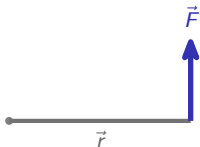


Torque & Rotational Dynamics

AP Physics C: Mechanics ■ Unit 5

AP Physics C: Mechanics



Rotation: build and test a physical model

Choose an axis and signs, connect angular motion to point motion, and account for each force at its application point.

Derive inertia from mass elements, solve a supported beam, then couple two translating masses to a rotating pulley. State the assumptions that make each equation valid.

1

Rotational Kinematics

Angular rates and signs

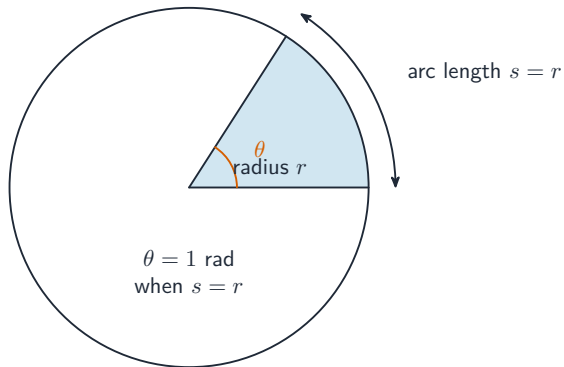
$$\Delta\theta = \theta - \theta_0, \quad \omega = \frac{d\theta}{dt}, \quad \alpha = \frac{d\omega}{dt}$$

Choose one fixed axis and a positive sense. A **rigid body** 刚体 holds its shape; points share angular motion about that axis.

Angular displacement 角位移 uses **radians** 弧度; **angular velocity** 角速度 and **angular acceleration** 角加速度 can have opposite signs while the body slows.

Kinematic directions here use clockwise/counterclockwise; torque-vector directions are considered later.

Radians connect angle and arc



One revolution is 2π radians. Use radians in the arc-length and linear-angular relations.

Constant angular acceleration: choose the model

$$\omega = \omega_0 + \alpha t$$

$$\Delta\theta = \omega_0 t + \frac{1}{2}\alpha t^2$$

$$\omega^2 = \omega_0^2 + 2\alpha\Delta\theta$$

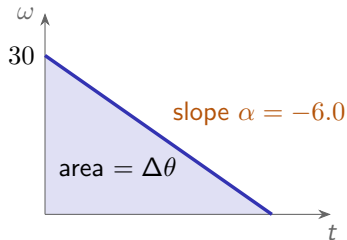
These equations require constant α . The final equation loses the sign of ω when squared: use the physical motion to choose a sign.

Stopping fan: original attempt

A fan rotates at 30 rad/s . It then slows with constant angular acceleration -6.0 rad/s^2 until it first stops.

Find the stopping time and angular displacement.
Convert the displacement to revolutions.

Does the model establish what happens after rest?



Stopping fan: time and displacement

$$\omega = \omega_0 + \alpha t \quad \Rightarrow \quad t = \frac{\omega - \omega_0}{\alpha}$$

$$t = \frac{0 - 30 \text{ rad/s}}{-6.0 \text{ rad/s}^2} = 5.0 \text{ s}$$

$$\Delta\theta = \omega_0 t + \frac{1}{2}\alpha t^2$$

$$\Delta\theta = (30 \text{ rad/s})(5.0 \text{ s}) + \frac{1}{2}(-6.0 \text{ rad/s}^2)(5.0 \text{ s})^2 = 75 \text{ rad}$$

About $75/(2\pi) = 12$ revolutions. The specified slowing phase ends at rest; subsequent motion needs a continued torque model.

Variable angular acceleration: original attempt

Let $\alpha = ct$, where $c = 4.0 \text{ rad/s}^3$. Initially $\omega_0 = 3.0 \text{ rad/s}$ and $\theta_0 = 1.0 \text{ rad}$.

Derive $\omega(t)$ and $\theta(t)$, then evaluate both at $t = 2.0 \text{ s}$.

Explain what initial state would be implied by writing only $\omega = 2t^2$.

Integrate with initial conditions

$$\omega(t) = \omega_0 + \int_0^t ct' dt' = \omega_0 + \frac{1}{2}ct^2$$

$$\theta(t) = \theta_0 + \int_0^t \left(\omega_0 + \frac{1}{2}ct'^2 \right) dt'$$

$$\theta(t) = \theta_0 + \omega_0 t + \frac{1}{6}ct^3$$

The constants represent the given starting angle and angular velocity. An indefinite integral needs equivalent constants.

Variable acceleration: numerical check

$$\omega = \omega_0 + \frac{1}{2}ct^2$$

$$\omega = 3.0 \text{ rad/s} + \frac{1}{2}(4.0 \text{ rad/s}^3)(2.0 \text{ s})^2 = 11 \text{ rad/s}$$

$$\theta = \theta_0 + \omega_0 t + \frac{1}{6}ct^3$$

$$\theta = 1.0 \text{ rad} + (3.0 \text{ rad/s})(2.0 \text{ s}) + \frac{1}{6}(4.0 \text{ rad/s}^3)(2.0 \text{ s})^3 \approx 12.3 \text{ rad}$$

Omitting ω_0 would incorrectly start from rest.

Angular graphs: original interpretation task

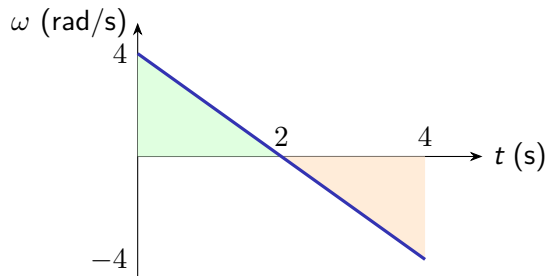
An angular-velocity graph is a straight line from $\omega = +4 \text{ rad/s}$ at $t = 0$ to $\omega = -4 \text{ rad/s}$ at $t = 4 \text{ s}$.

Find the angular acceleration, the instant of reversal, the net angular displacement and the total angle travelled.

Sketch the line and shade its positive and negative areas before calculating.

Angular graph: slope and signed areas

$$\alpha = \frac{\Delta\omega}{\Delta t} = \frac{-4 \text{ rad/s} - 4 \text{ rad/s}}{4 \text{ s}} = -2 \text{ rad/s}^2$$



Reversal occurs at 2 s. Each triangle has magnitude $\frac{1}{2}(2 \text{ s})(4 \text{ rad/s}) = 4 \text{ rad}$. Net displacement is $4 - 4 = 0 \text{ rad}$; total angle is $4 + 4 = 8 \text{ rad}$.

2

Connecting Linear and Rotational Motion

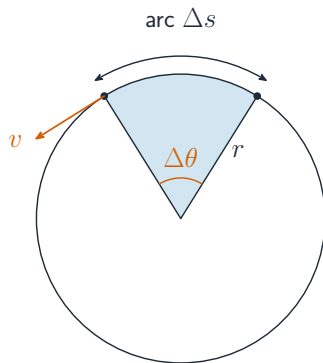
One angular motion, different linear motions

$$\Delta s = r\Delta\theta, \quad v = r|\omega|, \quad a_t = r\alpha, \quad a_c = r\omega^2$$

These relations describe points at perpendicular distance r from a fixed axis. All points of the rigid system share ω and α , but farther points have larger linear speed and acceleration components.

Tangential acceleration 切向加速度 can oppose motion. **Centripetal acceleration** 向心加速度 points inward, even at constant nonzero speed.

Angular speed and a point on the rim



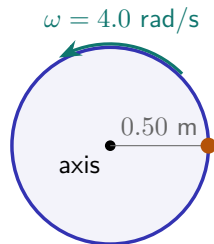
Doubling radius at the same angular speed doubles linear speed; it does not double angular speed.

Slowing wheel: original attempt

A wheel rotates counterclockwise at $\omega = 4.0 \text{ rad/s}$ and has $\alpha = -2.0 \text{ rad/s}^2$. A marked point lies 0.50 m from the fixed axis.

Find its speed, signed tangential acceleration, inward acceleration and total acceleration magnitude.

At the rightmost position, draw the directions of velocity and both accelerations.



Wheel: calculate the components

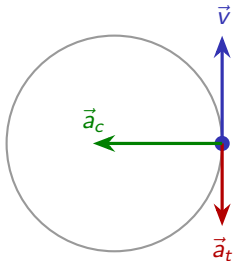
$$v = r|\omega| = (0.50 \text{ m})(4.0 \text{ rad/s}) = 2.0 \text{ m/s}$$

$$a_t = r\alpha = (0.50 \text{ m})(-2.0 \text{ rad/s}^2) = -1.0 \text{ m/s}^2$$

$$a_c = r\omega^2 = (0.50 \text{ m})(4.0 \text{ rad/s})^2 = 8.0 \text{ m/s}^2$$

$$a = \sqrt{a_t^2 + a_c^2} = \sqrt{(-1.0 \text{ m/s}^2)^2 + (8.0 \text{ m/s}^2)^2} \approx 8.1 \text{ m/s}^2$$

Wheel: directions at the rightmost point



Tangential acceleration opposes the upward velocity: the point is slowing. Inward acceleration turns the velocity. Arrow lengths are schematic, not a common numerical scale.

3

Torque

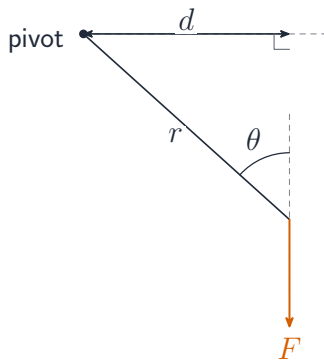
Torque about a chosen axis

$$\vec{\tau} = \vec{r} \times \vec{F}, \quad |\tau| = rF \sin \phi = Fr_{\perp}$$

Torque 力矩 is a **vector product** 矢量积. The **moment arm** 力臂 is the perpendicular distance to the force's line of action.

The **right-hand rule** 右手定则 points perpendicular to the plane: out of the page for counterclockwise torque, into it for clockwise. Units are N m; torque is not scalar work.

Locate the line of action



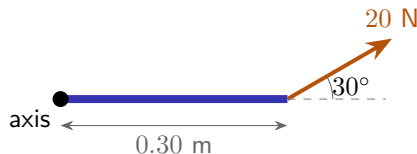
A force through the axis gives zero torque about that axis. It can still cause translation.

Angled lever: original attempt

A 20 N force acts 0.30 m from an axis, at 30° to the radius. It tends to turn the lever counterclockwise.

Find the signed torque and perpendicular moment arm. Compare with a perpendicular 10 N force at the same point.

Why is the distance to the application point not always the moment arm?



Angled lever: two equivalent calculations

$$\tau = rF \sin \phi$$

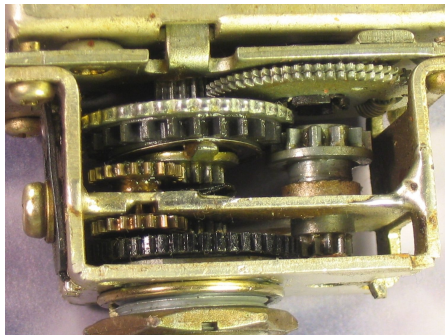
$$\tau = (0.30 \text{ m})(20 \text{ N}) \sin 30^\circ = +3.0 \text{ N m}$$

$$r_{\perp} = r \sin \phi = (0.30 \text{ m}) \sin 30^\circ = 0.15 \text{ m}$$

$$\tau = Fr_{\perp} = (20 \text{ N})(0.15 \text{ m}) = +3.0 \text{ N m}$$

A perpendicular 10 N force gives $(0.30 \text{ m})(10 \text{ N}) = 3.0 \text{ N m}$ too. Geometry and force both matter.

Gears: what the photo can establish



Visible geometry suggests different angular speeds. It supplies neither torque values nor inertias, efficiency or friction losses.

4

Rotational Inertia

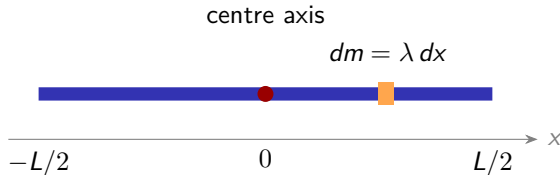
Inertia is defined relative to an axis

$$I = \sum_i m_i r_i^2, \quad I = \int r^2 dm$$

Rotational inertia 转动惯量 measures resistance to angular acceleration. Use perpendicular distance from each mass element to the chosen axis; units are kg m^2 .

For the same point mass, doubling distance multiplies its contribution by four. For several bodies, add inertias about the same axis.

Uniform rod: original derivation task



A uniform rod has mass M and length L . Derive its inertia about a perpendicular axis through its centre. State density, mass element, distance and integration limits.

Uniform rod: build the integral

$$I = \int x^2 dm, \quad \lambda = \frac{M}{L}, \quad dm = \frac{M}{L} dx$$

$$I_{\text{cm}} = \frac{M}{L} \int_{-L/2}^{L/2} x^2 dx$$

$$I_{\text{cm}} = \frac{M}{L} \left[\frac{x^3}{3} \right]_{-L/2}^{L/2} = \frac{ML^2}{12}$$

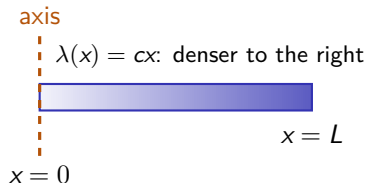
Linear density 线密度 has unit kg/m. Both halves contribute positively because distance is squared.

Nonuniform rod: original attempt

A thin rod extends from $x = 0$ to $x = L$. Its linear density is $\lambda(x) = cx$, with constant $c > 0$.

Find c in terms of total mass M and length L .
Then derive inertia about a perpendicular axis at $x = 0$.

Compare with a uniform rod of the same mass and length about one end.



Nonuniform rod: normalise before integrating

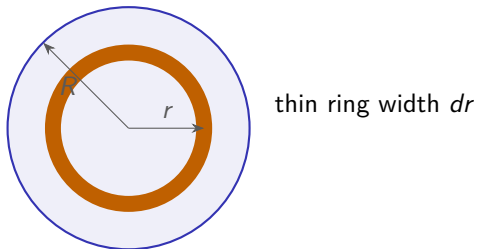
$$M = \int_0^L \lambda(x) dx = \int_0^L cx dx = \frac{1}{2}cL^2$$

$$c = \frac{2M}{L^2}, \quad dm = cx dx$$

$$I = \int_0^L x^2(cx) dx = \frac{1}{4}cL^4 = \frac{1}{2}ML^2$$

This exceeds the uniform end-axis result $ML^2/3$: more mass lies far from the axis. The coefficient c has unit kg/m^2 .

Disk from rings: original derivation task



Derive the inertia of a uniform thin disk of mass M and radius R about its central perpendicular axis. Begin with surface density and the mass of a ring.

Disk: integrate coaxial mass elements

$$\sigma = \frac{M}{\pi R^2}, \quad dm = \sigma dA = \sigma 2\pi r dr$$

$$I = \int_0^R r^2 dm = 2\pi\sigma \int_0^R r^3 dr$$

$$I = 2\pi \frac{M}{\pi R^2} \left[\frac{r^4}{4} \right]_0^R = \frac{1}{2} MR^2$$

A thin hoop with the same mass and radius has $I = MR^2$: all its mass lies at the outer radius. A disk is not a single ring at R .

Annular disk: transfer the method

$$\sigma = \frac{M}{\pi(b^2 - a^2)}, \quad dm = \sigma 2\pi r dr$$

$$I = 2\pi\sigma \int_a^b r^3 dr = \frac{\pi\sigma}{2}(b^4 - a^4)$$

$$I = \frac{M}{2} \frac{(b^2 - a^2)(b^2 + a^2)}{b^2 - a^2} = \frac{M}{2}(a^2 + b^2)$$

A uniform annular disk has inner radius a and outer radius b . Setting $a = 0$ recovers the solid disk; a very thin ring approaches Mb^2 .

Parallel axes: original attempt

$$I' = I_{\text{cm}} + Md^2$$

The **parallel-axis theorem** 平行轴定理 starts from an axis through the centre of mass and shifts to a parallel axis.

A uniform rod has mass 3.0 kg and length 2.0 m. Find its inertia about its centre and then about one end. Explain which distance is d .

Parallel axes: the rod comparison

$$I_{\text{cm}} = \frac{ML^2}{12} = \frac{(3.0 \text{ kg})(2.0 \text{ m})^2}{12} = 1.0 \text{ kg m}^2$$

$$I_{\text{end}} = I_{\text{cm}} + M(L/2)^2$$

$$I_{\text{end}} = 1.0 \text{ kg m}^2 + (3.0 \text{ kg})(1.0 \text{ m})^2 = 4.0 \text{ kg m}^2$$

The axis separation is $L/2$, not L . In symbols, $I_{\text{end}} = ML^2/12 + ML^2/4 = ML^2/3$.

2023 Q2(a): verified combined-inertia task

In the actual Set 1 question, N identical coaxial disks move together on a torsion wire. Each disk has mass M , radius R and inertia $MR^2/2$.

The question supplies $T = 2\pi\sqrt{I/\kappa}$. Derive $T(N)$ and predict the period change when the disk count quadruples.

Use the supplied period relation; deriving oscillations is a later-unit skill.

Combined disks: worked exam connection

$$I_{\text{total}} = \sum_{i=1}^N I_i = N \left(\frac{1}{2} MR^2 \right)$$

$$T = 2\pi \sqrt{\frac{I_{\text{total}}}{\kappa}} = 2\pi \sqrt{\frac{NMR^2}{2\kappa}}$$

$$\frac{T(4N)}{T(N)} = \sqrt{\frac{4N}{N}} = 2$$

The official scheme checks summed inertia and a consistent period expression. It does not support a universal checklist for every rotational question.

5

Rotational Equilibrium

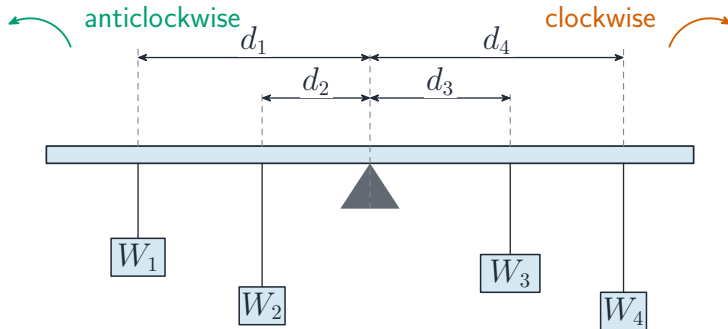
Zero torque is not necessarily rest

$$\sum \tau = 0 \quad \Rightarrow \quad \alpha = 0 \quad \text{for fixed-axis, constant-} I \text{ motion}$$

Rotational equilibrium 转动平衡 permits constant nonzero angular velocity. It does not alone establish zero net force.

Static equilibrium 静平衡 requires rest as well as $\sum \vec{F} = 0$ and $\sum \tau = 0$. A body can satisfy translational equilibrium without rotational equilibrium, or vice versa.

Torque balance and force balance



Equal clockwise and counterclockwise torques give zero net torque. Static equilibrium also requires force balance and rest.

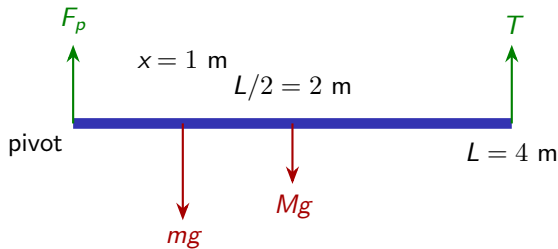
Beam and child: original attempt

A horizontal uniform 20 kg beam, 4.0 m long, is pivoted at its left end. A vertical rope holds its right end. A 40 kg child sits 1.0 m from the pivot. Take $g = 9.8 \text{ m/s}^2$.

Draw every external force at its application point. Find rope tension and the vertical pivot force.

Explain why the pivot force disappears from one equation but remains in the other.

Beam: place the forces before calculating



Taking torques about the pivot removes F_p because its moment arm is zero. The beam's weight acts at its centre; the child's weight acts at the stated position. Arrow lengths are schematic.

Beam: torque balance gives tension

$$\sum \tau = 0 \quad \Rightarrow \quad TL - Mg(L/2) - mgx = 0$$

$$T = \frac{Mg(L/2) + mgx}{L}$$

$$T = \frac{(20 \text{ kg})(9.8 \text{ m/s}^2)(2.0 \text{ m}) + (40 \text{ kg})(9.8 \text{ m/s}^2)(1.0 \text{ m})}{4.0 \text{ m}}$$

$$T = 196 \text{ N}$$

Counterclockwise is positive. Without the child the result would be 98 N; that is a different problem.

Beam: force balance gives the support force

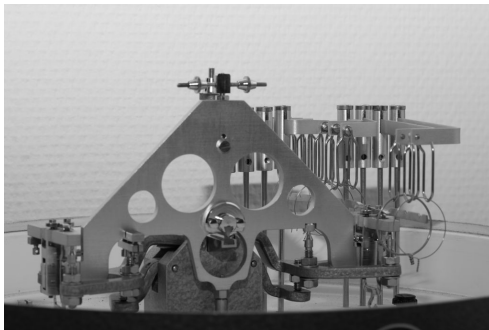
$$\sum F_y = 0 \quad \Rightarrow \quad F_p + T - (M + m)g = 0$$

$$F_p = (M + m)g - T$$

$$F_p = (20 \text{ kg} + 40 \text{ kg})(9.8 \text{ m/s}^2) - 196 \text{ N} = 392 \text{ N}$$

The pivot pushes upward. Torque balance alone did not determine this force; the separate translational condition was needed.

A real mechanical balance



The balance illustrates torque balance; its photograph provides no numerical mass or force. Use the stated data for calculations.

6

Newton's Second Law in Rotational Form

Net torque changes angular velocity

$$\sum \tau = I\alpha$$

For a rigid body with constant inertia about a fixed axis, the net torque component along that axis sets angular acceleration. The **net torque** 净力矩 can oppose the existing rotation.

If inertia changes, the constant- I model is insufficient; use the angular-momentum account developed in Unit 6.

Same torque, different inertia: original check

$$\alpha = \frac{\sum \tau}{I}$$

$$\alpha = \frac{6.0 \text{ N m}}{2.0 \text{ kg m}^2} = 3.0 \text{ rad/s}^2$$

At the same net torque, doubling I halves α .

Predict the motion if ω is initially negative but the net torque is positive. Explain why positive angular acceleration need not mean increasing angular speed.

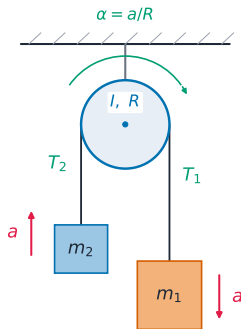
Massive pulley: original capstone attempt

A light taut inextensible rope passes over a pulley with negligible axle friction and no slip. The hanging masses are $m_1 = 4.0$ kg and $m_2 = 2.0$ kg. The pulley has $I = 0.50$ kg m² and $R = 0.20$ m. Use $g = 9.8$ m/s².

Find the acceleration and both tensions. Choose positive senses and draw the two mass diagrams and pulley diagram before forming equations.

Explain why the two tensions differ in this motion, and when a massive pulley could have equal tensions.

Pulley: connect translation and rotation



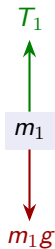
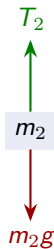
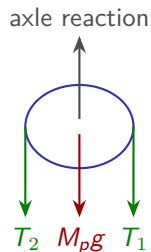
$$(T_1 - T_2)R = I\alpha$$

Ideal axle; taut rope without slip.

Unequal tensions when net torque is nonzero.

The rope does not slip: $a = R\alpha$. For this accelerating ideal-axle motion, the tension difference supplies the net torque.

Separate free-body diagrams



Forces of the rope on the pulley point downward, opposite to the tensions on the masses. Pulley weight and axle reaction pass through its centre, so give zero central torque. Diagrams are not to force scale.

Pulley: choose signs and state three laws

$$m_1 g - T_1 = m_1 a \quad (\text{mass 1: down positive})$$

$$T_2 - m_2 g = m_2 a \quad (\text{mass 2: up positive})$$

$$(T_1 - T_2)R = I\alpha \quad (\text{pulley: clockwise positive})$$

$$a = R\alpha \quad \Rightarrow \quad T_1 - T_2 = \frac{I}{R^2} a$$

The last equation is a kinematic constraint from the taut, non-slipping rope. Axle friction would require an extra torque term.

Pulley: eliminate the tensions symbolically

$$m_1 g - T_1 + T_2 - m_2 g = (m_1 + m_2) a$$

$$(m_1 - m_2) g - \frac{I}{R^2} a = (m_1 + m_2) a$$

$$a = \frac{(m_1 - m_2) g}{m_1 + m_2 + I/R^2}$$

I/R^2 has unit kg and acts as an additional inertial term in this expression. It is not an extra hanging weight.

Pulley: acceleration with units

$$\frac{I}{R^2} = \frac{0.50 \text{ kg m}^2}{(0.20 \text{ m})^2} = 12.5 \text{ kg}$$

$$a = \frac{(m_1 - m_2)g}{m_1 + m_2 + I/R^2}$$

$$a = \frac{(4.0 \text{ kg} - 2.0 \text{ kg})(9.8 \text{ m/s}^2)}{4.0 \text{ kg} + 2.0 \text{ kg} + 12.5 \text{ kg}} = 1.059 \dots \text{ m/s}^2$$

The positive result agrees with mass 1 moving downward. Retain the unrounded value for the tension calculations.

Pulley: calculate both tensions

$$T_1 = m_1(g - a)$$

$$T_1 = (4.0 \text{ kg})(9.8 \text{ m/s}^2 - 1.059 \dots \text{ m/s}^2) = 34.96 \dots \text{ N}$$

$$T_2 = m_2(g + a)$$

$$T_2 = (2.0 \text{ kg})(9.8 \text{ m/s}^2 + 1.059 \dots \text{ m/s}^2) = 21.72 \dots \text{ N}$$

The heavier mass has tension below its weight; the lighter mass has tension above its weight. Both checks agree with the acceleration directions.

Pulley: compare limits and reject a false rule

$$a_{I=0} = \frac{(m_1 - m_2)g}{m_1 + m_2} = \frac{(4.0 - 2.0) \text{ kg} (9.8 \text{ m/s}^2)}{(4.0 + 2.0) \text{ kg}} \approx 3.27 \text{ m/s}^2$$

With the same masses, finite pulley inertia reduces acceleration in this ideal model.

Finite inertia alone does not prove unequal tensions. For an ideal axle, $T_1 - T_2 = I\alpha/R$; if $\alpha = 0$, equal tensions are possible. With axle friction, include its torque before drawing a conclusion.

7

Exam tips

Rotation: retrieval

Why can angular acceleration oppose angular velocity?

Why is the moment arm not always the radius?

What must be stated before integrating for inertia?

Which condition allows the pulley equation $a = R\alpha$?

Rotation: retrieval checks

Angular signs	Opposite signs reduce angular speed until rest; later motion needs its own model.
Moment arm	It is perpendicular distance to the force's line of action.
Inertia integral	State the axis, perpendicular distance, mass element/density and limits.
Pulley constraint	The taut rope does not slip on the pulley rim.

Check yourself

The child moves farther toward the rope end of the supported beam. Predict how rope tension and pivot force change; justify with separate torque and force balances.

In the pulley experiment, replace the pulley by a heavier one of the same radius, increasing its inertia while leaving the hanging masses unchanged. Predict the acceleration and explain why tensions cannot simply be copied from the first run.