} = x.$$

$\ln x = \log_e x$ is the **natural logarithm**, inverse of e^x .



The logarithmic function, and its properties

$f(x) = \log_b x$: domain $x > 0$, range all reals, a **vertical asymptote at $x = 0$** – the mirror of the exponential's horizontal one.

It grows **very slowly** for large x .

$$\log_b(xy) = \log_b x + \log_b y$$

$$\log_b \frac{x}{y} = \log_b x - \log_b y$$

$$\log_b(x^n) = n \log_b x$$

change of base : $\log_b x = \frac{\ln x}{\ln b}$

Solving exponential and logarithmic equations

Use the fact that they are **inverses**:

- isolate the exponential, then **take a log** of both sides
- isolate the log, then **exponentiate** both sides

Check for **extraneous solutions** – a log's argument must stay **positive**.

Solve $2 \cdot 3^x = 54$

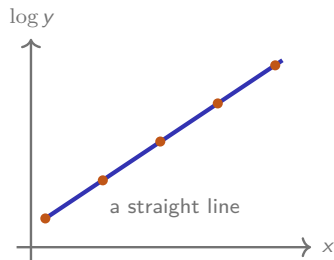
Divide by 2: $3^x = 27 = 3^3$, so $x = 3$.
When the sides are not tidy powers, take logs:

$$5^x = 20 \Rightarrow x = \frac{\ln 20}{\ln 5} \approx 1.86.$$

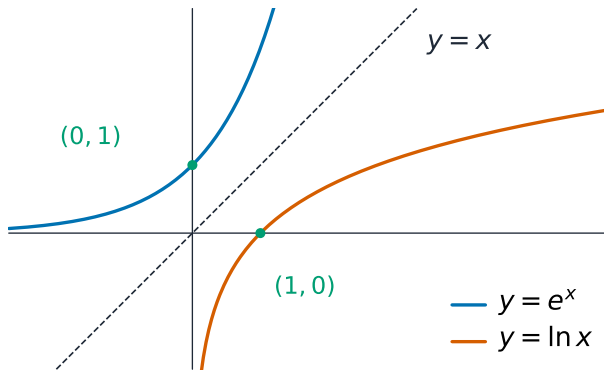
Semi-log plots

On a **logarithmic scale**, each unit is a **multiplicative** step ($10^0, 10^1, 10^2, \dots$).

Plot $\log y$ against x : an **exponential becomes a straight line**, whose slope and intercept recover the model.



Check yourself, in detail



e^x and $\ln x$ are reflections of each other in the line $y = x$

Exam tips

- Distinguish **arithmetic** (add a common difference) from **geometric** (multiply a common ratio) sequences and their linear/exponential function cousins.
- Exponential = repeated **multiplication**, so it eventually outgrows any linear or polynomial model.
- The **logarithm is the inverse** of the exponential ($\log_b c = a \Leftrightarrow b^a = c$); use it to solve $b^x = k$.
- Apply the log laws (product, quotient, power) and the change-of-base formula; the argument of a log must be **positive**.
- On a **semi-log plot** an exponential model becomes a straight line.



Unit 2 – key takeaways

1

Two kinds of change

linear repeats addition;
exponential repeats multiplication

2

Exponentials

$f = ab^x$: no extrema, no
inflection, proportional growth

3

Inverses

swap domain and range; reflect
over $y = x$

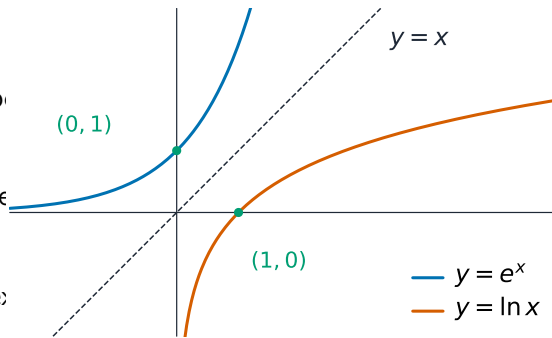
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Logarithms

the inverse of b^x ; log rules reverse
the exponent rules

Check yourself

- 1 Solve $2 \cdot 3^x = 54$.
- 2 What does $\log_b c = a$ mean, as an exponential?
- 3 Does an exponential function have inflection points?
- 4 On a semi-log plot, what shape is an exponential?



Answers 1. $x = 3$ 2. $b^a = c$ 3. no 4. a straight line