

Exercise walk-through

4.14

Matrices Modeling Contexts

AP Precalculus

Matrices, determinant and inverse

For $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, the **determinant** is

$$\det A = ad - bc.$$

A is **invertible** exactly when $\det A \neq 0$. The inverse **undoes** the transformation.

$$A = \begin{bmatrix} 3 & 1 \\ 2 & 4 \end{bmatrix}$$

$\det A = (3)(4) - (1)(2) = 10 \neq 0$, so A is invertible:

$$A^{-1} = \frac{1}{10} \begin{bmatrix} 4 & -1 \\ -2 & 3 \end{bmatrix}.$$

Swap the diagonal, negate the off-diagonal, divide by $\det$.

Matrix arithmetic

a matrix

a rectangular array of numbers

addition

entry by entry, so the two must be the **same size**

multiplication

combine **rows of the first with columns of the second**; A 's column count must equal B 's row count

not commutative

$AB \neq BA$ in general – and often only one of the two is even defined

the identity

I leaves any compatible matrix unchanged: $AI = IA = A$

the inverse

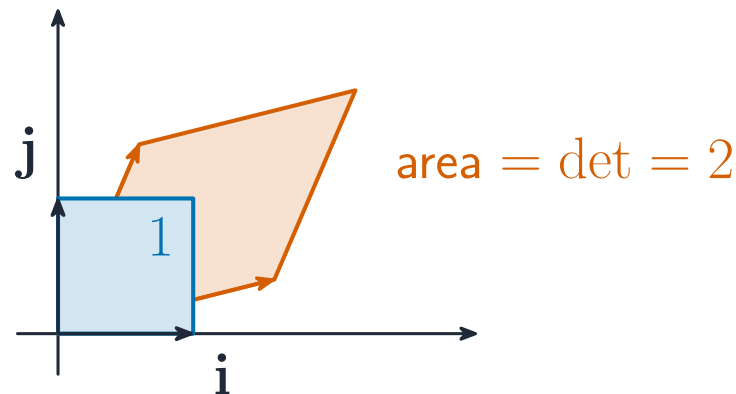
A^{-1} exists only when $\det A \neq 0$, and then $AA^{-1} = I$

As a transformation, $\det A$ is the **area scale factor**. A zero determinant collapses the plane onto a line, which is exactly why the inverse cannot exist.

Matrices as linear transformations

$\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$ stretches the plane

It sends $\begin{bmatrix} 1 \\ 1 \end{bmatrix} \mapsto \begin{bmatrix} 2 \\ 3 \end{bmatrix}$: $2\times$ horizontally, $3\times$ vertically. Its determinant $2 \times 3 = 6$ means every area is multiplied by 6 – the unit square becomes a 2×3 rectangle.

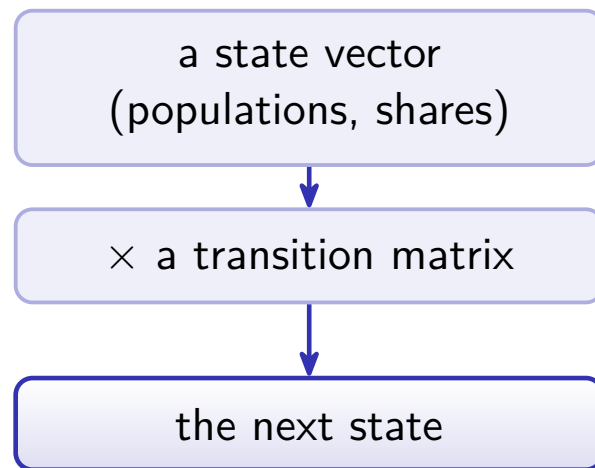


The determinant is an **area scale factor** – a negative one flips orientation.

Matrices as functions, and as models

A matrix **is** a function on vectors:
composition of transformations is matrix
multiplication, and the **identity matrix**
leaves every vector unchanged.

Order matters: AB and BA usually differ
– just like function composition.



A matrix is a function, and it models a system in stages

the claim

a matrix maps input vectors to output vectors, so it **is** a function on the plane

composition

applying one transformation then another is **matrix multiplication** – and the order matters

the inverse

A^{-1} undoes A , and exists exactly when $\det A \neq 0$

modelling in stages

matrices model systems that **step forward in stages** – populations moving between states

a transition matrix

each entry is the proportion moving from one state to another in one step

 n steps ahead

multiply by A n times: $\mathbf{v}_n = A^n \mathbf{v}_0$

Because composition is multiplication, “do this transformation n times” becomes “raise the matrix to the n th power” – which is what makes the stage model tractable.

Exercises 2.1-2.3

2.1 State what a transition matrix models. [1]

2.2 State how to get the next state from the current state. [1]

2.3 If the current state is $\begin{bmatrix} 10 \\ 20 \end{bmatrix}$ and the transition is the identity matrix, state the next state. [1]

Exercises 2.1-2.3

2.1 State what a transition matrix models. [1]

2.1

- a process that moves between states each step
-

2.2 State how to get the next state from the current state. [1]

2.2

- multiply the transition matrix by the current state vector
-

2.3 If the current state is $\begin{bmatrix} 10 \\ 20 \end{bmatrix}$ and the transition is the identity matrix, state the next state. [1]

2.3

- $\begin{bmatrix} 10 \\ 20 \end{bmatrix}$ (unchanged)

Exercises 3.1-3.2

3.1 A transition matrix models

[1]

☐

A a single number

B movement between states over steps

C a static picture

D an angle

3.2 The next state equals

[1]

☐

A current state minus the matrix

B matrix times current state

C matrix plus current state

D the current state alone

Exercises 3.1-3.2

3.1 A transition matrix models

[1]

☐

A a single number

B movement between states over steps

C a static picture

D an angle

3.1 B

- movement between states over steps.

3.2 The next state equals

[1]

☐

A current state minus the matrix

B matrix times current state

C matrix plus current state

D the current state alone

3.2 B

- matrix times current state.

Exercise 3.3 (a)

3.3 Current state $\begin{bmatrix} 100 \\ 0 \end{bmatrix}$; transition $\begin{bmatrix} 0.5 & 0 \\ 0.5 & 1 \end{bmatrix}$ (half of state 1 moves to state 2 each step).

(a) Compute the next state.

[1]

Exercise 3.3 (a)

3.3 Current state $\begin{bmatrix} 100 \\ 0 \end{bmatrix}$; transition $\begin{bmatrix} 0.5 & 0 \\ 0.5 & 1 \end{bmatrix}$ (half of state 1 moves to state 2 each step).

(a) Compute the next state. [1]

(a)

$$\bullet \quad \begin{bmatrix} 0.5(100) + 0 \\ 0.5(100) + 1(0) \end{bmatrix} = \begin{bmatrix} 50 \\ 50 \end{bmatrix}$$

Exercise 3.3 (b)

3.3 Current state $\begin{bmatrix} 100 \\ 0 \end{bmatrix}$; transition $\begin{bmatrix} 0.5 & 0 \\ 0.5 & 1 \end{bmatrix}$ (half of state 1 moves to state 2 each step).

(b) State the new value of state 1.

[1]

Exercise 3.3 (b)

3.3 Current state $\begin{bmatrix} 100 \\ 0 \end{bmatrix}$; transition $\begin{bmatrix} 0.5 & 0 \\ 0.5 & 1 \end{bmatrix}$ (half of state 1 moves to state 2 each step).

(b) State the new value of state 1. [1]

(b)

- 50

Exercise 3.3 (c)

3.3 Current state $\begin{bmatrix} 100 \\ 0 \end{bmatrix}$; transition $\begin{bmatrix} 0.5 & 0 \\ 0.5 & 1 \end{bmatrix}$ (half of state 1 moves to state 2 each step).

(c) State the new value of state 2.

[1]

Exercise 3.3 (c)

3.3 Current state $\begin{bmatrix} 100 \\ 0 \end{bmatrix}$; transition $\begin{bmatrix} 0.5 & 0 \\ 0.5 & 1 \end{bmatrix}$ (half of state 1 moves to state 2 each step).

(c) State the new value of state 2. [1]

(c)

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