

Exercise walk-through

1.9

Rational Functions and Vertical Asymptotes

AP Precalculus

Rational end behaviour – compare the degrees

top $>$ bottom
follows the **quotient** polynomial;
if linear, a **slant asymptote**

equal degrees
horizontal asymptote
at the **ratio of leading coefficients**

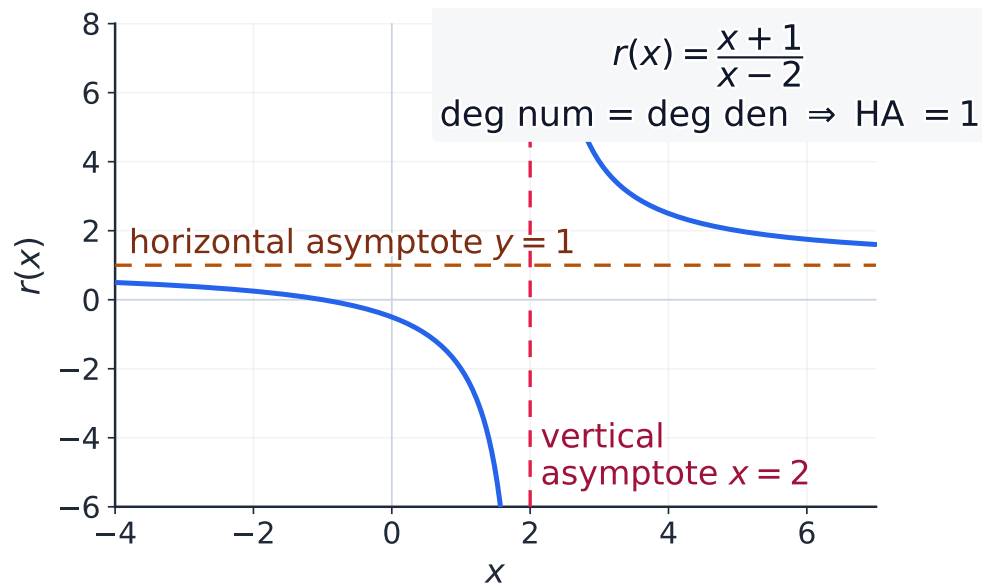
top $<$ bottom
the quotient
 $\rightarrow 0$, so $y = 0$

At a horizontal asymptote $y = b$, the outputs get **and stay** arbitrarily close to b .

Vertical asymptotes and holes

- a **vertical asymptote** at $x = a$: a is a zero of the **denominator** that does **not** cancel
- a **hole** at $x = c$: the factor **cancels** – a single missing point at (c, L)

Check **each side** of an asymptote – the two sides can go opposite ways.



Worked example – describe a rational function

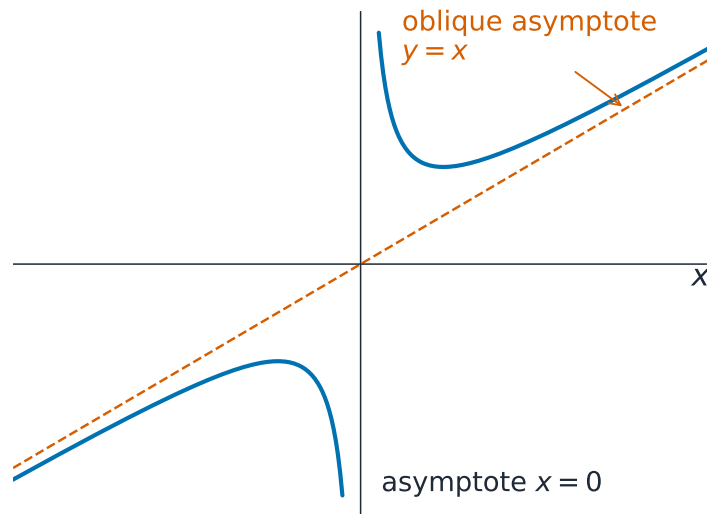
$$r(x) = \frac{2x^2 + 3}{x^2 - 1}$$

Equal degrees, so the horizontal asymptote is

$$y = \frac{2}{1} = 2.$$

$x^2 - 1 = 0$ at $x = \pm 1$, and neither cancels, so there are vertical asymptotes at

$$x = 1 \quad \text{and} \quad x = -1.$$



Top degree **one more** than the bottom: long division gives a **slant** asymptote.

Equivalent forms reveal different features

Factored form

shows real **zeros** – and hence intercepts, holes, vertical asymptotes, domain and range

Standard form

shows the **degree** and **leading term** – and hence end behaviour

Long division

rewrites $f = gq + r$; the **quotient** q is the **slant asymptote** when the degrees differ by one.

The same expression, written to reveal different things

factored form

shows the **zeros** immediately

standard form

shows the **degree** and the leading coefficient, so the end behaviour

vertex form

shows the **vertex** of a quadratic without any calculation

after polynomial long division

shows a rational function's **asymptote** as the quotient, plus a remainder that vanishes

the habit

before calculating, ask which form already *contains* the answer

Rewriting is not busywork; it is choosing the representation in which the question is already answered.

Exercises 2.1·2.3

2.1 State the condition for a vertical asymptote at $x = a$. [1]

2.2 For $r(x) = \frac{1}{x-3}$, state the vertical asymptote. [1]

2.3 State what happens to $r(x)$ near a vertical asymptote. [1]

Exercises 2.1·2.3

2.1 State the condition for a vertical asymptote at $x = a$. [1]

2.1

- a is a zero of the denominator but not of the numerator
-

2.2 For $r(x) = \frac{1}{x-3}$, state the vertical asymptote. [1]

2.2

- $x = 3$
-

2.3 State what happens to $r(x)$ near a vertical asymptote. [1]

2.3

- the values increase or decrease without bound

Exercises 3.1-3.2

3.1 A rational function has a vertical asymptote where the _____ is zero (but not the numerator). [1]

☐

- A numerator
- B denominator
- C leading term
- D degree

3.2 Near a vertical asymptote, the function values [1]

☐

- A approach 0
- B increase or decrease without bound
- C stay constant
- D equal the mean

Exercises 3.1-3.2

3.1 A rational function has a vertical asymptote where the _____ is zero (but not the numerator). [1]

☐

- A numerator
- B denominator
- C leading term
- D degree

3.1 B

- denominator.

3.2 Near a vertical asymptote, the function values [1]

☐

- A approach 0
- B increase or decrease without bound
- C stay constant
- D equal the mean

3.2 B

- increase or decrease without bound.

Exercise 3.3 (a)

3.3 $r(x) = \frac{x + 2}{x - 5}.$

(a) State the vertical asymptote.

[1]

Exercise 3.3 (a)

3.3 $r(x) = \frac{x + 2}{x - 5}.$

(a) State the vertical asymptote.

[1]

(a)

- $x = 5$

Exercise 3.3 (b)

3.3 $r(x) = \frac{x + 2}{x - 5}.$

(b) State the zero.

[1]

Exercise 3.3 (b)

3.3 $r(x) = \frac{x + 2}{x - 5}.$

(b) State the zero.

[1]

(b)

- $x = -2$

Exercise 3.3 (c)

3.3 $r(x) = \frac{x + 2}{x - 5}.$

(c) State the domain restriction.

[1]

Exercise 3.3 (c)

$$\mathbf{3.3} \quad r(x) = \frac{x + 2}{x - 5}.$$

(c) State the domain restriction.

[1]

(c)

- $x \neq 5$