Course Companion

AP Physics C: Mechanics

Every topic handout in one volume

全部专题讲义合集

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Kinematics

AP Physics C: Mechanics

Scalars and Vectors

In 1999 NASA lost the Mars Climate Orbiter, a 125-million-dollar spacecraft, because one team gave thruster data in pound-seconds and another read it as newton-seconds. The probe's calculated path was off by about 170 km, and it burned up in the Martian atmosphere. Kinematics – the careful description of where something is and how that changes – sounds simple, but every later unit in this course is built on getting it exactly right: the right quantities, the right directions, and the right units.

Kinematics 运动学 describes motion without asking what causes it. Two kinds of quantity:

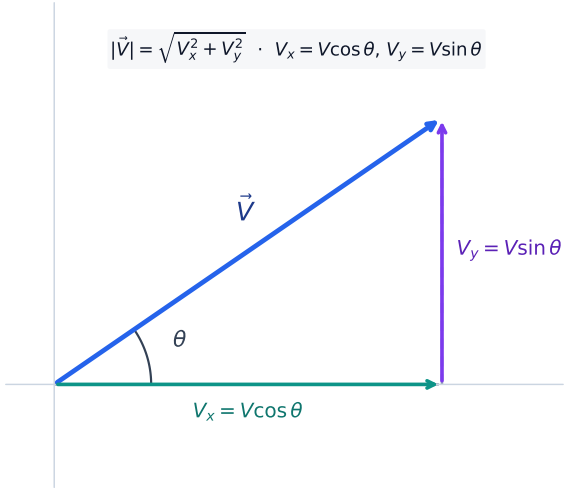
- A **scalar** 标量 has only size (**magnitude** 大小): distance, speed, mass, time, energy.
- A **vector** 矢量 has magnitude **and** direction: position, **displacement** 位移, **velocity** 速度, **acceleration** 加速度, force.

A vector is drawn as an arrow whose length is proportional to its magnitude. In one dimension, direction is just a **sign**: choose a positive direction, and anything pointing the other way is negative. So a velocity of -3 m/s means 3 m/s in the negative direction – not "less than zero speed".

In this calculus-based course you **resolve** 分解 vectors into **components** 分量 and add them component by component. Physics C writes vectors with **unit vectors** 单位矢量 $\hat{\imath}$, $\hat{\jmath}$ and $\hat{k}$, which have length one and point along x, y and z:

$$\vec{A} = A_x\hat{\imath} + A_y\hat{\jmath} + A_z\hat{k}, \qquad |\vec{A}| = \sqrt{A_x^2 + A_y^2 + A_z^2}.$$

A **resultant** vector is the sum of the components: $\vec{C} = \vec{A}+\vec{B} = (A_x+B_x)\hat{\imath}+(A_y+B_y)\hat{\jmath}$. A vector can also be given as a magnitude and a direction; convert with $A_x = A\cos\theta$, $A_y = A\sin\theta$ and $\tan\theta = A_y/A_x$. The position vector of a point is $\vec{r}$, and $\hat{r}$ is the unit vector pointing the same way.



Resolving a vector into its x and y components

Worked example (adding vectors). $\vec{A} = 3\hat{i} + 4\hat{j}$ m and $\vec{B} = 6$ m at $30°$ below the $+x$ axis. Find $\vec{A} + \vec{B}$. $\vec{B} = 6\cos 30°\,\hat{i} - 6\sin 30°\,\hat{j} = 5.2\hat{i} - 3.0\hat{j}$ m. $\vec{C} = (3 + 5.2)\hat{i} + (4 - 3.0)\hat{j} = 8.2\hat{i} + 1.0\hat{j}$ m. $|\vec{C}| = \sqrt{8.2^2 + 1.0^2} = 8.3$ m at $\tan^{-1}(1.0/8.2) = 7°$ above $+x$. Adding the magnitudes ($5 + 6 = 11$ m) would be wrong.



A car speedometer: speed is the magnitude of velocity – a scalar rate of motion

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Displacement, Velocity, and Acceleration

In the **object model**, we ignore an object's size, shape and internal parts and treat it as a single point that carries its mass (and charge). A car, a planet or a ball can all be objects when only their overall motion matters.

Displacement is the change in position, $\Delta x = x - x_0$. It depends only on the start and end points: run once round a 400 m track and your distance is 400 m but your displacement is zero.

Average values use only the start and end of a time interval:

$$\vec{v}_{\text{avg}} = \frac{\Delta\vec{x}}{\Delta t}, \qquad \vec{a}_{\text{avg}} = \frac{\Delta\vec{v}}{\Delta t}.$$

An object is **accelerating** whenever its velocity changes in **magnitude, direction, or both** – a car going round a bend at a steady 50 km/h is accelerating.

As the time interval shrinks toward zero, the average value approaches the value at one instant. The **instantaneous** velocity and acceleration are therefore **derivatives** 导数:

$$v = \frac{dx}{dt}, \qquad a = \frac{dv}{dt} = \frac{d^2x}{dt^2}.$$

Reversing this, **integrate** 积分 to recover velocity and **position** 位置:

$$v(t) = v_0 + \int_0^t a\,dt, \qquad x(t) = x_0 + \int_0^t v\,dt.$$

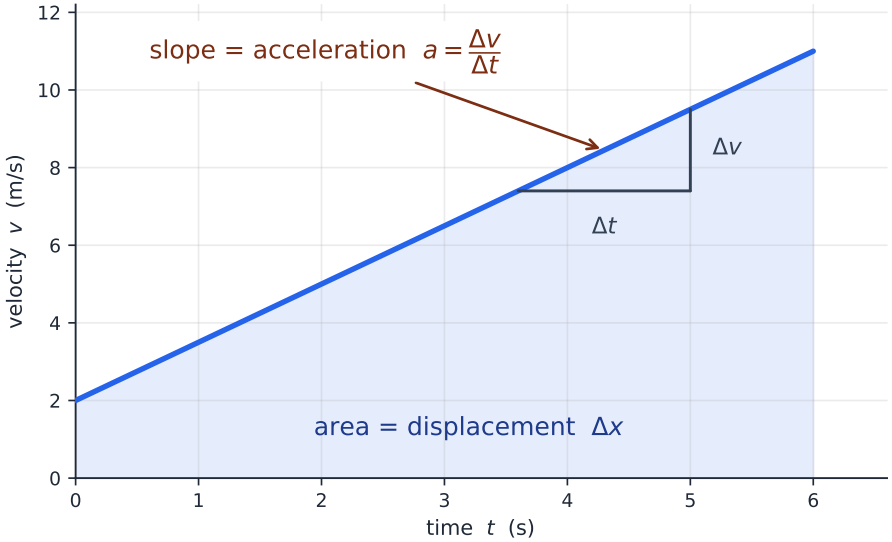
So **average velocity** 平均速度 is $\Delta x/\Delta t$ over an interval, while **instantaneous velocity** 瞬时速度 is the derivative at one moment – the **slope** 斜率 of the **tangent line** 切线 on an x–t graph, and what a **speedometer** 速度计 shows (as a magnitude).

Worked example (differentiating). A particle moves with $x(t) = 2t^3 - 3t^2$ (metres). Differentiate for velocity and acceleration:

$$v = \frac{dx}{dt} = 6t^2 - 6t, \qquad a = \frac{dv}{dt} = 12t - 6.$$

At $t = 2$ s, $v = 24 - 12 = 12$ m/s and $a = 24 - 6 = 18$ m/s^2. The particle is momentarily at rest when $v = 0$: $6t(t-1) = 0$, so at $t = 0$ and $t = 1$ s. Between them $v < 0$, so it moves backwards. Its *average* velocity from $t = 0$ to $t = 2$ s is $\dfrac{x(2) - x(0)}{2} = \dfrac{4 - 0}{2} = 2$ m/s – very different from the instantaneous 12 m/s at the end. This calculus link – not just the constant-a formulas – is what distinguishes Physics C.

Worked example (integrating). A particle starts from rest at the origin with $a(t) = 6t$. Integrate: $v = \int_0^t 6t\,dt = 3t^2$ and $x = \int_0^t 3t^2\,dt = t^3$. At $t = 2$ s, $v = 12$ m/s and $x = 8$ m. If instead it had started at $x_0 = 5$ m with $v_0 = -4$ m/s, then $v = 3t^2 - 4$ and $x = 5 - 4t + t^3$: the initial conditions are the constants of integration.



On a velocity-time graph the gradient is the acceleration and the area is the displacement

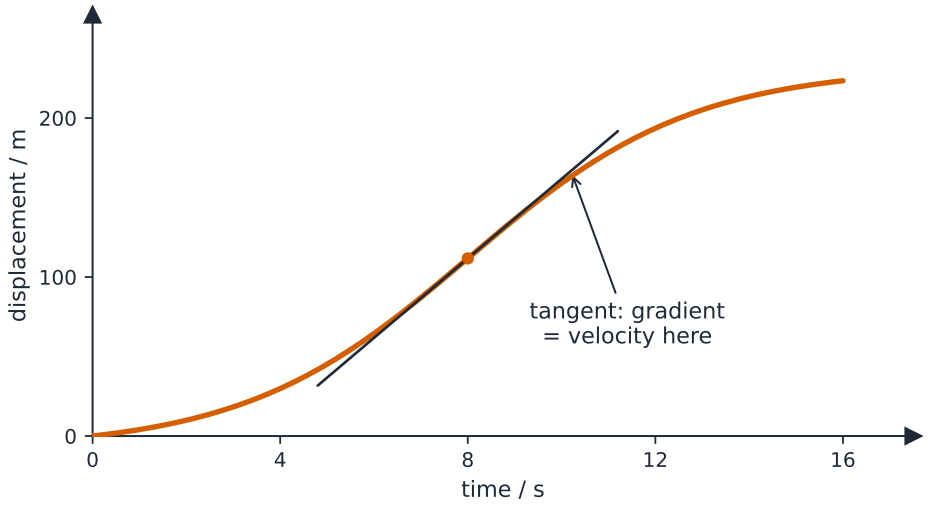


A high-speed train: kinematics describes displacement, velocity and acceleration over time

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Representing Motion

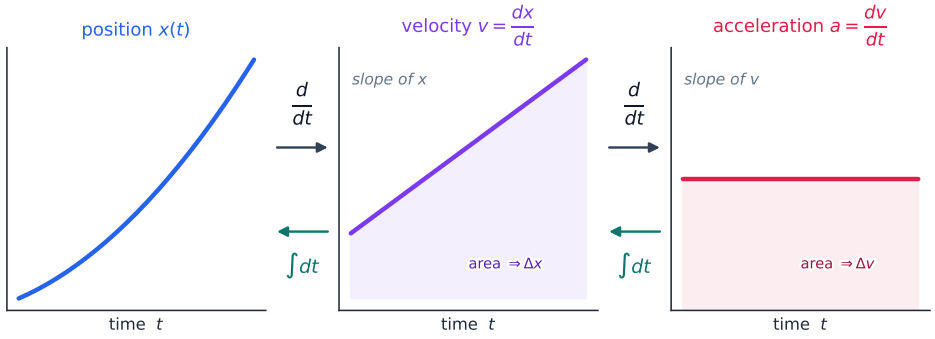
Motion can be shown with motion diagrams, graphs, equations, tables and words; free-response questions ask you to move between them.



A displacement-time graph: the slope at any instant is the velocity

- On a **position–time** graph, the slope of the tangent (dx/dt) is the instantaneous velocity.
- On a **velocity–time** graph, the slope is acceleration and the **area** under the curve is displacement: $\Delta x = \int_{t_1}^{t_2} v\,dt$. Area below the time axis counts as negative displacement.
- On an **acceleration–time** graph, the area is the change in velocity: $\Delta v = \int_{t_1}^{t_2} a\,dt$.

Slopes are derivatives; areas are integrals – the two are **inverse operations** 逆运算 here.



Position, velocity, and acceleration are linked by differentiation

For **constant** acceleration, integrating a twice gives the three **kinematic equations** 运动学方程, which work in any single direction:

$$v_x = v_{x0} + a_x t, \qquad x = x_0 + v_{x0}t + \tfrac{1}{2}a_x t^2, \qquad v_x^2 = v_{x0}^2 + 2a_x(x - x_0).$$

Choose the equation that leaves out the quantity you neither know nor need. Use them **only** when a is constant; if a depends on time, integrate.

The most important constant-a case is **free fall** 自由落体: near Earth's surface every object, heavy or light, accelerates downward at g once **air resistance** 空气阻力 is **negligible** 可忽略的. **On the AP exam use** $g \approx 10$ **m/s**^{2} whenever a number is needed (9.8 is also accepted).

> **Worked example (area under a graph).** A cart starts at rest. Its acceleration is $a = 2.0$ m/s^2 for 4.0 s, then $a = -1.0$ m/s^2 for 6.0 s. Area under the a–t graph: $\Delta v = 2.0(4.0) + (-1.0)(6.0) = 8.0 - 6.0 = 2.0$ m/s, so the final velocity is 2.0 m/s. For displacement, use the v–t graph: v rises from 0 to 8.0 m/s, then falls to 2.0 m/s. Area $= \frac{1}{2}(4.0)(8.0) + \frac{1}{2}(8.0 + 2.0)(6.0) = 16 + 30 = 46$ m.

> **Worked example (free fall).** A stone is thrown straight up at 15 m/s from the edge of a cliff 20 m high and lands at the base. Take up as positive, $a = -10$ m/s^2, and $\Delta y = -20$ m.

$$-20 = 15t - 5t^2 \;\Rightarrow\; t^2 - 3t - 4 = 0 \;\Rightarrow\; (t - 4)(t + 1) = 0,$$

so $t = 4.0$ s (the negative root is not physical). Landing speed: $v^2 = 15^2 + 2(-10)(-20) = 625$, so $v = -25$ m/s – 25 m/s downward.

> **Worked example (when a is not constant).** Students drop a coffee filter and fit its speed to $v = A\left(1 - e^{-Bt}\right)$, with $A = 2.0$ m/s and $B = 5.0$ s^{-1}.

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1. Acceleration: $a = \dfrac{dv}{dt} = ABe^{-Bt}$. At $t = 0$, $a = AB = 10$ m/s^2: at the start it is in free fall. At $t = 0.20$ s, $a = 10e^{-1} \approx 3.7$ m/s^2.
2. As $t \to \infty$, $e^{-Bt} \to 0$, so $v \to A = 2.0$ m/s: A is the **terminal speed**, and the acceleration falls to zero.
3. Position (starting at $x = 0$): $x = \displaystyle\int_0^t A\left(1 - e^{-Bt'}\right) dt' = At - \frac{A}{B}\left(1 - e^{-Bt}\right)$.

The kinematic equations would give wrong answers here, because a changes. If you only have a v–t **graph**, find a at an instant by drawing the tangent there and taking its slope, using two points far apart on the tangent line.

Measuring g with a straight-line graph. A ball is dropped from rest from several heights h, and the fall time t is measured. Since $h = \frac{1}{2}gt^2$, a graph of h against t^2 is a **straight line through the origin** with slope $\frac{1}{2}g$.

h (m)	0.40	0.80	1.20	1.60
t (s)	0.286	0.404	0.495	0.571
t^2 (s^2)	0.082	0.163	0.245	0.326

1. Plot h (vertical) against t^2 (horizontal) and draw the **best-fit line**. Do not join the dots.
2. Slope from two points **on the line**, far apart: $\dfrac{1.60 - 0.40}{0.326 - 0.082} \approx 4.92$ m/s^2.
3. $g = 2 \times$ slope ≈ 9.84 m/s^2.
4. Percent error $= \dfrac{|\text{experimental} - \text{accepted}|}{\text{accepted}} \times 100 = \dfrac{|9.84 - 9.81|}{9.81} \times 100 \approx 0.3\%$.

Choose the axes so the expected relation becomes $y = mx$ (or $y = mx + c$); then say which physical quantity the slope stands for. To reduce uncertainty, take several readings at each height and average them, and use a wide range of heights.

Reference Frames and Relative Motion

All motion is measured relative to a **reference frame** 参考系 – a point of view with its own origin and axes. The frame you choose decides the direction and magnitude of the velocities you measure: a passenger walking forward at 1 m/s in a train moving at 30 m/s moves at 31 m/s relative to the ground.

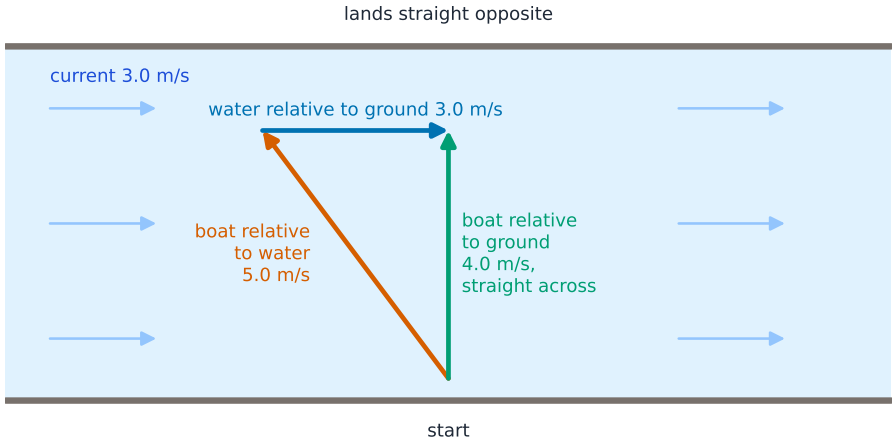
Measurements in one frame are converted to another by **vector addition** 矢量相加 or subtraction:

$$\vec{v}_{A/C} = \vec{v}_{A/B} + \vec{v}_{B/C}.$$

Read the subscripts as a chain: "A relative to B" plus "B relative to C" gives "A relative to C". Also, $\vec{v}_{B/A} = -\vec{v}_{A/B}$. This handles boats crossing rivers, planes in wind and passengers on moving vehicles – **relative motion** 相对运动 problems.

An **inertial reference frame** is one that is not accelerating. Observers in different inertial frames measure **different velocities but the same acceleration** for an object, because a constant frame velocity disappears when you differentiate. Unless told otherwise, assume your frame is inertial.

> **Worked example (heading straight across).** A boat moves at 5.0 m/s relative to the water. The river flows at 3.0 m/s and is 80 m wide. The pilot wants to land directly opposite. The boat must point partly upstream so that the upstream component of its velocity cancels the current: $5.0\sin\theta = 3.0$, so $\theta = 37°$ upstream of straight across. Its across-river speed is $5.0\cos 37° = 4.0$ m/s, and the crossing takes $\dfrac{80}{4.0} = 20$ s. If instead it pointed straight across, it would cross in $\dfrac{80}{5.0} = 16$ s but land $3.0 \times 16 = 48$ m downstream.



To land straight across, the boat heads upstream: its velocity relative to the water plus the water's velocity relative to the ground points straight across

Motion in Two or Three Dimensions

In two or three dimensions, position is a vector $\vec{r}(t) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}$, and velocity and acceleration are its successive derivatives. Motion is analysed by splitting it into components and treating each with one-dimensional kinematics **independently** 独立地:

- the velocity and acceleration may be different in each direction, and need not be constant;

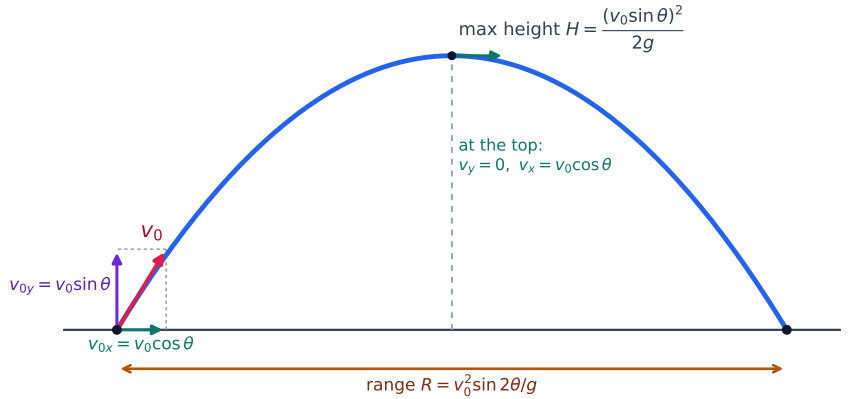
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- a change in one direction does not change the motion in a perpendicular direction – a bullet dropped and a bullet fired horizontally from the same height hit the ground at the same time.

> **Worked example (calculus in 2D).** $\vec{r}(t) = \left(2t^2\right)\hat{i} + \left(4t - t^3\right)\hat{j}$ (metres). Differentiating each component: $\vec{v} = 4t\,\hat{i} + \left(4 - 3t^2\right)\hat{j}$ and $\vec{a} = 4\,\hat{i} - 6t\,\hat{j}$. At $t = 1$ s: $\vec{v} = 4\hat{i} + 1\hat{j}$, so the speed is $\sqrt{17} \approx 4.1$ m/s – no new physics, just one derivative per component.

For ideal **projectile motion** 抛体运动 under uniform gravity with air resistance neglected, the acceleration is zero horizontally and constant vertically: $a_x = 0$ (constant horizontal velocity) and $a_y = -g$. The two motions are linked only by the time t, and the path is a **parabola** 抛物线.

> **Worked example (launch on level ground).** A ball is launched at 20 m/s, 30° above the horizontal ($g = 10$ m/s^2). The components are $v_{0x} = 20\cos 30° = 17.3$ m/s and $v_{0y} = 20\sin 30° = 10$ m/s. The vertical motion sets the time: total flight $= \dfrac{2v_{0y}}{g} = \dfrac{20}{10} = 2.0$ s, maximum height $= \dfrac{v_{0y}^2}{2g} = 5.0$ m, and range $= v_{0x} \times 2.0 = 35$ m.



A projectile launched at an angle: velocity components, maximum height, and range

> **Worked example (launch from a height).** A ball rolls off a table 1.25 m high at 3.0 m/s. How far from the table does it land, and how fast is it moving? Vertical: $1.25 = \frac{1}{2}(10)t^2$, so $t = 0.50$ s. Horizontal: $x = 3.0 \times 0.50 = 1.5$ m. At landing $v_x = 3.0$ m/s and $v_y = -10 \times 0.50 = -5.0$ m/s, so $v = \sqrt{3.0^2 + 5.0^2} = 5.8$ m/s at $\tan^{-1}(5.0/3.0) = 59°$ below the horizontal.

Practice

Answer each before reading the answers below.

1. A particle moves with $x(t) = 2t^3 - 6t^2 + 4$ (m, s). Find $v(t)$ and $a(t)$, the times when it is at rest, and its acceleration at the later of those times.
2. A particle has $a(t) = 6t$ m/s^2, $v(0) = 2$ m/s and $x(0) = 0$. Find v and x at $t = 2$ s.
3. A ball is thrown horizontally at 15 m/s from a cliff 20 m high. Find the time of flight and the horizontal distance.
4. A boat heads due north at 4 m/s relative to the water across a river 80 m wide that flows east at 3 m/s. Find its speed relative to the ground, the crossing time and how far downstream it lands.
5. A cart's velocity rises steadily from 0 to 10 m/s in 5 s, then stays constant for 5 s. Find its displacement.

Answers

1. $v = 6t^2 - 12t$, $a = 12t - 12$. At rest when $6t(t-2) = 0$: $t = 0$ and $t = 2$ s. $a(2) = 12$ m/s^2.
2. $v = 3t^2 + 2$, so $v(2) = 14$ m/s. $x = t^3 + 2t$, so $x(2) = 12$ m.
3. $t = \sqrt{2(20)/9.8} = 2.02$ s; $x = 15 \times 2.02 = 30.3$ m.
4. $\sqrt{4^2 + 3^2} = 5$ m/s; $t = 80 \div 4 = 20$ s; drift $= 3 \times 20 = 60$ m.
5. Area under the v-t graph: $\frac{1}{2}(5)(10) + 5(10) = 75$ m.

Exam tips

- Resolve every vector into components before adding – never add magnitudes at an angle directly.
- On Physics C you are expected to use **calculus**: velocity is $\vec{v} = \frac{d\vec{r}}{dt}$ and acceleration $\vec{a} = \frac{d\vec{v}}{dt}$; reverse with integration and include the initial conditions.
- Use $g \approx 10$ m/s^2 unless told otherwise.
- Carry and check **units** and treat direction with signs (choose a positive **axis** 坐标轴 and stick to it).
- Read graphs both ways: **slope** gives the next derivative, **area** gives the change in the previous quantity.
- In relative-motion problems, write the subscript chain $\vec{v}_{A/C} = \vec{v}_{A/B} + \vec{v}_{B/C}$ before substituting.
- Use the **dot product** 点积 for work-type quantities and the **cross product** 叉积 for **torque** 力矩 and **angular momentum** 角动量 (later units).
- Sketch the vectors – a diagram catches sign and direction errors the algebra hides.

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Common mistakes

- Dropping the constant of integration. $v(t) = v_0 + \int_0^t a(t')\,dt'$ and $x(t) = x_0 + \int_0^t v(t')\,dt'$: the initial conditions belong in the answer.
- Finding average velocity from total distance. Average velocity is displacement over time; distance gives average *speed*.
- Saying a projectile has zero velocity at the top. Only v_y is zero; v_x is unchanged throughout.
- Using the constant-acceleration equations when a depends on time. Differentiate or integrate instead.
- Adding relative velocities as numbers. $\vec{v}_{A/G} = \vec{v}_{A/B} + \vec{v}_{B/G}$ is a vector sum; draw the triangle.
- Thinking zero velocity means zero acceleration. At the top of a vertical throw $v = 0$ but $a = -g$.

Force and Translational Dynamics

AP Physics C: Mechanics

Systems and Center of Mass

On 2 August 1971, astronaut David Scott stood on the Moon and dropped a hammer and a falcon feather at the same moment. With no air to slow the feather, both hit the dusty ground together – exactly as Galileo had predicted nearly 400 years earlier. That one test captures this unit: the same few force laws, applied with care through Newton's laws, explain falling feathers, cars on curves, skydivers and orbiting moons.

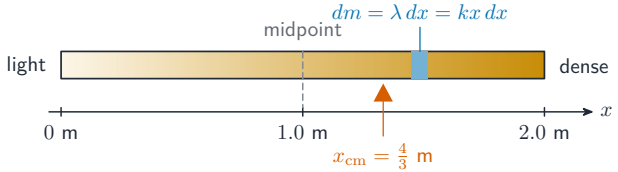
A **system** 系统 is the object or objects you analyze, treated as a point at its **center of mass** 质心. For a set of masses, $x_{\text{cm}} = \dfrac{\sum m_i x_i}{\sum m_i}$; for a continuous body, $x_{\text{cm}} = \dfrac{1}{M}\int x\,dm$. Only a net **external** force accelerates the center of mass of a fixed-mass system.

Choosing the system is your decision, and it changes the analysis. If the parts' internal details do not matter, treat the whole system as one object at its center of mass. If they do – two blocks joined by a string, a person in an elevator – you can look inside it, and the parts may move differently from each other and from the system as a whole. A system can also exchange energy or mass with its surroundings (a rocket losing fuel, a car losing heat). For any body with a **symmetrical mass distribution**, the center of mass lies on every line of symmetry – the center of a uniform disk or sphere, the midpoint of a uniform rod.

> **Worked example (a non-uniform rod).** A thin rod of length $L = 2.0$ m lies along the x-axis from $x = 0$. Its linear mass density increases along it, $\lambda = kx$ with $k = 3.0$ kg/m^2. Find its mass and its center of mass.

- A slice of width dx at position x has mass $dm = \lambda\,dx = kx\,dx$.
- Mass: $M = \displaystyle\int_0^L kx\,dx = \frac{kL^2}{2} = \frac{(3.0)(2.0)^2}{2} = 6.0$ kg.
- Center of mass: $x_{\text{cm}} = \dfrac{1}{M}\displaystyle\int_0^L x\,(kx\,dx) = \frac{1}{M}\cdot\frac{kL^3}{3} = \frac{(3.0)(2.0)^3/3}{6.0} = 1.33$ m, which is $\frac{2}{3}L$.

Check the answer against the physics: a uniform rod would balance at its midpoint, 1.0 m. More mass sits toward the right end, so the center of mass must be to the right of the midpoint.

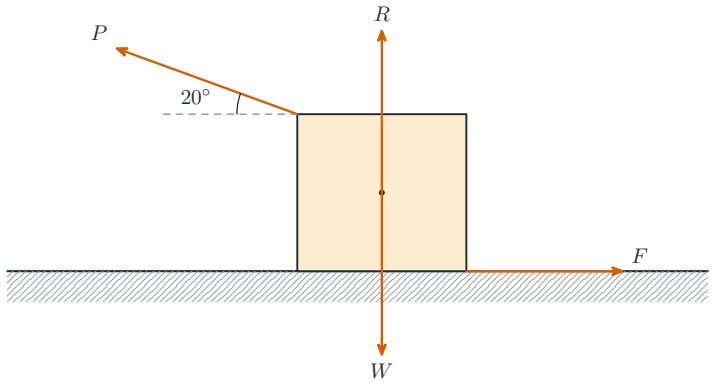


The rod gets denser to the right, so its center of mass sits at 2L/3, beyond the midpoint

Forces and Free-Body Diagrams

A **force** 力 is a push or pull (a vector, in newtons). A **free-body diagram** 受力图 draws one object with an arrow for **every** force on it – weight, normal, tension, friction, applied, drag. Draw it first; it sets up every dynamics equation.

- Draw the object as a **dot** representing its center of mass, with every force arrow starting from that dot.
- Show only forces exerted **on** the object **by** its environment – never forces the object exerts on other things, and never "ma" or "centripetal force".
- **Contact forces** (normal force, friction, tension) are the large-scale result of the electric forces between atoms at the surfaces in contact.
- Choose one axis **parallel to the acceleration** (along an incline, or toward the centre of a circle). Then only one equation contains a.



A free-body diagram shows every force acting on one object

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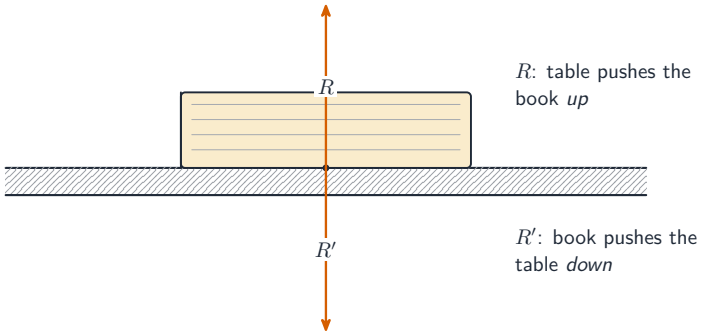
Newton's Third Law

Newton's third law 牛顿第三定律: A pushes B, B pushes back equally and oppositely. The pair acts on **different** objects, so it never cancels within one free-body diagram:

$$\vec{F}_{\text{A on B}} = -\vec{F}_{\text{B on A}}.$$

The two forces are always the same type (both gravitational, both contact) and act at the same moment. When a truck hits a fly, the force on the fly is exactly as large as the force on the truck; the fly suffers more because its mass is tiny. Inside a system, third-law pairs are **internal forces**: they cancel in pairs, so they cannot change the motion of the system's center of mass.

Tension in a string or cable is the net result of each tiny segment of the string pulling on its neighbours. For an ideal (massless) string over a frictionless pulley, the tension is the same all the way along.



A Newton's third-law pair: equal and opposite forces on two different objects

Newton's First Law

Newton's first law (**inertia** 惯性): with zero **net force** 合力, velocity stays constant – the object is in **translational equilibrium** 平动平衡. The net force is the **vector sum** of all forces on the system, so equilibrium means $\sum F_x = 0$ and $\sum F_y = 0$ separately.

Forces can be balanced in one direction but not another: a hockey puck sliding on frictionless ice has balanced vertical forces (weight and normal force), so its vertical velocity stays zero, while any unbalanced horizontal force changes only its horizontal velocity. An **inertial reference frame** is exactly a frame in which Newton's first law is observed to hold – one that is not accelerating.

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Worked example (equilibrium). A 4.0 kg lamp hangs from two cords, each at 30° above the horizontal. Vertical: $2T\sin 30° = mg = 40$ N, so $T = 40$ N. Each cord carries a tension equal to the lamp's whole weight – shallow cords need large tensions.

Newton's Second Law

The general form uses momentum:

$$\sum \vec{F} = \frac{d\vec{p}}{dt} = m\vec{a} \quad \text{(for constant mass)}.$$

Apply it one axis at a time. When forces depend on velocity or position, this becomes a **differential equation** 微分方程 to solve.

Worked example. A 2.0 kg block slides down a frictionless **incline** 斜面 at 30°. Only the along-ramp component of gravity drives it, so $a = g\sin 30° = 9.8(0.5) = 4.9$ m/s^2, independent of the mass.

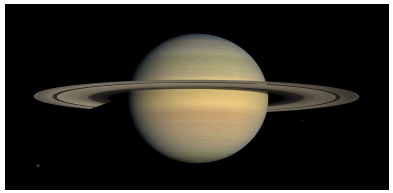
Worked example (two bodies). A 6.0 kg cart on a frictionless table is pulled by a string over a light pulley to a hanging 2.0 kg mass. Treat the pair as one system: only the hanging weight drives it, so $a = \dfrac{2.0(9.8)}{8.0} = 2.45$ m/s^2. Then isolate the cart alone to find the string **tension** 张力: $T = 6.0(2.45) \approx 15$ N. System first for a, single body for internal forces – that two-step is the standard pattern.



A rocket launch: net force changes momentum – F_net = ma for translational dynamics

Worked example (elevator and apparent weight). A 60 kg student stands on a scale in an elevator that accelerates upward at 2.0 m/s^2. Taking up as positive, $N - mg = ma$, so $N = 60(10 + 2.0) = 720$ N. The scale reads the normal force, so the student's **apparent weight** is 720 N, though the gravitational force is still 600 N. Accelerating downward at 2.0 m/s^2 gives $N = 480$ N; in free fall, $N = 0$.

Gravitational Force



Saturn, its rings and moons: every particle is held in orbit by the same inverse-square gravitational force

Near a surface, weight is $F_g = mg$. In general, **Newton's law of gravitation** 万有引力定律:

$$F_g = \frac{Gm_1m_2}{r^2},$$

attractive and inverse-square, for point masses or nonoverlapping spherically symmetric bodies, where r is the distance between their centres. The gravitational field is $g = \frac{GM}{r^2}$: a **field** describes the force per kilogram that a mass would feel at each point in space. The gravitational force from a planet on a small nearby object is its **weight**. Near Earth's surface the field hardly changes with height, so it is treated as constant: $g \approx 10$ N/kg (the same as 10 m/s^2).

Apparent weight is the magnitude of the **normal force** on a system (what a bathroom scale measures). When the system accelerates, apparent weight differs from mg (see the elevator example). An object appears **weightless** when gravity is the only force on it, as for astronauts in an orbiting space station – they are in free fall around Earth, not beyond its gravity. The **equivalence principle** says that an observer inside a closed accelerating room cannot tell the difference between that acceleration and a gravitational field.

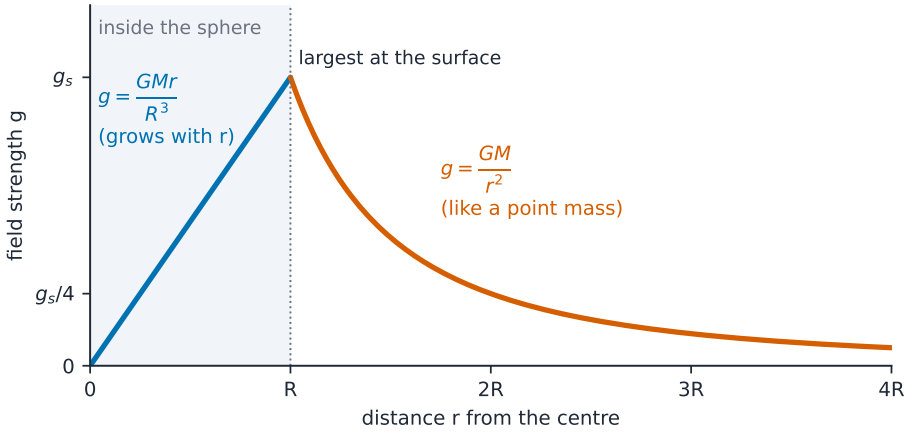
Mass appears in two roles. **Inertial mass** measures how strongly an object resists changes in its motion (m in $F = ma$); **gravitational mass** measures how strongly it attracts and is attracted by other masses (m in Gm_1m_2/r^2). Experiments show the two are **equal** to better than one part in 10^{14} – which is why all objects fall with the same acceleration, as the hammer and feather showed.

Spheres and the shell theorem. The force from a large body is the sum of the forces from all its small pieces dm. For a uniform spherical shell, Newton showed that:

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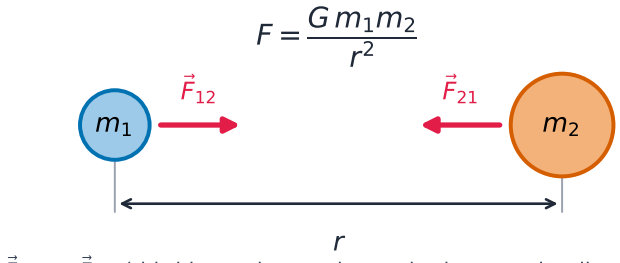
- outside the shell, it attracts as if all its mass were at its centre;
- inside the shell, the forces from all its parts cancel – the net force is **zero**.

So outside a uniform solid sphere, $g = GM/r^2$. Inside it, at radius r, only the mass closer to the centre pulls; that mass is $M\frac{r^3}{R^3}$, so $g = \frac{GMr}{R^3}$ – **proportional to** r, and zero at the centre.



Field strength of a uniform sphere: inside it grows in proportion to r; outside it falls as 1/r^2, like a point mass; it is largest at the surface

Worked example (field at altitude). At what height above Earth's surface ($R = 6.4 \times 10^6$ m) is g one quarter of its surface value? $g \propto 1/r^2$, so $r = 2R$ and the height is $R = 6.4 \times 10^6$ m. Halfway to the centre of a uniform-density sphere, by contrast, g is one half of its surface value.



$\vec{F}_{12} = -\vec{F}_{21}$ (third-law pair; equal magnitude, opposite direction)

$F = Gm_1m_2/r^2$ · $g = GM/r^2$ outside sphere

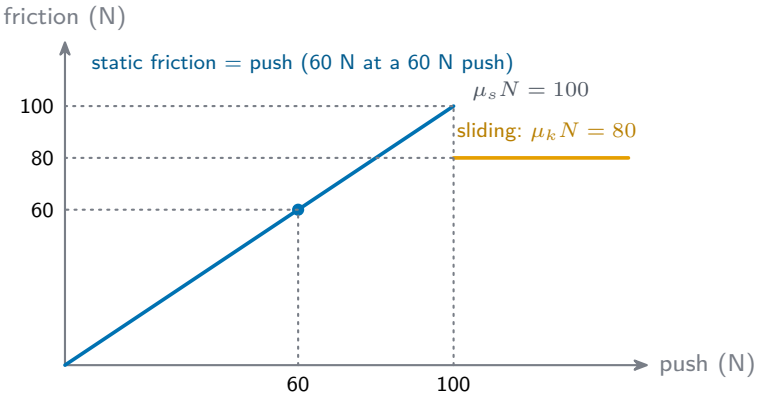
Two masses attract each other with equal, opposite, inverse-square forces along the line joining them

Kinetic and Static Friction

Friction 摩擦力 opposes sliding along a surface: **kinetic** 动摩擦 $f_k = \mu_k N$ while sliding, and **static** 静摩擦 $f_s \le \mu_s N$ up to a maximum before sliding. N is the **normal force** 法向力. Note the inequality: static friction takes whatever value and direction are needed to stop the surfaces sliding, and $\mu_s N$ is its *ceiling*, not its value. For a given pair of surfaces μ_s is usually **larger** than μ_k, which is why a heavy box is harder to start moving than to keep moving.

Worked example (static friction). A 20 kg crate rests on a floor with $\mu_s = 0.50$ and $\mu_k = 0.40$. A worker pushes horizontally with 60 N. The maximum static friction is $0.50(200) = 100$ N, more than 60 N, so the crate stays put and static friction is exactly **60 N**, not 100 N. The crate starts moving only above 100 N; once it slides, friction drops to $0.40(200) = 80$ N.

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Static friction matches the push until it reaches its ceiling of 100 N; after that the crate slides and friction drops to 80 N

Worked example. The same block on the same 30° incline, now with $\mu_k = 0.20$. Perpendicular to the ramp: $N = mg\cos 30°$. Along the ramp: $ma = mg\sin 30° - \mu_k mg\cos 30°$, so $a = g(\sin 30° - 0.20\cos 30°) = 9.8(0.500 - 0.173) = 3.2$ m/s^2 – the mass cancels again.

Worked example (two blocks at rest). Block 1 (mass $M = 2.0$ kg) sits on a table, joined by a string over a light, frictionless pulley to block 2 (mass $m = 0.60$ kg) hanging over the edge. What is the smallest μ_s that keeps them at rest?

1. At rest, the tension equals the hanging weight: $T = mg$.
2. Block 1 stays put if static friction can reach T: $\mu_s Mg \ge mg$.
3. So $\mu_s \ge \dfrac{m}{M} = \dfrac{0.60}{2.0} = 0.30$. The answer does not depend on g.

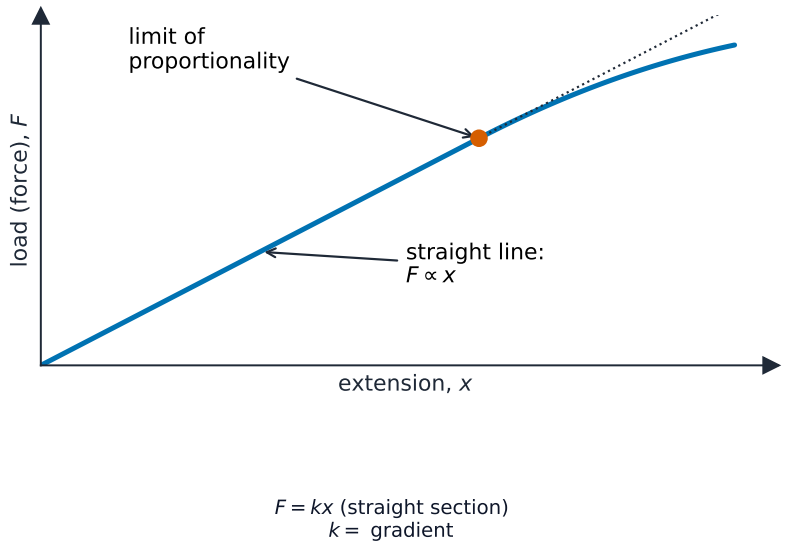
Finding μ_k from a graph. In the same set-up, students keep the **total** mass M_{tot} fixed but move small masses from block 1 to the hanger, so the hanging mass m changes. Newton's second law for the whole system:

$$M_{\text{tot}}\,a = mg - \mu_k\,(M_{\text{tot}} - m)\,g \;\Rightarrow\; a = \frac{(1+\mu_k)g}{M_{\text{tot}}}\,m - \mu_k g.$$

A graph of a against m is a straight line. Its vertical **intercept** is $-\mu_k g$, so $\mu_k = -\dfrac{\text{intercept}}{g}$. If an intercept of -2.0 m/s^2 is read off, $\mu_k \approx 0.20$. If the pulley actually has friction, the measured a values are too small, so this method gives a μ_k that is **too large**.

Spring Forces

An **ideal spring** has negligible mass and exerts a force proportional to the change in its length from its relaxed length. (A real spring may have significant mass, or stop obeying the proportional rule when stretched too far.) It obeys **Hooke's law** 胡克定律 $\vec{F}_s = -k\Delta\vec{x}$, always directed toward the spring's relaxed length – a restoring force set by the **spring constant** 弹簧劲度系数 k. Its stored energy is $U = \frac{1}{2}kx^2$.



Hooke's law: extension is proportional to load up to the limit of proportionality

Combined springs act as one **equivalent spring** of constant k_{eq}. In **parallel** (side by side, sharing the load) the constants add, $k_{\text{eq}} = k_1 + k_2$ – stiffer than either. In **series** (end to end) they combine as $\dfrac{1}{k_{\text{eq}}} = \dfrac{1}{k_1} + \dfrac{1}{k_2}$ – softer than the softest one. Two springs pulling a mass from opposite sides also act in parallel, giving a restoring constant $k_1 + k_2$. Whichever you have, the SHM period uses k_{eq}: $T = 2\pi\sqrt{m/k_{\text{eq}}}$.

Resistive Forces

A **resistive force** 阻力 (drag) opposes motion through a fluid and **grows with speed**, often modeled as $F = -bv$ or $F = -cv^2$. Newton's second law then gives a differential equation, e.g. $m\dfrac{dv}{dt} = mg - bv$ for a falling object. As speed rises, drag builds until it balances the driving force; the object then stops accelerating and moves at a constant **terminal velocity** 终极速度, found by setting the net force (and dv/dt) to zero.

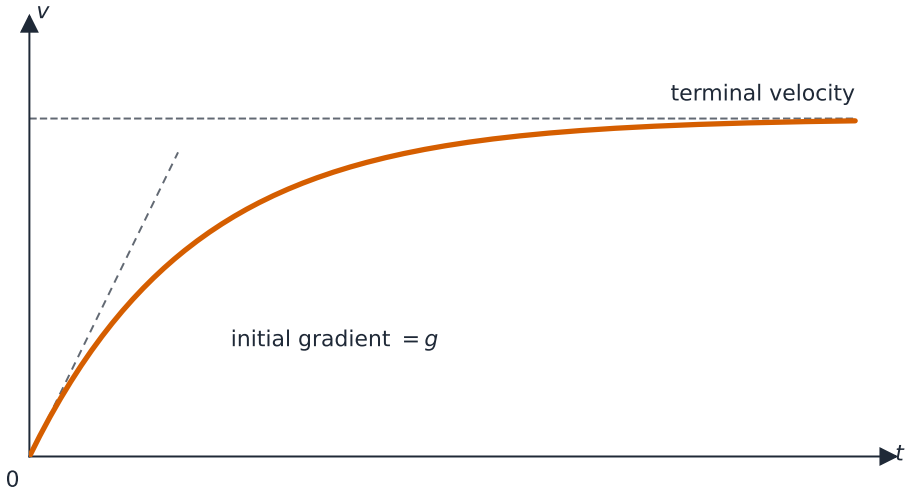
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Worked example. For a falling object with $m\dfrac{dv}{dt} = mg - bv$, the terminal velocity is where $\dfrac{dv}{dt} = 0$: $mg = bv_T$, so $v_T = \dfrac{mg}{b}$. With $m = 0.10$ kg and $b = 0.50$ kg/s, $v_T = \dfrac{0.10(9.8)}{0.50} = 2.0$ m/s.

To get the full motion $v(t)$, **separate the variables** 分离变量 and integrate from rest:

$$\int_0^v \frac{dv'}{mg - bv'} = \int_0^t \frac{dt'}{m} \;\Rightarrow\; -\frac{1}{b}\ln\frac{mg - bv}{mg} = \frac{t}{m},$$

which rearranges to $v(t) = v_T\left(1 - e^{-bt/m}\right)$ – the speed rising exponentially toward v_T. Showing this integration is the calculus skill a Physics C free-response rewards, not just quoting the final formula.



An object falling through a fluid speeds up to a terminal velocity



Skydivers in freefall: drag grows with speed until it balances weight at terminal velocity

Worked example (a translation-between-representations question). The second free-response question, *Translation Between Representations*, asks for one situation as a diagram, an equation and a graph, and then for the links between them. A block of mass m is pushed so that it slides at speed v_0 across a frictionless horizontal surface, then released at $t = 0$. Air resistance $F = -bv$ is the only horizontal force.

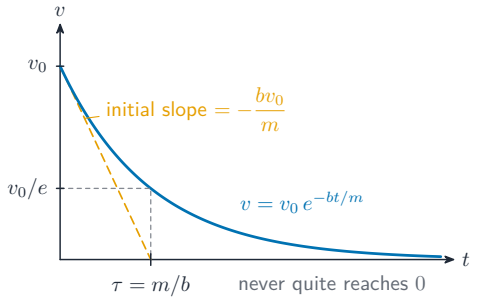
(a) *Draw a free-body diagram for the block after release.* Three arrows from a dot: the weight mg down, the normal force F_N up (equal length), and the drag bv pointing **opposite** to the velocity. There is no forward force: once released, nothing pushes the block.

(b) *Derive an expression for $v(t)$.* Newton's second law along the motion:

$$m\frac{dv}{dt} = -bv \;\Rightarrow\; \int_{v_0}^{v}\frac{dv'}{v'} = -\frac{b}{m}\int_0^t dt' \;\Rightarrow\; \ln\frac{v}{v_0} = -\frac{bt}{m} \;\Rightarrow\; v = v_0\,e^{-bt/m}.$$

(c) *Sketch v against t.* The curve starts at v_0, falls steeply at first and then more and more slowly, approaching zero but never reaching it.

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Drag alone slows the block: the speed decays exponentially, fastest at first, and the initial tangent meets the t-axis at the time constant m/b

(d) *Explain how the shape of your graph follows from your free-body diagram.* The slope of the graph is the acceleration, $-\frac{bv}{m}$. The drag arrow is longest at the start, when v is largest, so the graph is steepest there, with slope $-\frac{bv_0}{m}$. As the block slows, the drag arrow shrinks, so the slope gets less steep: the curve flattens. Because the drag is never zero while the block moves, but becomes very small, the speed keeps falling without ever reaching zero in a finite time.

Integrating $v(t)$ from 0 to ∞ gives the total distance: the block covers only a finite distance, $\frac{mv_0}{b}$, even though its speed never quite becomes zero.

Circular Motion

Uniform circular motion has a **centripetal acceleration** 向心加速度 $a_c = \frac{v^2}{r}$ pointing to the center, requiring a **net inward force** $F_c = \frac{mv^2}{r}$ supplied by a real force (tension, gravity, friction, normal). There is no outward force. Centripetal acceleration can come from a single force, from several forces, or from components of forces. For vertical circles and banked curves, decompose the real forces to find which provides the centripetal requirement.

Worked example. A 0.50 kg ball slides on a frictionless horizontal table, attached to a 1.0 m string fixed at the circle's centre, at 3.0 m/s. The string tension supplies the whole centripetal force: $T = \frac{mv^2}{r} = \frac{0.50(3.0)^2}{1.0} = 4.5$ N.

Worked example (vertical circle). At the top of a vertical loop of radius r, gravity and the normal force *both* point toward the center: $N + mg = \dfrac{mv^2}{r}$. The slowest possible speed at the top is where the track pushes with nothing at all ($N = 0$): $v_{\min} = \sqrt{gr}$. For $r = 2.5$ m: $v_{\min} = \sqrt{9.8(2.5)} = 4.9$ m/s. Any slower and the cart leaves the track before the top.

If the speed is also changing, there is a **tangential acceleration** $a_t = \dfrac{dv}{dt}$ along the path as well. The object's total acceleration is the vector sum $\vec{a} = \vec{a}_c + \vec{a}_t$, with magnitude $\sqrt{a_c^2 + a_t^2}$.

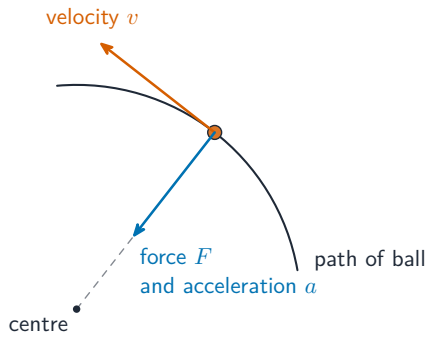
Uniform circular motion repeats, so it is described by its **period** T (time for one revolution) and **frequency** $f = 1/T$; the speed is $v = \dfrac{2\pi r}{T}$.

For a **satellite in a circular orbit**, gravity alone supplies the centripetal force:

$$\frac{GMm}{r^2} = \frac{m(2\pi r/T)^2}{r} \quad \Rightarrow \quad T^2 = \frac{4\pi^2}{GM} r^3.$$

This is **Kepler's third law**: the square of the period is proportional to the cube of the orbit radius, and the satellite's own mass cancels.

Worked example (geostationary orbit). A satellite that stays above one point on the equator needs $T = 24$ h $= 8.64 \times 10^4$ s. With $GM = 4.0 \times 10^{14}$ m^3/s^2: $r^3 = \dfrac{GMT^2}{4\pi^2} = \dfrac{4.0 \times 10^{14}(8.64 \times 10^4)^2}{39.5} = 7.6 \times 10^{22}$, so $r = 4.2 \times 10^7$ m – about 6.6 Earth radii from Earth's centre.



The velocity points along the tangent; the centripetal force points to the centre

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Practice

Answer each before reading the answers below.

1. A 2.0 kg block is pulled across a floor by a 20 N horizontal force; $\mu_k = 0.30$. Find its acceleration.
2. An Atwood machine has masses of 3.0 kg and 5.0 kg on a light, frictionless pulley. Find the acceleration and the tension.
3. An object of mass m falls from rest with a drag force $-bv$. Find the terminal speed and derive $v(t)$.
4. Find the greatest speed at which a car can round a flat curve of radius 50 m if $\mu_s = 0.80$.
5. The Earth pulls a book down with a 10 N force. What is the Newton's-third-law partner of this force?
6. A block slides down a frictionless 30° incline. Find its acceleration.

Answers

1. $f = 0.30(2.0)(9.8) = 5.9$ N; $a = (20 - 5.9) \div 2.0 = 7.1$ m/s^2.
2. $a = \dfrac{(5-3)(9.8)}{8} = 2.45$ m/s^2; $T = 3(9.8 + 2.45) = 36.8$ N.
3. Terminal speed when $mg = bv$: $v_T = mg/b$. $m\dfrac{dv}{dt} = mg - bv$; separating and integrating from $v = 0$ gives $v(t) = \dfrac{mg}{b}\left(1 - e^{-bt/m}\right)$.
4. Friction provides the centripetal force: $\mu_s mg = mv^2/r$, so $v = \sqrt{\mu_s gr} = \sqrt{0.80(9.8)(50)} = 19.8$ m/s.
5. The book pulls the Earth up with a 10 N force. (The normal force on the book is not the partner: it acts on the same object and is a different kind of force.)
6. $a = g \sin 30° = 4.9$ m/s^2.

Exam tips

- Locate the **center of mass** with $x_{cm} = \frac{1}{M}\int x\,dm$ (or $\frac{\sum m_i x_i}{\sum m_i}$ for point masses) and use λ, σ, ρ for the mass element.
- The net external force moves the center of mass as if all mass sat there: $\vec{F}_{net} = M\vec{a}_{cm}$.
- Define your **system** clearly – internal forces cancel, so only external forces change its momentum.
- Exploit symmetry to shortcut a center-of-mass integral.
- Distinguish center of mass from center of gravity (identical in a uniform field).

Common mistakes

- Writing $N = mg$ on every surface. On a ramp $N = mg\cos\theta$ (if nothing else pushes), and in an elevator accelerating upward at a, $N = m(g + a)$.
- Using $f_s = \mu_s N$ as the static friction. $\mu_s N$ is only the maximum; static friction is whatever the equilibrium needs, up to that limit.
- Drawing a "centripetal force" or a "centrifugal force" on a free-body diagram. Draw only real forces; their net inward component equals mv^2/r.
- Forgetting that N and mg both point to the centre at the top of a vertical loop: $N + mg = mv^2/r$.
- Quoting $v(t) = v_T(1-e^{-bt/m})$ without the separation of variables when a derivation is asked for.

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Work, Energy, and Power

AP Physics C: Mechanics

Translational Kinetic Energy

Driving instructors say: "twice as fast needs four times the stopping distance." Energy explains why. The brakes must remove all the car's kinetic energy, and kinetic energy grows with the *square* of speed. This unit gives you one bookkeeping tool, energy, that often replaces a long force-and-acceleration calculation with two lines.

Kinetic energy 动能 describes motion. For a particle of mass m and speed v,

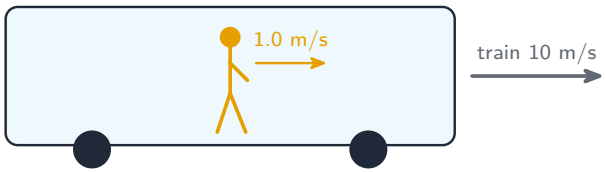
$$K = \frac{1}{2}mv^2.$$

It is a **scalar** 标量 measured in **joules** 焦耳 (J), and is nonnegative. At fixed mass, doubling speed quadruples kinetic energy. Velocity direction does not change K when speed is unchanged.

The observer's **reference frame** 参考系 matters. A passenger at rest relative to a train moving at 10 metres per second has zero translational kinetic energy in the train frame but nonzero kinetic energy in the platform frame. Use one consistent frame throughout a calculation.

Comparison. For a 2.0-kilogram object, $K = \frac{1}{2}mv^2 = \frac{1}{2}(2.0\text{ kg})(3.0\text{ m/s})^2 = 9.0$ J. At 6.0 metres per second, $K = \frac{1}{2}mv^2 = \frac{1}{2}(2.0\text{ kg})(6.0\text{ m/s})^2 = 36$ J. Reversing velocity from positive 3 to negative 3 metres per second would leave the energy at 9 joules.

> **Worked example (frame dependence).** A 60-kilogram passenger walks forward at 1.0 metre per second inside a train moving at 10 metres per second. In the train frame, $K = \frac{1}{2}(60\text{ kg})(1.0\text{ m/s})^2 = 30$ J. In the platform frame the speed is 11 metres per second, so $K = \frac{1}{2}(60\text{ kg})(11\text{ m/s})^2 = 3630$ J. Both answers are correct; they belong to different frames. The *change* in kinetic energy in one process can also differ between frames, so never mix frames in one energy equation.



train frame: $K = \frac{1}{2}(60)(1.0)^2 = 30$ J

platform frame: $K = \frac{1}{2}(60)(11)^2 = 3630$ J

both correct: pick one frame and keep it

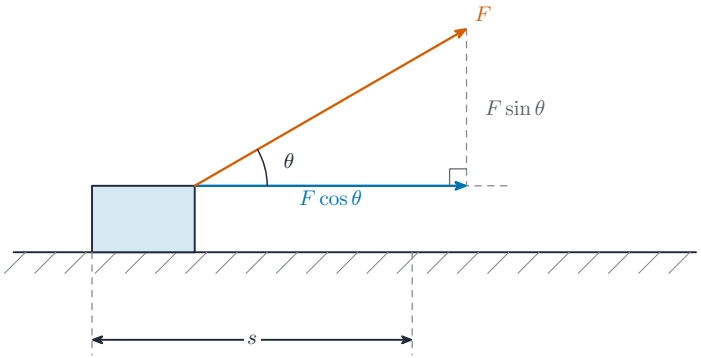
Kinetic energy depends on the frame: 30 J relative to the train, 3630 J relative to the platform

Work

Work 功 is energy transfer by a force acting through displacement of its point of application. For a variable force, use an **integral** 积分 along the actual path:

$$W = \int_a^b \vec{F} \cdot d\vec{r}.$$

For a constant force, $W = Fd\cos\theta$, with θ between force and displacement. In one dimension, signed area under $F_x(x)$ gives work. A region below the axis contributes negative area; the full force magnitude is not always the component along the path.



For the shown horizontal displacement, only the parallel component contributes to work

The **work–energy theorem** 动能定理 states

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$$W_{\text{net}} = \sum_i W_i = \Delta K.$$

Positive work by one force can coexist with negative net work. **Friction** 摩擦力 does negative work on a block sliding over a stationary rough surface, but can do positive work on an object carried along by a moving belt. A **normal force** 法向力 does zero work only when perpendicular to the displacement of its application point. Do not classify work from the force's name alone.

Signed-work task. Over 2.0 metres rightward, a constant 10-newton pull acts rightward and a 12-newton resistance acts leftward. $W_{\text{pull}} = Fd\cos\theta = (10\text{ N})(2.0\text{ m})\cos 0 = 20$ J; $W_{\text{res}} = Fd\cos\theta = (12\text{ N})(2.0\text{ m})\cos 180^\circ = -24$ J. Therefore $W_{\text{net}} = W_{\text{pull}} + W_{\text{res}} = 20\text{ J} - 24\text{ J} = -4$ J. The pull does positive work while kinetic energy falls.

Variable-force task. Let $F_x = ax^2$, with $a = 3.0\text{ N/m}^2$, from 0 to 2.0 metres. Then

$$W = \int_0^L F_x\,dx = \frac{aL^3}{3}.$$

$$W = \frac{aL^3}{3} = \frac{(3.0\text{ N/m}^2)(2.0\text{ m})^3}{3} = 8.0\text{ J}.$$

If this is the net force on a 1.0-kilogram object initially moving at $v_i = 3.0$ m/s in the positive x direction, $W_{\text{net}} = \frac{1}{2}m(v_f^2 - v_i^2)$, giving $v_f = \sqrt{v_i^2 + 2W_{\text{net}}/m} = 5.0$ m/s. If other forces also do work, include them before finding the speed.

> **Worked example (work done by a spring).** An ideal spring pulls back with $F_x = -kx$. When you stretch it from $x = 0$ to $x = d$, the spring does
>
> $$W_s = \int_0^d (-kx)\,dx = -\tfrac{1}{2}kd^2.$$
>
> On an F_x–x graph this is the triangle of base d and height kd, below the axis. For $k = 200$ N/m and $d = 0.10$ m, $W_s = -\frac{1}{2}(200\text{ N/m})(0.10\text{ m})^2 = -1.0$ J. Your hand does $+1.0$ J if the spring ends at rest. Stretching from 0.10 metre to 0.20 metre takes $\frac{1}{2}k(0.20^2 - 0.10^2) = 3.0$ J, three times as much, because the force keeps growing.

> **Worked example (braking distance).** A car of mass m brakes on a level road with a constant friction force f until it stops. The work–energy theorem gives $-fd = 0 - \frac{1}{2}mv^2$, so
>
> $$d = \frac{mv^2}{2f}.$$
>
> The stopping distance is proportional to v^2. At 20 metres per second a car stops in 25 metres; at 40 metres per second it needs $25\text{ m} \times 4 = 100$ m. Mass cancels if $f = \mu mg$, which is why the result holds for small and large cars on the same road.

Potential Energy

Potential energy 势能 belongs to the configuration of an interacting system. **Conservative forces** 保守力 have path-independent work and an associated potential:

$$\Delta U = -\int_a^b \vec{F}_c \cdot d\vec{r}.$$

A conservative force does zero net work around a closed path. Choose a convenient zero of U; adding a constant does not change forces or energy differences.

Interaction	Potential relation
Near-surface gravity	$\Delta U_g = mg\Delta y$ for nearly constant g
Two spherical masses	$U_g = -Gm_1m_2/r$ with zero at infinity
Ideal spring	$U_s = \frac{1}{2}kx^2$, x from natural length

Worked example (escape speed). Earth has $M = 6.0 \times 10^{24}$ kg and $R = 6.4 \times 10^6$ m. A launched object just escapes if it reaches infinity, where $U = 0$, with zero speed. For the object–Earth system,

$$\tfrac{1}{2}mv_e^2 - \frac{GMm}{R} = 0, \qquad v_e = \sqrt{\frac{2GM}{R}}.$$

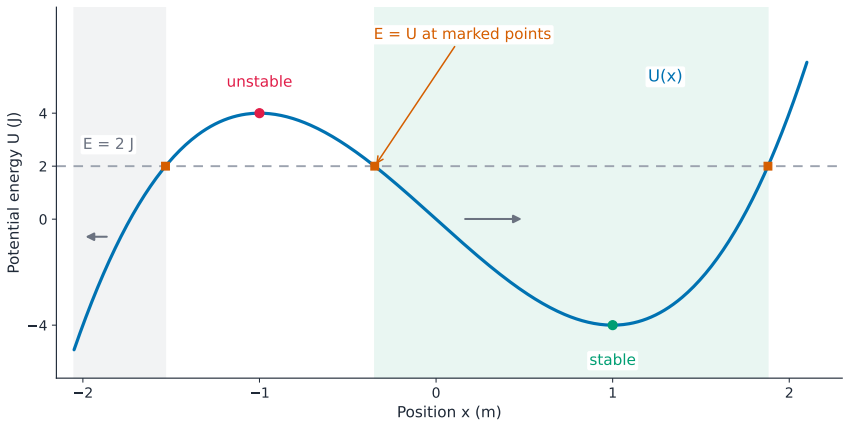
Then $v_e = \sqrt{2(6.67 \times 10^{-11})(6.0 \times 10^{24})/(6.4 \times 10^6)} \approx 1.1 \times 10^4$ m/s, about 11 kilometres per second. The **escape speed** 逃逸速度 does not depend on the object's mass or launch direction (air resistance ignored). mgh would be wrong here, because g is not constant over such large distances.

Sum each interacting pair once in a multi-object system. In one dimension, the conservative force is minus the **derivative** 导数:

$$F_x = -\frac{dU}{dx}.$$

A positive slope gives force in the negative x direction; a negative slope gives positive force. Zero slope gives equilibrium, not necessarily zero speed. A **stable equilibrium** 稳定平衡 has a restoring force after a small displacement; an **unstable equilibrium** 不稳定平衡 has a force away from it. Local minima and maxima provide these classifications in the usual smooth one-dimensional cases.

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Shading: U ≤ E. Separate allowed intervals need not connect.
Force arrows follow F = −dU/dx; the curve uses a = 2 J/m³ and b = 6 J/m.

For the plotted cubic potential, the same total energy admits disconnected position intervals

Differentiation task. Take $U(x) = ax^3 - bx$, where $a = 2.0$ J/m^3 and $b = 6.0$ J/m. Then $F_x = b - 3ax^2$. Equilibria satisfy $x = \pm\sqrt{b/(3a)} = \pm 1.0$ m. Since $U'' = 6ax$, the positive position is a minimum and stable; the negative position is a maximum and unstable. At $x = 0$, $F_x = b = 6.0$ N in the positive direction. The coefficients carry units; writing only $2x^3 - 6x$ without specifying the coordinate units hides that information.

Conservation of Energy

Mechanical energy 机械能 is $E_{\text{mech}} = K + U$. In the course's particle model, a single object has kinetic energy; include Earth or the spring to account for the relevant interaction potential. This is a mechanical model, not a claim that a real body has no internal energy.

For an isolated ideal mechanical system with no dissipative interactions,

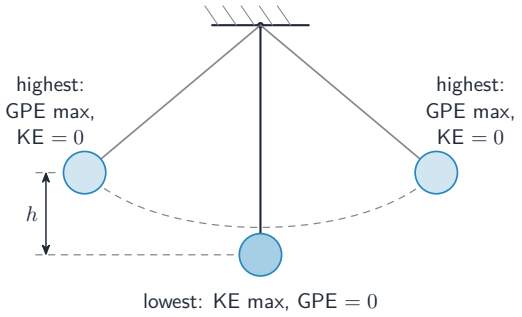
$$K_i + U_i = K_f + U_f.$$

Nonconservative forces 非保守力 can transfer or dissipate mechanical energy; they do not invariably remove it. Sliding friction over a fixed rough track converts mechanical energy into **thermal energy** 热能. For constant kinetic friction and path length s, $\Delta E_{\text{mech}} = -f_k s$ under the stated model. Variable friction requires an integral. Choose the system before classifying forces as internal or external.

With block, Earth and track inside the system, negligible energy transfer to the surroundings, and only kinetic, potential and thermal changes relevant,

$$0 = \Delta K + \Delta U + \Delta E_{\text{thermal}}.$$

External work changes system energy when work is the only transfer; heat or other transfers require additional terms. Do not count gravitational work and the same gravitational potential change twice. AP mechanics recognises dissipation as thermal energy or sound without requiring a thermodynamics treatment.



For release from rest with negligible dissipation, the extreme positions have zero speed

For fixed mechanical energy in one-dimensional motion, $K = E - U(x) \geq 0$. Positions with $U > E$ are inaccessible. The **turning points** 转折点 of a bounded motion occur where $E = U$ and the particle reverses. An object at rest at a potential minimum with $E = U_{\min}$ stays at equilibrium; equality alone does not prove reversal. Distinct accessible intervals need not connect.

Graph task. In the plotted cubic potential, $E = 2.0$ J. At $x = 1.0$ m, $U = -4.0$ J, so $K = E - U = 2.0\ \text{J} - (-4.0\ \text{J}) = 6.0$ J. For a 2.0-kilogram particle, $v = \sqrt{2K/m} = \sqrt{(2)(6.0\ \text{J})/(2.0\ \text{kg})} \approx 2.45$ m/s. It is fastest at this minimum within the bounded well, not necessarily over every disconnected allowed interval.

Ramp task. A 2.0-kilogram block starts from rest 1.5 metres above the bottom and arrives at 4.0 metres per second. Use $g = 9.8\ \text{m/s}^2$. The initial gravitational energy is $U_i = mgh = 29.4$ J; the final kinetic energy is $K_f = \frac{1}{2}mv_f^2 = 16.0$ J. For the block–Earth–track system with no other transfer, $\Delta E_{\text{thermal}} = U_i - K_f = 29.4\ \text{J} - 16.0\ \text{J} = 13.4$ J. The conclusion depends on the stated energy model.

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Height constrains gravitational energy; a photograph alone does not establish speed or dissipation

Verified exam connection – 2025 Q2(A/B), Version J. Two wall-mounted springs of constants k and $2k$ are relaxed at $x = 0$ and both change length by magnitude $|x|$. At release from rest at $x = x_1$, the spring energies have ratio $U_A : U_B = 1 : 2$ and block kinetic energy is zero. At $x = x_1/2$, conservation gives $\frac{1}{2}mv^2 = \frac{9}{8}kx_1^2$, hence $v = \frac{3}{2}x_1\sqrt{k/m}$ for positive x_1. Both spring energies must be included. The official question explicitly requests a fundamental principle at the start of part B; this is a demand of this question, not a universal scoring guarantee.

Energy bar charts. FRQs often ask for a bar chart: one bar for each kind of energy (K, U_g, U_s, $\Delta E_{\text{thermal}}$) at two moments, drawn to scale. Check three things: the bars at each moment add to the same total (if the system is closed); a bar that must be zero is drawn as zero, not left out; and kinetic energy is never negative, while gravitational energy can be below zero if the zero level is above the object.

> **Worked example (spring and rough patch).** A block of mass $m = 0.50$ kg is pressed against a spring ($k = 200$ N/m), compressing it by $x_c = 0.10$ m, and released on a horizontal track. After leaving the spring it crosses a rough patch of length $D = 0.40$ m with $\mu_k = 0.25$ (use $g = 10\ \text{m/s}^2$). Find its speed after the patch.

1. Spring energy: $U_s = \frac{1}{2}kx_c^2 = \frac{1}{2}(200)(0.10)^2 = 1.0$ J, all turned into kinetic energy on the smooth part.
2. Energy lost on the patch: $\mu_k mgD = 0.25(0.50)(10)(0.40) = 0.50$ J.
3. Kinetic energy after: $1.0 - 0.50 = 0.50$ J, so $v = \sqrt{2(0.50)/0.50} \approx 1.4$ m/s.

If the block instead stops on the patch, set the final kinetic energy to zero and solve for the distance: $\frac{1}{2}kx_c^2 = \mu_k mg\,d$. To find k from data, plot the launch speed v against

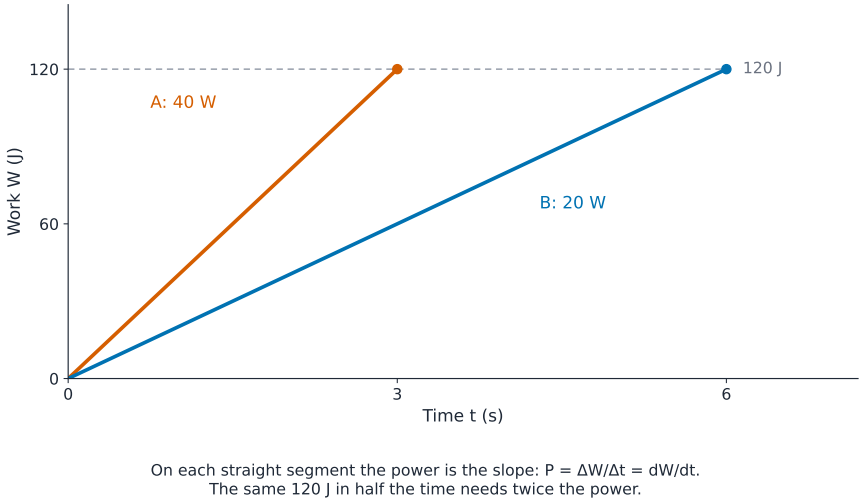
x_c: since $\frac{1}{2}mv^2 = \frac{1}{2}kx_c^2$, $v = \sqrt{k/m}\,x_c$, a straight line of slope $\sqrt{k/m}$.

Power

Power 功率 is the rate of energy transfer or conversion, measured in **watts** 瓦特 (one joule per second):

$$P_{\text{avg}} = \frac{\Delta E}{\Delta t}, \qquad P_{\text{inst}} = \frac{dW}{dt} = \vec{F}\cdot\vec{v}.$$

Average and instantaneous power need not agree. A tangent slope on a work–time graph gives instantaneous power; a secant over an interval gives average power. The displayed straight lines have constant slopes and therefore equal instantaneous and interval-average powers on each line.



Transfers A and B deliver 120 joules in 3 and 6 seconds; slope rather than height determines the rate

For A, $P = W/\Delta t = 120\text{ J}/3.0\text{ s} = 40\text{ W}$; for B, $P = W/\Delta t = 120\text{ J}/6.0\text{ s} = 20\text{ W}$. Each straight segment has constant instantaneous power equal to its interval average.

Ramp-to-motor task. A particle released from rest drops through 3.0 metres on a fixed frictionless ramp. If it stays on the track with no other energy transfer, $v = \sqrt{2gh} = \sqrt{(2)(9.8\text{ m/s}^2)(3.0\text{ m})} \approx 7.7$ m/s. A motor maintaining that speed against a 20-newton resistance needs $P = Fv = (20\text{ N})(7.7\text{ m/s}) \approx 150$ W. These ideal results exclude other losses.

Lift task. Lifting 50 kilograms steadily upward at 2.0 metres per second, with negligible losses, requires $F = mg = (50\text{ kg})(9.8\text{ m/s}^2) = 490$ N. Then $P = Fv = (490\text{ N})(2.0\text{ m/s}) = 980$ W. Doubling the speed doubles this ideal power, not the work required to reach the same height.

Hill task. A 1,200-kilogram car climbs at 15 metres per second along a road rising 1.0 metre per 20 metres travelled. The vertical speed is $v_y = v\Delta y/\Delta s = (15\text{ m/s})(1.0\text{ m}/20\text{ m}) = 0.75$ m/s. Against gravity, $P = mgv_y = (1200\text{ kg})(9.8\text{ m/s}^2)(0.75\text{ m/s}) = 8820\text{ W} \approx 8.8$ kW. Air resistance, rolling losses and efficiency would increase the required input power.

Varying-power task. Let $W(t) = ct^2$, with $c = 2.0\text{ J/s}^2$. From 0 to 3.0 seconds, $P_{\text{avg}} = [W(3) - W(0)]/(3.0\text{ s}) = 6.0$ W. At 3.0 seconds, $P_{\text{inst}} = dW/dt = 2ct = 12$ W. The final power is not the interval average.

Practice

Answer each before reading the answers below.

1. A force $F(x) = 3x^2$ N acts along the x-axis. Find the work done from $x = 0$ to $x = 2$ m.
2. A spring ($k = 400$ N/m) compressed 0.10 m launches a 0.50 kg block on a frictionless surface. Find the block's speed.
3. A 50 kg crate is lifted 10 m at constant speed in 5.0 s. Find the power.
4. A potential energy function is $U(x) = 3x^2 - x^3$ (J, m). Find $F(x)$ and the equilibrium positions, and classify each as stable or unstable.
5. A 2.0 kg block sliding at 6.0 m/s on a rough floor comes to rest. How much energy is dissipated by friction?

Answers

1. $W = \displaystyle\int_0^2 3x^2\,dx = x^3\Big|_0^2 = 8$ J.
2. $\frac{1}{2}kx^2 = \frac{1}{2}(400)(0.10)^2 = 2.0\text{ J} = \frac{1}{2}(0.50)v^2$, so $v = 2.8$ m/s.
3. $P = \dfrac{mgh}{t} = \dfrac{50(9.8)(10)}{5.0} = 980$ W.
4. $F = -\dfrac{dU}{dx} = -6x + 3x^2$. $F = 0$ at $x = 0$ and $x = 2$ m. $U''(x) = 6 - 6x$: positive at 0 (minimum, stable) and negative at 2 (maximum, unstable).
5. All the kinetic energy: $\frac{1}{2}(2.0)(6.0)^2 = 36$ J.

Exam tips

Name the system and energy transfers. Start from the relevant principle, preserve signs and units, and distinguish one force's work from net work. Read signed area, slope and energy gaps as different graph operations. State assumptions before using conservation or a friction model; follow the actual prompt rather than a memorised

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scoring claim.

Common mistakes

- Counting gravity twice: either include the work done by gravity *or* the change in U_g, never both.
- Losing the minus sign in $F_x = -\dfrac{dU}{dx}$. Where U slopes upwards, the force points towards smaller x.
- Treating a zero force as zero speed. At an equilibrium point $F = 0$, but the particle may be moving fastest there.
- Using $P = W/t$ for an instantaneous power that changes. Use $P = \vec{F} \cdot \vec{v}$ or dW/dt at that instant.
- Saying friction always does negative work. It does negative work on a block sliding over a fixed surface, but can do positive work on an object carried by a moving belt.

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Linear Momentum

AP Physics C: Mechanics

In 2005 NASA's Deep Impact spacecraft fired a 370 kg copper block into comet Tempel 1 at about 10 km/s. The comet has a mass of about 7×10^{13} kg, so the impactor's momentum alone gives an estimated velocity change of only about 0.00005 mm/s (ignoring the ejected material) – yet the impact blasted out thousands of tonnes of dust and ice for scientists to study. In 2022 the DART mission went further and deliberately changed the orbit of the small asteroid Dimorphos by ramming it. Both missions were planned with one idea: momentum is conserved. This unit shows how to use it.

Linear momentum 动量 is **mass** 质量 times **velocity** 速度:

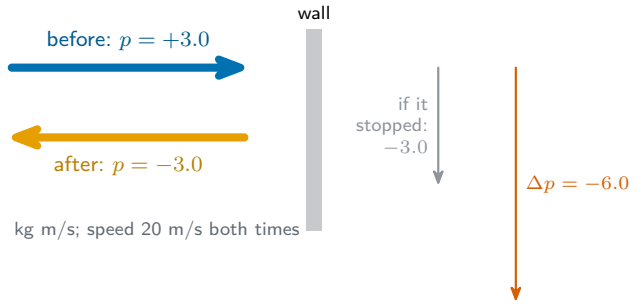
$$\vec{p} = m\vec{v}.$$

It is a **vector** 矢量 pointing along the velocity, and it is *the* quantity for analysing **collisions** 碰撞 and **explosions** 爆炸. A collection of objects can be treated as one **system** 系统 moving with the velocity of its **center of mass** 质心:

$$\vec{v}_{\text{cm}} = \frac{\sum m_i \vec{v}_i}{\sum m_i},$$

so the system's total momentum is its total mass times $\vec{v}_{\text{cm}}$ – one object's worth of bookkeeping for any number of parts. Its unit is the kilogram metre per second (kg · m/s), which is the same as the newton second (N · s).

Because momentum is a vector, direction matters. A 0.15 kg ball moving at 20 m/s toward a wall has $p = +3.0$ kg · m/s; if it bounces straight back at 20 m/s, $p = -3.0$ kg · m/s. Its speed did not change, but its momentum changed by -6.0 kg · m/s – twice what it would be if the ball had simply stopped.



Same speed, opposite direction: the momentum change on bouncing is -6.0 kg m/s, twice that of stopping

Change in Momentum and Impulse

Newton's second law 牛顿第二定律 for a system containing the same particles is:

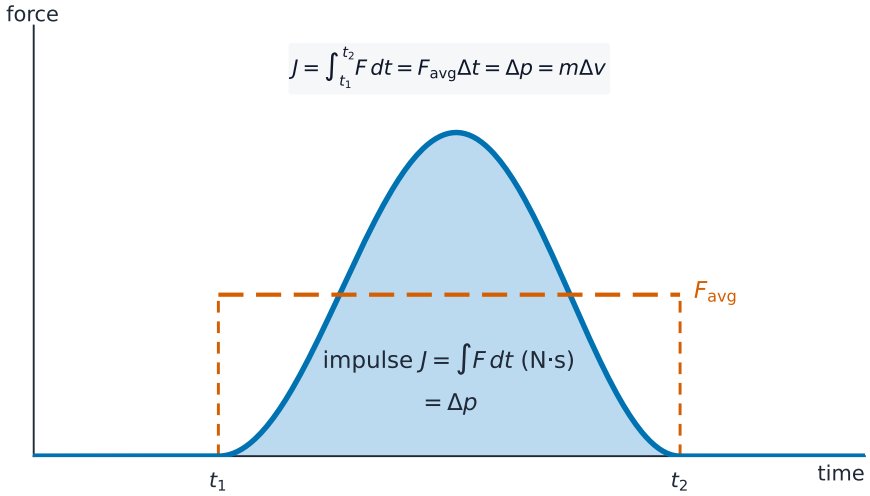
$$\vec{F}_{\text{net}} = \frac{d\vec{p}}{dt},$$

This reduces to $m\vec{a}$ for constant total mass. If matter crosses the chosen boundary, include the momentum it carries. Sand arriving with zero horizontal speed and leaving on a **conveyor belt** 传送带 at speed v needs horizontal force $F = \dot{m}v$, where $\dot{m}$ is the mass arriving per second. A **rocket** 火箭 gains momentum as exhaust carries momentum away; applying $d(m\vec{v})/dt$ to the rocket alone without this momentum flow is incorrect. The **impulse** 冲量 delivered by a force is its **time integral** 对时间的积分, and the **impulse–momentum theorem** 冲量-动量定理 says it equals the change in momentum:

$$\vec{J} = \int_{t_1}^{t_2} \vec{F}_{\text{net}}\, dt = \Delta\vec{p}.$$

For a constant **net force** 合力, the impulse points in its direction. Read it off graphs both ways:

- On a **force–time graph** 力 – 时间图, impulse is the **area** 面积 under the curve.
- On a momentum–time graph, the net force is the **slope** 斜率 at each **instant** 时刻.



Impulse is the area under the force-time curve, equal to the average force times the contact time

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Spreading the same Δp over a longer time lowers the force – that is why **airbags** 安全气囊, **crumple zones** 溃缩区, and soft landings work, and why you bend your knees when you land.

> **Worked example (impulse from a graph).** A tennis racket exerts a force that rises steadily from 0 to 1500 N in 2.0 ms and falls back to 0 in another 2.0 ms on a 0.060 kg ball arriving at 30 m/s. Take the ball's outgoing direction as positive. The force–time graph is a triangle, so $J = \frac{1}{2}(4.0 \times 10^{-3})(1500) = 3.0$ N · s. Then $mv - mv_0 = J$, so $0.060v - 0.060(-30) = 3.0$, which gives $v = +20$ m/s. The ball leaves at 20 m/s in the positive direction. The average force is $J/\Delta t = 3.0/0.0040 = 750$ N – half the peak.

The slope of a momentum–time graph is the net force. If a cart's momentum rises in a straight line from 2.0 to 8.0 kg · m/s in 3.0 s, the net force on it is $\dfrac{6.0}{3.0} = 2.0$ N.

> **Worked example (variable force).** A time-varying force $F(t) = 10t$ N acts on a 2.0 kg object for 2.0 s. The impulse is $J = \displaystyle\int_0^2 10t\,dt = \left[5t^2\right]_0^2 = 20$ N · s, so the velocity changes by $\Delta v = J/m = 10$ m/s.

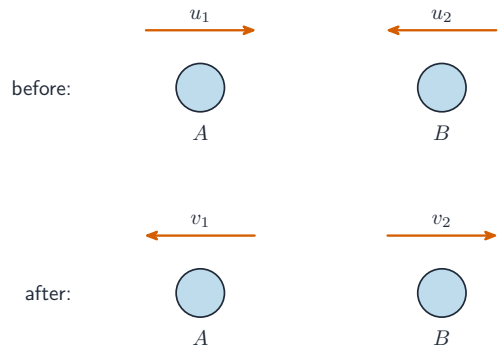
Conservation of Linear Momentum

Internal forces 内力 come in Newton's-third-law pairs, so they cancel inside any system: they can shuffle momentum *between* parts but never change the total. For a fixed set of particles, total momentum changes only through a net *external* force ($\vec{J} = \Delta\vec{p}$). So, with zero **net external force** 合外力, total momentum is **conserved** 守恒:

$$\sum \vec{p}_{\text{before}} = \sum \vec{p}_{\text{after}}.$$

Strictly, **momentum is conserved in all interactions**: momentum is never created or destroyed, only transferred. What changes is the *system* you choose. If the net external force on the system is **zero**, the system's total momentum is constant. If it is **not zero**, momentum is transferred between the system and its surroundings – a ball falling toward Earth gains momentum because Earth's gravity is external to the ball; in the ball–Earth system, Earth gains an equal and opposite (and undetectably small) momentum.

Momentum is conserved over a collision when the net external impulse is negligible; choose a system containing both colliding objects. Apply conservation separately along each **axis** 坐标轴; AP asks for quantitative work in one or two dimensions.



A head-on collision: total momentum before equals total momentum after

> **Worked example (explosion).** A 6.0 kg shell at rest splits into a 2.0 kg piece moving at 9.0 m/s east and a 4.0 kg piece. Total momentum stays zero, so the heavy piece moves west at $v = \dfrac{2.0(9.0)}{4.0} = 4.5$ m/s. The kinetic energy came from stored (chemical or spring) energy – momentum conservation does not require kinetic-energy conservation. The centre of mass of the two pieces stays at rest, exactly where the shell was.

> **Worked example (recoil).** A 60 kg skater standing on ice throws a 5.0 kg ball forward at 12 m/s. Total momentum is zero before and after, so $60v = -5.0(12)$ and the skater slides backward at 1.0 m/s. The same idea drives a rocket: it pushes exhaust backward, and the exhaust pushes the rocket forward.

Elastic and Inelastic Collisions

When the net external impulse is negligible, collisions conserve momentum; they differ in what happens to the **kinetic energy** 动能:

type	momentum	kinetic energy
elastic collision 弹性碰撞	conserved	total conserved (individual shares may change)
inelastic collision 非弹性碰撞	conserved	decreases – some becomes heat, sound, **deformation** 形变
perfectly inelastic collision 完全非弹性碰撞	conserved	largest possible loss – the objects stick and share one velocity

Strategy: *always* write momentum conservation first; add the kinetic-energy equation only when the problem says "elastic". Two useful elastic facts: equal masses in a 1D elastic collision simply exchange velocities, and in the **center-of-mass frame** 质心参考系 each object reverses its velocity in a 1D elastic collision.

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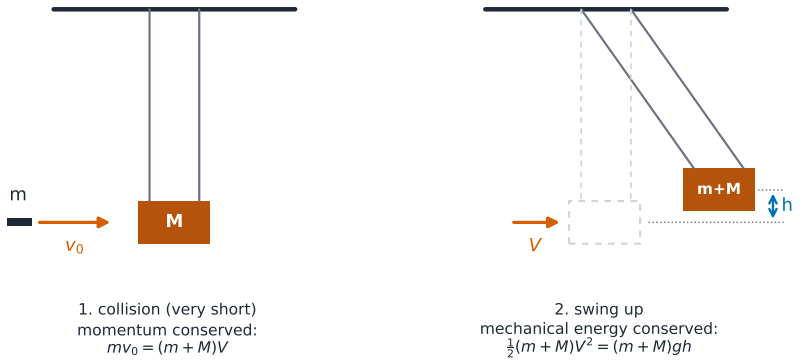
> **Worked example.** A 1000 kg car at 20 m/s strikes a stationary 1500 kg car and they lock together – perfectly inelastic. Momentum: $v = \dfrac{1000(20)}{2500} = 8.0$ m/s. Kinetic energy falls from 2.0×10^5 J to $\frac{1}{2}(2500)(8.0)^2 = 8.0 \times 10^4$ J: about 60% is lost, even though momentum is exactly conserved.

Elastic collisions in one dimension. When a mass m_1 moving at v_1 hits a stationary m_2 elastically, solving the momentum and kinetic-energy equations together gives

$$v_1' = \frac{m_1 - m_2}{m_1 + m_2}\, v_1, \qquad v_2' = \frac{2m_1}{m_1 + m_2}\, v_1.$$

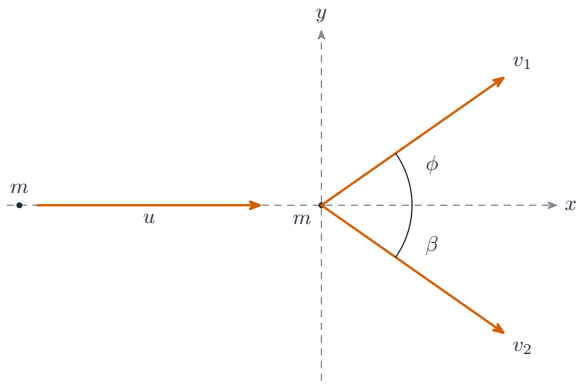
Check the limits: equal masses exchange velocities ($v_1' = 0$, $v_2' = v_1$); a light ball hitting a heavy wall bounces back with nearly its original speed; a heavy ball hitting a light one carries on almost unchanged and the light one leaves at nearly $2v_1$. In any 1D elastic collision the **relative speed of approach equals the relative speed of separation**: $v_1 - v_2 = -(v_1' - v_2')$, a quicker equation to use than kinetic energy.

> **Worked example (ballistic pendulum).** A 0.010 kg bullet embeds in a 2.0 kg wooden block hanging on strings, and the block swings up 0.20 m. Find the bullet's speed. The problem has two stages, and each needs a different conservation law. **Swing (energy):** $\frac{1}{2}(m+M)V^2 = (m+M)gh$, so $V = \sqrt{2gh} = \sqrt{2(10)(0.20)} = 2.0$ m/s. **Collision (momentum):** $mv_0 = (m+M)V$, so $v_0 = \dfrac{2.01(2.0)}{0.010} \approx 400$ m/s. Only $\frac{0.5(2.01)(2.0)^2}{0.5(0.010)(400)^2} \approx 0.5\%$ of the bullet's kinetic energy remains after the collision; the rest became heat and deformation of the wood.



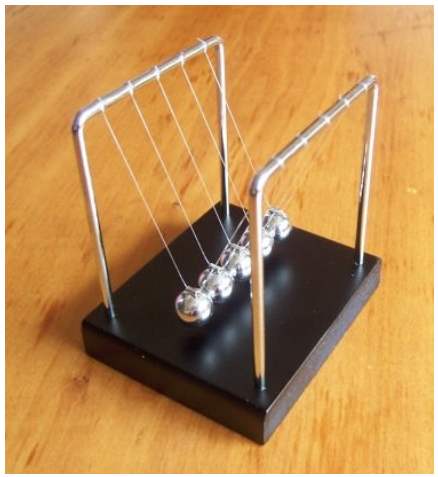
A ballistic pendulum: the collision conserves momentum, then the swing conserves mechanical energy

Worked example (2D). A puck moving east at 4.0 m/s strikes an identical puck at rest; after the glancing hit, one moves at 2.0 m/s at 60° north of east. Conserve each axis: east–west, $m(4.0) = m(2.0)\cos 60° + mv_x$, so $v_x = 3.0$ m/s; north–south, $0 = m(2.0)\sin 60° - mv_y$, so $v_y = 1.7$ m/s. The second puck moves at $\sqrt{3.0^2 + 1.7^2} = 3.5$ m/s, about 30° south of east.



A glancing collision, resolved along two perpendicular axes

Exam skill. On FRQs, justify with the *condition*, not the slogan: "the net external force on the two-puck system is zero during the collision, so its total momentum is constant." If asked whether the collision is elastic, *compute* the kinetic energy before and after and compare – never assume.



A Newton's cradle shows momentum and kinetic energy passing through a near-elastic collision

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Practice

Answer each before reading the answers below.

1. A force $F(t) = 100t$ N acts for $0 \le t \le 0.20$ s. Find the impulse.
2. A 2.0 kg cart at 6.0 m/s hits and sticks to a 4.0 kg cart at rest. Find the final speed and the kinetic energy lost.
3. Two equal masses collide head-on elastically, one moving and one at rest. Describe the result.
4. A rod of length L along the x-axis has linear density $\lambda = kx$. Find its center of mass.
5. A 3.0 kg object at rest explodes into a 1.0 kg piece moving east at 6.0 m/s and a 2.0 kg piece. Find the velocity of the second piece.
6. A 0.15 kg ball hits a wall at 20 m/s and rebounds at 15 m/s. Contact lasts 0.010 s. Find the average force.

Answers

1. $J = \displaystyle\int_0^{0.20} 100t\,dt = 50t^2\Big|_0^{0.20} = 2.0$ N · s.
2. $v = \dfrac{2.0(6.0)}{6.0} = 2.0$ m/s. KE: 36 J before, 12 J after; 24 J lost.
3. The moving mass stops, and the other moves off with the first one's velocity: they exchange velocities.
4. $x_{cm} = \dfrac{\int_0^L x(kx)\,dx}{\int_0^L kx\,dx} = \dfrac{L^3/3}{L^2/2} = \dfrac{2L}{3}$.
5. Total momentum stays zero: $1.0(6.0) + 2.0v = 0$, so $v = 3.0$ m/s west.
6. $\Delta p = 0.15(15 - (-20)) = 5.25$ kg · m/s; $F = 5.25 \div 0.010 = 525$ N, away from the wall.

Exam tips

- Impulse equals the momentum change: $\vec{J} = \int \vec{F}\,dt = \Delta\vec{p}$, and it is the area under a force–time graph.
- **Momentum is conserved** whenever the net external force is zero – always the go-to for collisions.
- Distinguish **elastic** (kinetic energy conserved) from **inelastic** (kinetic energy decreases) collisions; in a **perfectly inelastic** collision the objects stick and move with one velocity.
- Apply conservation to each **component** 分量 (x and y) separately in 2-D.
- Connect to center of mass: total momentum $= M\vec{v}_{cm}$.

Common mistakes

- Conserving momentum while an external impulse is large, for example during a long push from the ground. Conservation needs a negligible net external impulse over the interval.
- Calling a collision elastic because momentum is conserved. Check kinetic energy too.
- Reading impulse as the peak force times the time. Impulse is the *area* under the F–t graph.
- Adding momenta in two dimensions as magnitudes. Conserve the x and y components separately.
- Forgetting that internal forces cannot accelerate the centre of mass: an explosion leaves $\vec{v}_{cm}$ unchanged.
- Using energy conservation across a perfectly inelastic collision (as in a ballistic pendulum). Use momentum for the collision, then energy for the swing.

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Torque and Rotational Dynamics

AP Physics C: Mechanics

Rotational Kinematics

Why is a door handle on the side far from the hinges? Why does a tightrope walker carry a long pole? Why does a tall chimney often break in the middle as it falls? All three answers come from this unit: where a force acts matters as much as how big it is, and where the mass sits matters as much as how much there is.

A fan can keep turning while it slows down. Its velocity and acceleration then have opposite signs. Choose one fixed axis and a positive sense, such as counterclockwise. A **rigid body** 刚体 holds its shape; its points share the same angular motion about that axis.

Angular displacement 角位移 is $\Delta\theta = \theta - \theta_0$, measured in **radians** 弧度. One revolution is 2π radians. **Angular velocity** 角速度 and **angular acceleration** 角加速度 are rates:

$$\omega = \frac{d\theta}{dt}, \qquad \alpha = \frac{d\omega}{dt}.$$

The AP rotational-kinematics descriptions use clockwise/counterclockwise signs; three-dimensional directions of these kinematic vectors are outside this unit's assessment boundary. Torque directions are considered separately below.

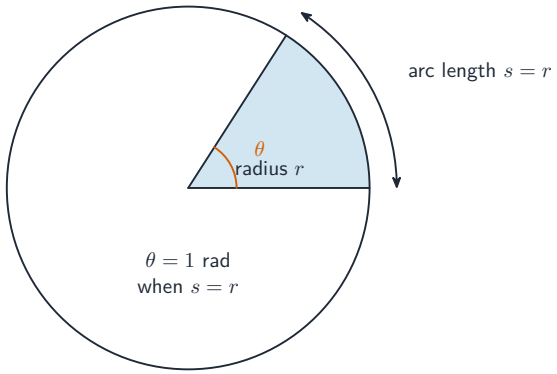
For constant α only,

$$\omega = \omega_0 + \alpha t, \qquad \Delta\theta = \omega_0 t + \frac{1}{2}\alpha t^2, \qquad \omega^2 = \omega_0^2 + 2\alpha\Delta\theta.$$

For varying acceleration, retain the initial conditions:

$$\omega(t) = \omega_0 + \int_0^t \alpha(t')\,dt', \qquad \theta(t) = \theta_0 + \int_0^t \omega(t')\,dt'.$$

Slope of a θ–t graph gives ω; slope of an ω–t graph gives α. Signed area under $\omega(t)$ gives angular displacement; it need not equal total angle travelled after reversal.



Use radians in the arc-length and linear–angular relations

Constant-acceleration task. A fan slows from 30 rad/s to rest at constant $\alpha = -6.0$ rad/s^2. From $\omega = \omega_0 + \alpha t$, $t = (0 - 30)/(-6.0) = 5.0$ s. Then $\Delta\theta = \omega_0 t + \frac{1}{2}\alpha t^2 = 75$ rad, or $75/(2\pi) \approx 12$ revolutions. Stop the model at rest unless the driving torque is specified to continue.

Variable-acceleration task. Let $\alpha = ct$, with $c = 4.0$ rad/s^3, $\omega_0 = 3.0$ rad/s and $\theta_0 = 1.0$ rad. Integration gives $\omega = \omega_0 + \frac{1}{2}ct^2$ and $\theta = \theta_0 + \omega_0 t + ct^3/6$. At $t = 2.0$ s, these are 11 rad/s and 12.3 rad. Omitting the integration constants changes the physical initial state.

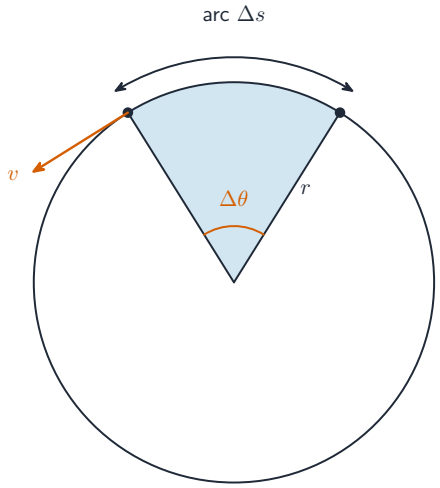
Connecting Linear and Rotational Motion

For a point at perpendicular distance r from a fixed axis,

$$\Delta s = r\Delta\theta, \qquad v = r|\omega|, \qquad a_t = r\alpha, \qquad a_c = r\omega^2.$$

Here v is speed; a_t is signed relative to the chosen tangential direction. The **tangential acceleration** 切向加速度 changes speed and may oppose motion. The **centripetal acceleration** 向心加速度 points toward the axis and changes direction. Their perpendicular magnitudes give $a = \sqrt{a_t^2 + a_c^2}$. Constant nonzero angular speed still gives centripetal acceleration.

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Points at different radii share angular velocity but have different linear speeds

Wheel task. At one instant $\omega = 4.0$ rad/s and $\alpha = -2.0$ rad/s^2. At $r = 0.50$ m, $v = r|\omega| = 2.0$ m/s, $a_t = r\alpha = -1.0$ m/s^2 and $a_c = r\omega^2 = 8.0$ m/s^2. Thus $a = \sqrt{(-1.0)^2 + 8.0^2} = 8.1$ m/s^2. At twice the radius all three component magnitudes double, although ω and α are unchanged.

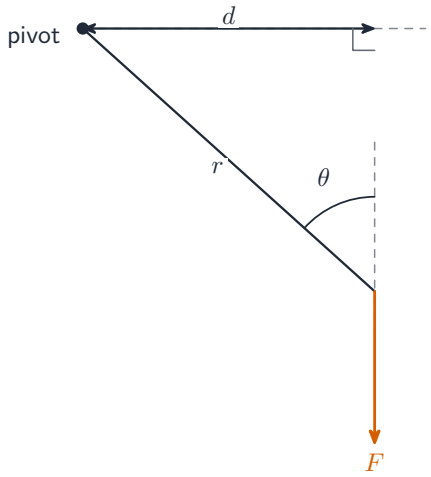
Torque

Torque 力矩 describes a force's turning effect about a chosen axis. It is a **vector product** 矢量积:

$$\vec{\tau} = \vec{r} \times \vec{F}, \qquad |\tau| = rF\sin\phi = Fr_\perp.$$

The **moment arm** 力臂 $r_\perp$ is the perpendicular distance from the axis to the force's line of action, not necessarily the distance to its application point. A force whose line of action passes through the axis has zero torque about that axis; it can still exert a force and cause translation.

The **right-hand rule** 右手定则 gives the torque direction perpendicular to the plane of $\vec{r}$ and $\vec{F}$: curl from $\vec{r}$ toward $\vec{F}$. Counterclockwise torque is out of the page; clockwise is into it. For planar calculations choose signs consistently. Torque has unit N m, but it is not the scalar work measured in joules.

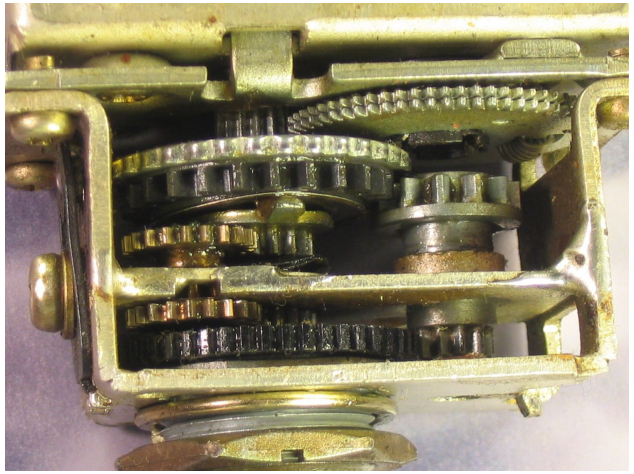


Measure the perpendicular distance to the force's line of action

Lever task. A 20 N force acts at $r = 0.30$ m and angle 30° to the radius, tending counterclockwise. Its torque is $\tau = rF\sin\phi = (0.30)(20)\sin 30° = +3.0$ N m. Equivalently, $r_\perp = r\sin\phi = 0.15$ m. A perpendicular 10 N force at the same radius gives the same torque. More force alone does not determine more torque.

> **Worked example (the door handle).** A stiff door needs a torque of 2.0 N m about its hinges to start opening. Pushing at right angles at the handle, 0.80 m from the hinge line, needs $F = \tau/r_\perp = 2.0/0.80 = 2.5$ N. Pushing at 0.10 m from the hinges needs $2.0/0.10 = 20$ N, eight times as much. Pushing straight toward the hinges, along the door, gives $r_\perp = 0$ and no torque at all, however hard you push.

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Visible gear sizes suggest different angular speeds; the photo does not supply forces, inertias or losses

Rotational Inertia

Rotational inertia 转动惯量 depends on the mass distribution relative to a specified axis:

$$I = \sum_i m_i r_i^2, \qquad I = \int r^2\,dm.$$

Use perpendicular distance from each mass element to that axis. The unit is kg m^2. For a point-mass model, moving the same mass from r to $2r$ multiplies its contribution by four. This is the tightrope walker's trick: a long pole puts mass far from the walker's axis, so a small sideways torque gives only a small α, and the walker has time to correct a wobble. Inertias add only when defined about the same axis.

Uniform rod about its centre. For mass M, length L and **linear density** 线密度 $\lambda = M/L$, take $x = 0$ at the centre. Then $dm = (M/L)dx$ and

$$I_{\text{cm}} = \int_{-L/2}^{L/2} x^2 \frac{M}{L}\,dx = \frac{ML^2}{12}.$$

Nonuniform rod about its light end. If $\lambda(x) = cx$ on $0 \le x \le L$, c has unit kg/m^2. First normalise the density:

$$M = \int_0^L cx\,dx = \frac{1}{2}cL^2, \qquad c = \frac{2M}{L^2}.$$

Then $I = \int_0^L x^2(cx)dx = cL^4/4 = ML^2/2$. This is not the uniform-rod result; the mass is concentrated farther from the axis.

Disk from coaxial rings. For a uniform thin disk, surface density is $\sigma = M/(\pi R^2)$. A thin ring has $dm = \sigma 2\pi r\, dr$, so

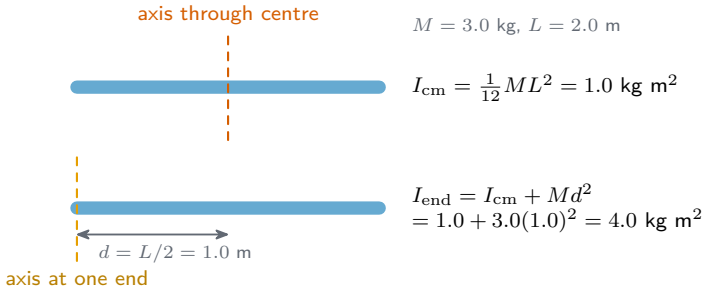
$$I = \int_0^R r^2 \sigma 2\pi r\, dr = \frac{1}{2}MR^2.$$

A thin hoop of the same mass and radius has $I = MR^2$. For a uniform annular disk with inner radius a and outer radius b, use $\sigma = M/[\pi(b^2 - a^2)]$ and limits a to b: $I = M(a^2 + b^2)/2$. AP's calculus scope includes rods and bodies built from coaxial rings/shells; arbitrary three-dimensional shapes are not required here.

The **parallel-axis theorem** 平行轴定理 relates an axis through the centre of mass to a parallel axis a distance d away:

$$I' = I_{\text{cm}} + Md^2.$$

For the uniform rod about one end, $I = ML^2/12 + M(L/2)^2 = ML^2/3$. The starting axis must pass through the centre of mass. For $M = 3.0$ kg and $L = 2.0$ m, $I_{\text{cm}} = 1.0$ kg m^2 and $I_{\text{end}} = 4.0$ kg m^2.



Moving the axis from the centre to the end adds Md squared: the same rod is four times as hard to turn about its end

Verified exam connection – 2023 Set 1 Q2(a). The question gives $T = 2\pi\sqrt{I/\kappa}$ for a torsional pendulum and $I_{\text{disk}} = MR^2/2$. For N identical coaxial disks moving together, $I_{\text{total}} = NMR^2/2$, so $T = 2\pi\sqrt{NMR^2/(2\kappa)}$. Four times as many disks doubles the period, not quadruples it. The official scheme checks the summed inertia and consistent substitution. The period relation is supplied here; its oscillation derivation belongs to the oscillations unit.

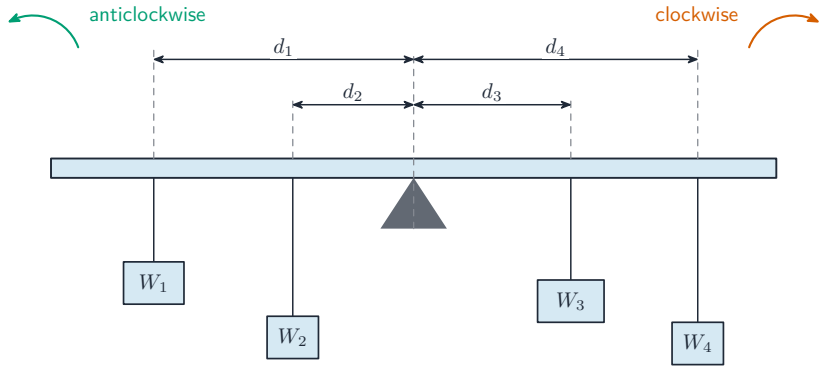
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Rotational Equilibrium

For a rigid system with constant inertia about a fixed axis, zero **net torque** 净力矩 gives $\alpha = 0$, so angular velocity is constant. This is **rotational equilibrium** 转动平衡, which can include steady rotation. It does not by itself establish zero net force or rest. Full **static equilibrium** 静平衡 also requires rest and

$$\sum \vec{F} = 0, \qquad \sum \tau = 0.$$

Draw forces at their actual application points. In a static beam problem, choosing the axis through an unknown support removes that force's torque, but its force remains in the force-balance equation. Equilibrium torque balance can be taken about any axis.

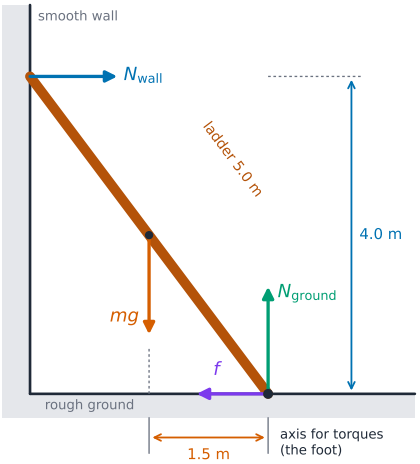


Torque balance is one condition; force balance is also needed for static equilibrium

Beam task. A horizontal uniform 20 kg beam of length 4.0 m is pivoted at its left end and held by a vertical rope at its right end. A 40 kg child sits 1.0 m from the pivot. With $g = 9.8$ m/s^2, counterclockwise positive, torque balance gives

$$0 = TL - Mg(L/2) - mgx, \qquad T = \frac{Mg(L/2) + mgx}{L}.$$

Thus $T = [(20)(9.8)(2.0) + (40)(9.8)(1.0)]/4.0 = 196$ N. Vertical force balance gives $F_p + T - (M + m)g = 0$, so $F_p = (M + m)g - T = 392$ N upward. Without the child, the tension would be 98 N; these are different physical problems.

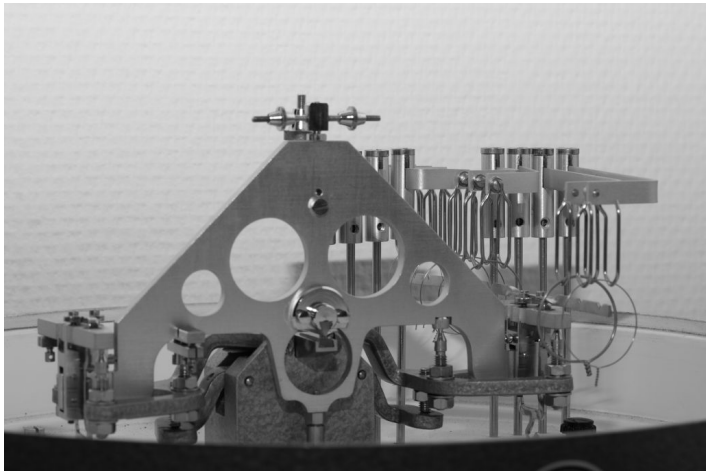


Taking torques about the foot removes both foot forces; the lever arms are 1.5 m for the weight and 4.0 m for the wall force

Worked example (the ladder). A uniform 5.0 m ladder of mass 20 kg leans against a smooth (frictionless) wall. Its top is 4.0 m up the wall and its foot is 3.0 m out. What friction must the ground supply? The wall can only push horizontally, so the foot needs friction toward the wall. Take torques about the foot, where two unknowns act:

$$N_{\text{wall}}(4.0\ \text{m}) - mg(1.5\ \text{m}) = 0, \qquad N_{\text{wall}} = \frac{(196\ \text{N})(1.5\ \text{m})}{4.0\ \text{m}} = 73.5\ \text{N}.$$

Horizontal force balance gives $f = N_{\text{wall}} = 73.5$ N. Vertical balance gives $N_{\text{ground}} = mg = 196$ N. The ladder stays put only if $\mu_s \geq f/N_{\text{ground}} = 0.375$. A person climbing higher adds torque with a longer lever arm, so the friction needed grows as they climb. That is why ladders slip with someone near the top.



A mechanical balance uses torque balance; the photograph alone gives no numerical load

Newton's Second Law in Rotational Form

For a rigid body with constant I about a fixed axis, the component along that axis obeys

$$\sum \tau = I\alpha.$$

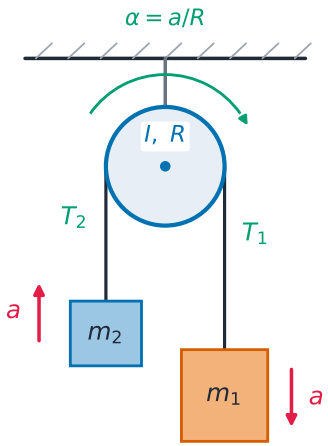
The torque changes angular velocity, not necessarily in the direction of existing rotation. At fixed net torque, doubling I halves α. For $\sum\tau = 6.0$ N m and $I = 2.0$ kg m^2, $\alpha = 3.0$ rad/s^2. If I changes, the simple constant-I model is insufficient; angular-momentum accounting is developed in Unit 6.

Worked example (a falling rod). A uniform rod of mass M and length L is pivoted at one end and released from rest while horizontal. Gravity acts at the centre, $L/2$ from the pivot, and $I = ML^2/3$ about the end. So

$$\alpha = \frac{\tau}{I} = \frac{Mg(L/2)}{ML^2/3} = \frac{3g}{2L}.$$

The free end has $a_t = L\alpha = \frac{3}{2}g \approx 14.7$ m/s^2, more than free fall. A coin resting on that end is left behind as the rod drops away from it. Points closer than $2L/3$ from the pivot accelerate at less than g. This is also part of why a falling chimney cracks: its top would need to fall faster than free fall, so the bricks there must be pulled down by the rest of the chimney.

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$(T_1 - T_2)R = I\alpha$

Ideal axle; taut rope without slip.

Unequal tensions when net torque is nonzero.

For the accelerating ideal-axle pulley shown, the tension difference provides its net torque

Massive-pulley task. Masses $m_1 = 4.0$ kg and $m_2 = 2.0$ kg hang from a light inextensible rope over a **pulley** 滑轮 with $I = 0.50$ kg m^2 and $R = 0.20$ m. The axle has negligible friction, the rope remains taut and does not slip, and m_1 accelerates downward. Use down positive for m_1, up positive for m_2 and clockwise positive for the pulley. Draw each body's forces separately. The pulley has downward rope forces at opposite rims, its weight at the centre and an axle reaction; the last two give zero torque about its centre.

$$m_1 g - T_1 = m_1 a, \qquad T_2 - m_2 g = m_2 a.$$

$$\sum \tau = (T_1 - T_2)R = I\alpha, \qquad a = R\alpha.$$

$$a = \frac{(m_1 - m_2)g}{m_1 + m_2 + I/R^2}.$$

Here $I/R^2 = 12.5$ kg, so $a = 19.6/18.5 = 1.059\ldots$ m/s^2, $T_1 = m_1(g - a) = 34.96\ldots$ N and $T_2 = m_2(g + a) = 21.72\ldots$ N. Keep unrounded a until the tensions are found. The acceleration is less than the ideal massless-pulley value 3.27 m/s^2. For this model, $T_1 - T_2 = Ia/R^2 > 0$. A massive pulley at zero angular acceleration can have equal tensions; finite inertia alone does not prove unequal tension. Axle friction would add a torque term.

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> **Worked example (just after release).** A uniform rod of mass M and length L is pivoted at one end and held horizontal, then released. Find its angular acceleration immediately after release.

1. Torque from gravity acts at the centre of mass: $\tau = Mg\dfrac{L}{2}$.
2. Rotational inertia about the end: $I = \frac{1}{3}ML^2$.
3. $\alpha = \dfrac{\tau}{I} = \dfrac{MgL/2}{ML^2/3} = \dfrac{3g}{2L}$.

The free end then has linear acceleration $a = \alpha L = \frac{3}{2}g$, **more** than g. As the rod swings down, the lever arm of gravity shrinks, so α falls to zero when the rod is vertical. "Immediately after" means the speed is still zero, so only the torque matters.

Measuring rotational inertia. A string is wrapped round a spindle of radius r on a platform and passes over a pulley to a hanging mass m. The mass falls from rest with measured acceleration a.

1. For the mass: $mg - T = ma$, so $T = m(g - a)$.
2. For the platform: $Tr = I\alpha = I\dfrac{a}{r}$.
3. So $I = \dfrac{Tr^2}{a} = mr^2\left(\dfrac{g}{a} - 1\right)$.

To find the inertia of an object added to the platform, measure I with and without it and subtract. The acceleration a comes from timing the fall through a known height: $a = \dfrac{2h}{t^2}$.

Practice

Answer each before reading the answers below.

1. A disk turns with $\theta(t) = 2t^3$ rad. Find ω and α at $t = 1.0$ s.
2. A wheel starts from rest with $\alpha = 4.0$ rad/s^2 for 5.0 s. Find ω, the angle turned, and the speed and centripetal acceleration of a point 0.30 m from the axis.
3. A 20 N force is applied 0.40 m from a bolt at 30° to the wrench. Find the torque.
4. Use the parallel-axis theorem to find the rotational inertia of a uniform rod about one end, given $I_{cm} = \frac{1}{12}ML^2$.
5. A uniform 4.0 m plank of mass 10 kg rests on supports at its two ends. A 20 kg child stands 1.0 m from the left end. Find both support forces.
6. A 1.0 kg mass hangs from a string wrapped around a uniform disk pulley ($M = 2.0$ kg, $I = \frac{1}{2}MR^2$). Find the acceleration of the mass.

Answers

1. $\omega = 6t^2 = 6.0$ rad/s; $\alpha = 12t = 12$ rad/s^2.
2. $\omega = 20$ rad/s; $\theta = \frac{1}{2}(4.0)(5.0)^2 = 50$ rad; $v = \omega r = 6.0$ m/s; $a_c = \omega^2 r = 120$ m/s^2.
3. $\tau = rF\sin\theta = 0.40(20)(0.50) = 4.0$ N $\cdot$ m.
4. $I = \frac{1}{12}ML^2 + M\left(\frac{L}{2}\right)^2 = \frac{1}{3}ML^2$.
5. Torques about the left end: $N_R(4.0) = 10(9.8)(2.0) + 20(9.8)(1.0)$, so $N_R = 98$ N. $N_L = 30(9.8) - 98 = 196$ N.
6. $mg - T = ma$ and $TR = \frac{1}{2}MR^2(a/R)$, so $T = \frac{1}{2}Ma$ and $a = \dfrac{mg}{m + M/2} = \dfrac{9.8}{2.0} = 4.9$ m/s^2.

Exam tips

Choose the axis, object/system and signs before writing equations. Use initial conditions with integrals, and constant-acceleration formulas only when justified. For inertia, state dm, the perpendicular distance and integration limits; use the centre-of-mass axis correctly in the parallel-axis theorem. For combined translation and rotation, write separate laws and the no-slip constraint before eliminating unknowns. Read the actual exam demand; a universal free-body-diagram or full-mark checklist is not a substitute for its prompt.

■ Common mistakes

- Using the full lever arm when the force is at an angle. $\tau = rF\sin\phi$, or F times the perpendicular distance $r_\perp$.
- Using $I = \frac{1}{12}ML^2$ for a rod pivoted at its end. Rotational inertia depends on the axis: about the end it is $\frac{1}{3}ML^2$ (parallel-axis theorem: $I = I_{cm} + Md^2$).
- Writing equal tensions on both sides of a pulley with mass. A net torque needs $T_1 \neq T_2$ when $\alpha \neq 0$.
- Using $a = \alpha R$ when the rope slips or the wheel skids. The link needs rolling or rope motion without slipping.
- Mixing degrees and radians. ω and α must be in rad/s and rad/s^2 before $v = \omega r$ or $a_t = \alpha r$.

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Energy and Momentum of Rotating Systems

AP Physics C: Mechanics

Rotational Kinetic Energy

In 1967 the astronomer Jocelyn Bell noticed radio pulses arriving every 1.34 seconds, as regular as a clock. They came from a **pulsar** – a neutron star only about 20 km across, left behind when a massive star collapsed. A star that turned once every few weeks shrank to roughly a thousandth of its radius, and because its angular momentum stayed the same, it now spins about once every few seconds; some pulsars spin more than 700 times a second. This unit extends energy and momentum – the two great conservation laws – to things that rotate.

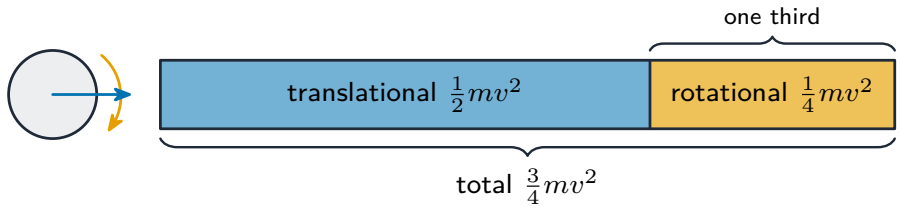
A spinning body has **rotational kinetic energy** 转动动能

$$K_{\text{rot}} = \tfrac{1}{2}I\omega^2,$$

the rotational twin of $\frac{1}{2}mv^2$, with the **rotational inertia** 转动惯量 I playing the role of mass. A body that both moves and spins carries **both** terms:

$$K = \tfrac{1}{2}mv_{\text{cm}}^2 + \tfrac{1}{2}I_{\text{cm}}\omega^2.$$

> **Worked example.** A uniform cylinder ($I = \frac{1}{2}mr^2$) rolls at speed v. Its kinetic energy is $K = \frac{1}{2}mv^2 + \frac{1}{2}\left(\frac{1}{2}mr^2\right)\left(\frac{v}{r}\right)^2 = \frac{3}{4}mv^2$ – one third of it is rotational. The same energy bookkeeping decides every rolling problem.



A rolling cylinder carries both kinds of kinetic energy: one third of its 3/4 mv squared is rotational

Here I_{cm} is about the rotation axis through the center of mass.

A rigid body can have rotational kinetic energy even when its center of mass is **at rest** – a spinning flywheel stays in one place, yet every point on it moves and carries

kinetic energy. Like all energy, K_{rot} is a **scalar**: it has no direction, even though ω does.

Torque and Work

A **torque** 力矩 acting through an **angular displacement** 角位移 does work, and power is torque times angular velocity:

$$W = \int_{\theta_1}^{\theta_2} \tau \, d\theta, \qquad P = \tau\omega.$$

These are the rotational forms of $W = \int F\,dx$ and $P = Fv$ – the whole translational energy toolkit carries over with $F \to \tau$, $x \to \theta$, $v \to \omega$.

Work done by a torque is the **area under a torque–angle graph**, just as work done by a force is the area under a force–position graph.

> **Worked example (variable torque).** A wind-up spring motor exerts a torque that falls steadily from 10 N · m to zero as it unwinds through one full turn (2π rad). The graph is a triangle, so $W = \frac{1}{2}(2\pi)(10) = 31$ J.

> **Worked example.** A motor applies a constant 8.0 N·m torque to a flywheel for 5.0 full turns: $W = \tau\,\Delta\theta = 8.0(5.0)(2\pi) = 250$ J, which appears as rotational kinetic energy if friction is negligible.

Angular Momentum and Angular Impulse

Angular momentum 角动量 for a particle is

$$\vec{L} = \vec{r} \times \vec{p}, \qquad |L| = mvr\sin\theta,$$

so even a particle moving in a straight line has angular momentum about any point not on that line ($L = mv\,d$, with d the perpendicular distance). For a rigid body spinning about a fixed axis, $L = I\omega$. Newton's second law in rotational form is

$$\vec{\tau}_{\text{net}} = \frac{d\vec{L}}{dt} \quad (= I\alpha \text{ when } I \text{ is constant}),$$

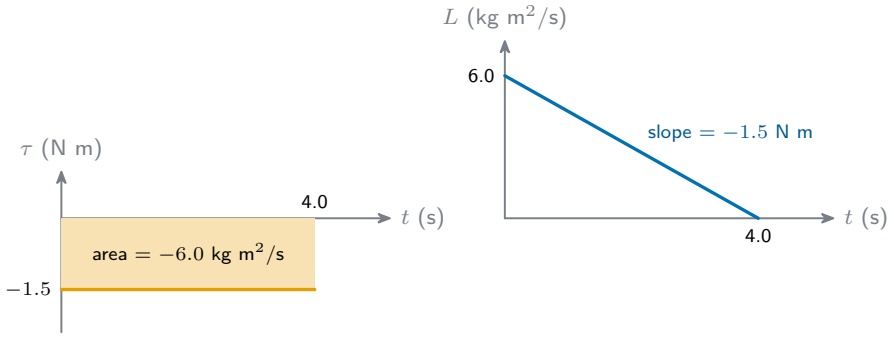
and a net torque acting over time delivers an **angular impulse** 角冲量 $\int \tau\,dt = \Delta L$ – the rotational impulse–momentum theorem. Angular impulse points the same way as the torque that delivers it.

The graphs work exactly like their linear versions:

- the **area under a torque–time graph** is the angular impulse, and so the change in L;
- the **slope of an angular momentum–time graph** is the net torque.

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> **Worked example (angular impulse).** A 0.50 kg · m^2 wheel spinning at 12 rad/s is braked by a constant friction torque of 1.5 N · m. The angular impulse needed to stop it is $\Delta L = 0 - 0.50(12) = -6.0$ kg · m^2/s, so the time is $\dfrac{6.0}{1.5} = 4.0$ s. During this time it turns through $\frac{1}{2}(12)(4.0) = 24$ rad, and the brake does $W = -1.5(24) = -36$ J of work – exactly the wheel's starting kinetic energy, $\frac{1}{2}(0.50)(12)^2 = 36$ J.



Area under the torque-time graph is the change in angular momentum; slope of the angular momentum-time graph is the torque

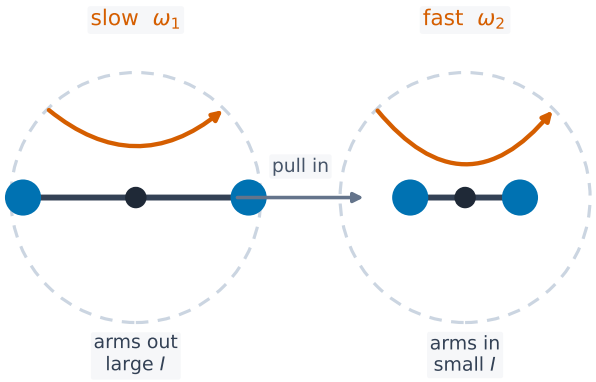
Conservation of Angular Momentum

The angular momentum of a system about an axis is the **sum** of the angular momenta of its parts about that axis. As with linear momentum, angular momentum is conserved in all interactions: any change in a system's L must come from an interaction with its surroundings, through an external torque. With **zero net external torque**, total angular momentum is **conserved** 守恒:

$$L_{\text{before}} = L_{\text{after}}, \qquad I_1\omega_1 = I_2\omega_2 \text{ (one rigid body reshaping).}$$

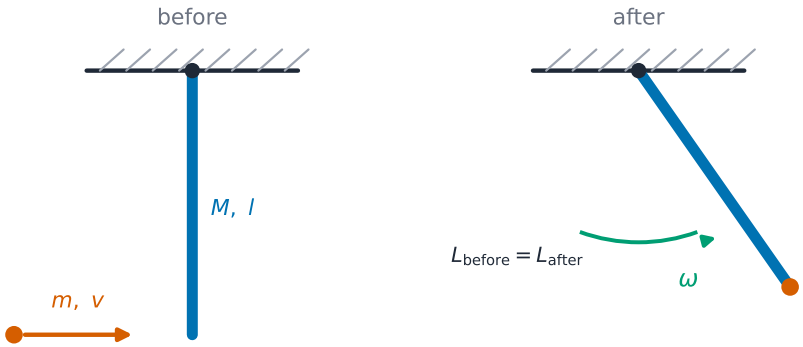
If I shrinks, ω grows – a spinning skater speeds up pulling her arms in. The rule survives collisions and shape changes, which is what makes it so useful: pick the axis so that every external force (gravity, the pivot force) exerts no torque about it.

> **Worked example.** A skater spins at 2.0 rev/s with $I_1 = 4.0$ kg · m^2. Pulling in her arms drops it to $I_2 = 1.6$ kg · m^2: $\omega_2 = \dfrac{4.0}{1.6}(2.0) = 5.0$ rev/s. Her kinetic energy *rises* – the extra energy is the work her muscles do pulling her arms inward.



$L = I\omega$ conserved (no external torque): $I_1\omega_1 = I_2\omega_2$

Pulling mass inward lowers I, so ω rises to conserve $L = I\omega$



$m v l = I_{\text{sys}}\,\omega$ (about pivot; KE not conserved — inelastic)

Angular momentum about the pivot is conserved as the bullet embeds in the rod

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Worked example (rotational collision). A 0.020 kg bullet at 300 m/s strikes the tip of a uniform rod ($M = 1.5$ kg, length $l = 0.60$ m) hanging from a pivot, and embeds. About the pivot, gravity and the pivot force exert no torque during the strike, so L is conserved: $L = mvl = 0.020(300)(0.60) = 3.6\ \text{kg}\cdot\text{m}^2/\text{s}$. Afterwards $I = \frac{1}{3}Ml^2 + ml^2 = 0.18 + 0.0072 = 0.187\ \text{kg}\cdot\text{m}^2$, so $\omega = \dfrac{3.6}{0.187} \approx 19$ rad/s. (Linear momentum is *not* conserved here – the pivot pushes on the rod – and kinetic energy certainly is not: check both before claiming them.)

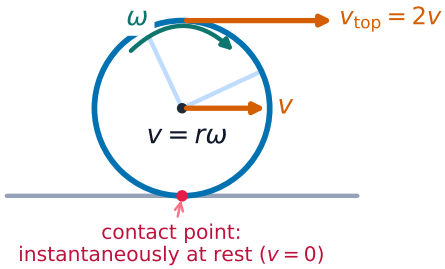


Figure skating and angular momentum: a skater who pulls their arms in during a spin lowers their rotational inertia, so they spin faster

Rolling

Rolling without slipping 无滑滚动 ties translation to rotation: on a stationary surface the contact point is momentarily at rest, so

$$\Delta x_{\text{cm}} = r\,\Delta\theta, \qquad v_{\text{cm}} = r\omega, \qquad a_{\text{cm}} = r\alpha.$$



$$K = \tfrac{1}{2}mv^2 + \tfrac{1}{2}I\omega^2 = \tfrac{1}{2}mv^2\left(1 + \tfrac{I}{mr^2}\right)$$

rolling no slip: $v = r\omega$ · $a = r\alpha$

In rolling without slipping the contact point is at rest, so $v = r\omega$

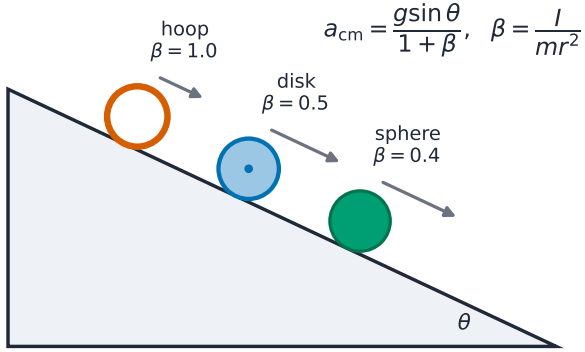
Static friction supplies the torque that keeps the spin matched to the motion, but at a point that is not sliding – so for pure rolling, friction does **no work**, and energy conservation is safe to use. (Rolling friction is beyond the AP course.)

Racing shapes down an incline shows the energy split. With $I = \beta mr^2$, energy conservation gives

$$a_{\text{cm}} = \frac{g\sin\theta}{1+\beta}:$$

a **sphere** ($\beta = \frac{2}{5}$) beats a **disk** 圆盘 ($\beta = \frac{1}{2}$), which beats a **hoop** 圆环 ($\beta = 1$) – mass and radius cancel completely. More of the hoop's energy is locked in rotation, so its center moves slower.

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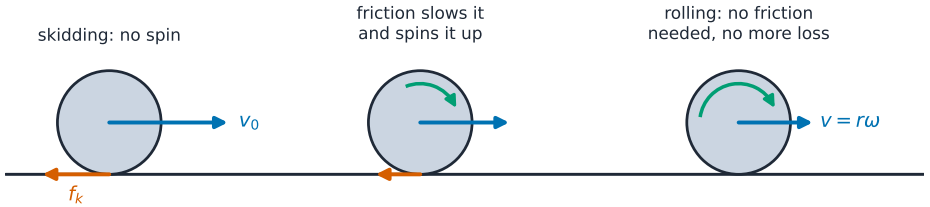


smallest β ⇒ largest a (sphere wins)

rolling race: smaller I reaches bottom first

Rolling race: the shape with the smallest I/mr^2 reaches the bottom first

Slipping. When an object slides as well as spins, $v_{\text{cm}} \neq r\omega$ and the two motions cannot be linked directly. The contact point moves over the surface, so **kinetic friction does dissipate energy** – it turns kinetic energy into heat. For a ball released without spin, kinetic friction then works on both motions at once: it slows the center of mass and its torque changes the spin, until $v = r\omega$ and pure rolling begins.



while it slips, kinetic friction turns kinetic energy into heat

A bowling ball released skidding: kinetic friction slows it and spins it up until it rolls without slipping, after which no more energy is lost

Worked example (skid to roll). A bowling ball (solid sphere, $I = \frac{2}{5}mr^2$) is released at $v_0 = 7.0$ m/s with no spin. The coefficient of kinetic friction is 0.20. Linear: $a = -\mu g = -2.0$ m/s^2, so $v = 7.0 - 2.0t$. Rotational: $\alpha = \dfrac{\mu mgr}{\frac{2}{5}mr^2} = \dfrac{5\mu g}{2r}$, so $r\omega = \frac{5}{2}\mu g\,t = 5.0t$. Rolling starts when $v = r\omega$: $7.0 - 2.0t = 5.0t$, so $t = 1.0$ s and $v = 5.0$ m/s ($= \frac{5}{7}v_0$). Kinetic energy falls from $\frac{1}{2}m(7.0)^2 = 24.5m$ to $\frac{1}{2}m(5.0)^2\left(1 + \frac{2}{5}\right) = 17.5m$: about 29% was dissipated by friction while slipping.

Worked example. A solid sphere rolls from rest down a 30° incline: $a = \dfrac{g\sin 30°}{1 + \frac{2}{5}} = \dfrac{9.8(0.50)}{1.4} = 3.5$ m/s^2, versus 4.9 m/s^2 for a frictionless slider – from the same starting height and rest, rolling objects lose the race against frictionless sliders.

How the energy is shared. For an object rolling without slipping, $v = \omega R$ and $I = \beta MR^2$, so

$$\frac{K_{\text{rot}}}{K_{\text{total}}} = \frac{\frac{1}{2}I\omega^2}{\frac{1}{2}Mv^2 + \frac{1}{2}I\omega^2} = \frac{\beta}{1+\beta}.$$

For a solid cylinder ($\beta = \frac{1}{2}$) the ratio is $\frac{1}{3}$; for a solid sphere ($\beta = \frac{2}{5}$) it is $\frac{2}{7}$; for a hoop ($\beta = 1$) it is $\frac{1}{2}$. The larger the share that goes into rotation, the slower the object moves down a ramp, so a sphere beats a cylinder, which beats a hoop.

Worked example (rolling round a loop). A small solid sphere rolls without slipping down a track and round a vertical loop of radius R (sphere radius much smaller than R). From what height h must it start to just stay on the track at the top?

1. At the top, gravity alone gives the centripetal force: $mg = \dfrac{mv^2}{R}$, so $v^2 = gR$.
2. Energy: $mgh = mg(2R) + \frac{1}{2}mv^2 + \frac{1}{2}\left(\frac{2}{5}mr^2\right)\left(\frac{v}{r}\right)^2 = 2mgR + \frac{7}{10}mgR$.
3. So $h = 2.7R$, higher than the $2.5R$ needed by a block that slides without friction, because some energy goes into spinning.

Worked example (sliding, then rolling). A bowling ball (solid sphere) is thrown along a lane with speed v_0 and no spin. Kinetic friction (μ) slows its centre and makes it spin, until $v = \omega R$ and it rolls.

1. Linear: $v = v_0 - \mu g t$.
2. Rotation: friction's torque $\mu MgR = \frac{2}{5}MR^2\alpha$, so $\alpha = \dfrac{5\mu g}{2R}$ and $\omega R = \frac{5}{2}\mu g t$.
3. Rolling starts when $v_0 - \mu g t = \frac{5}{2}\mu g t$: $t = \dfrac{2v_0}{7\mu g}$, and then $v = \frac{5}{7}v_0$.

Angular momentum about the contact line with the lane is conserved too (friction acts through that line), which gives the same $\frac{5}{7}v_0$ more quickly. Kinetic energy is lost while it slides.

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Motion of Orbiting Satellites

When a satellite orbits a much more massive central object (a planet around the Sun, a spacecraft around Earth), the central object's own motion is negligible, so we treat it as fixed. The satellite's motion is then fixed by the conservation laws.

Gravity supplies the **centripetal force** 向心力 for a **satellite** 卫星 in a circular orbit:

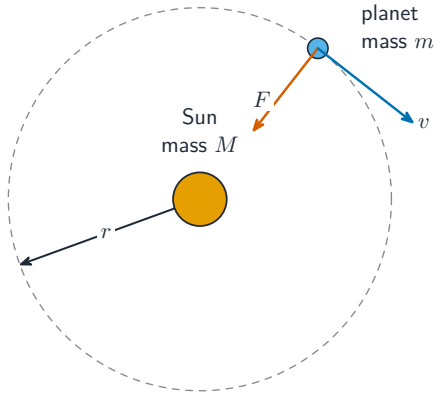
$$\frac{GMm}{r^2} = \frac{mv^2}{r} \quad \Rightarrow \quad v = \sqrt{\frac{GM}{r}}$$

– larger orbits are slower. With **gravitational potential energy** 引力势能 $U_g = -\dfrac{GMm}{r}$, a circular orbit obeys $K = -\frac{1}{2}U$, so the **total mechanical energy** 总机械能 is

$$E = K + U = \frac{U}{2} = -\frac{GMm}{2r},$$

negative because the satellite is bound. To escape from radius r, total energy must reach zero, giving the **escape velocity** 逃逸速度

$$v_{\text{esc}} = \sqrt{\frac{2GM}{r}}.$$



Gravity provides the centripetal force that keeps a satellite in orbit

In an **elliptical orbit** 椭圆轨道, E and L are fixed: gravity points at the focus, so it exerts no torque about it. Conserved L means the satellite sweeps equal areas in

equal times – **Kepler's second law** 开普勒第二定律 – and therefore moves fastest at closest approach, slowest at the far point. Energy conservation connects speeds at the two ends.

> **Worked example.** For a satellite at $r = 7.0 \times 10^6$ m around Earth ($GM = 4.0 \times 10^{14}$ m^3/s^2): orbital speed $v = \sqrt{GM/r} = 7.6 \times 10^3$ m/s, while escaping from that radius needs $v_{\text{esc}} = \sqrt{2GM/r} = 1.1 \times 10^4$ m/s – exactly $\sqrt{2}$ times the circular speed.

> **Worked example (elliptical orbit).** A satellite moves on an ellipse around Earth ($GM = 4.0 \times 10^{14}$ m^3/s^2). At closest approach it is 7.0×10^6 m from Earth's centre, moving at 8.7×10^3 m/s; at its farthest it is 1.4×10^7 m away. Find its speed there. At both ends the velocity is perpendicular to the radius, so $L = mvr$ and $v_2 = \dfrac{v_1 r_1}{r_2} = \dfrac{8.7 \times 10^3 (7.0 \times 10^6)}{1.4 \times 10^7} = 4.4 \times 10^3$ m/s. Check with energy per kilogram: $\tfrac{1}{2}(8.7 \times 10^3)^2 - \dfrac{4.0 \times 10^{14}}{7.0 \times 10^6} = -1.9 \times 10^7$ J/kg, and $\tfrac{1}{2}(4.4 \times 10^3)^2 - \dfrac{4.0 \times 10^{14}}{1.4 \times 10^7} = -1.9 \times 10^7$ J/kg. The two agree.

Exam skill. Orbit FRQs are energy-and-angular-momentum problems in disguise: write $E = \tfrac{1}{2}mv^2 - \dfrac{GMm}{r}$ and $L = mvr\sin\theta$ at the two points of interest and solve the pair – never assume the orbit formulae for circles apply to an ellipse.

Practice

Answer each before reading the answers below.

1. A solid sphere ($I = \tfrac{2}{5}MR^2$) rolls without slipping from rest down a height of 1.4 m. Find its speed at the bottom.
2. A skater spinning at 2.0 rad/s with $I = 4.0$ kg $\cdot$ m^2 pulls in her arms to $I = 1.6$ kg $\cdot$ m^2. Find the new angular speed and the change in kinetic energy. Where does the energy come from?
3. A 2.0 kg particle moves at 3.0 m/s along a line that passes 4.0 m from a point P. Find its angular momentum about P.
4. A satellite's orbital radius is made four times larger. By what factor does its period change?
5. A bullet embeds in a rod hanging from a pivot. Which is conserved during the collision: linear momentum, angular momentum about the pivot, or kinetic energy? Explain.
6. A constant torque of 5.0 N $\cdot$ m turns a wheel through 10 rad. Find the work done.

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Answers

1. $mgh = \frac{1}{2}mv^2 + \frac{1}{2}\left(\frac{2}{5}mR^2\right)\left(\frac{v}{R}\right)^2 = \frac{7}{10}mv^2$, so $v = \sqrt{\frac{10}{7}gh} = 4.4$ m/s.
2. $\omega = \dfrac{4.0(2.0)}{1.6} = 5.0$ rad/s. KE rises from 8.0 J to 20 J (+12 J), from the work her muscles do pulling her arms in.
3. $L = mvr_\perp = 2.0(3.0)(4.0) = 24$ kg $\cdot$ m^2/s.
4. $T^2 \propto r^3$, so T increases by $4^{3/2} = 8$.
5. Only angular momentum about the pivot: the pivot exerts a large external force (so linear momentum changes) but no torque about itself. The collision is inelastic, so kinetic energy is not conserved.
6. $W = \tau\theta = 50$ J.

Exam tips

- Compute the **moment of inertia** $I = \int r^2\,dm$ and shift axes with the **parallel-axis theorem** $I = I_{cm} + Md^2$.
- Rotational kinetic energy is $\frac{1}{2}I\omega^2$; a rolling body has both translational and rotational KE.
- Conserve **angular momentum** $L = I\omega$ when net external torque is zero (a spinning skater pulling in).
- Use energy conservation for rolling-without-slipping problems ($v = r\omega$ ties the two motions).
- Know standard I values (hoop, disk, rod, sphere) and where the axis is.

Common mistakes

- Conserving linear momentum when a rod is hit while hanging from a pivot. The pivot pushes during the collision; conserve **angular** momentum about the pivot instead.
- Leaving out rotational kinetic energy for a rolling object. $K = \frac{1}{2}Mv^2 + \frac{1}{2}I\omega^2$ with $\omega = v/R$.
- Saying static friction on a rolling wheel wastes energy. With no slipping on a stationary surface, the contact point is at rest, so static friction does no work.
- Using circular-orbit formulas for an ellipse. Use conservation of energy and of angular momentum between the two points.
- Forgetting that a skater who pulls in her arms *gains* kinetic energy: L is conserved, not K.
- Using $v = r\omega$ while an object is still skidding. It holds only once rolling without slipping has begun.

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Oscillations

AP Physics C: Mechanics

Defining Simple Harmonic Motion

Near the top of Taipei 101, once the world's tallest building, a steel ball weighing 660 tonnes hangs from cables between the 87th and 92nd floors. When typhoon winds push the tower, the ball swings the opposite way and its dampers soak up the energy, cutting the building's sway by up to 40%. Engineers tuned the ball's swing to match the tower's own natural period. The tower, the ball and the tiny quartz crystal that keeps time in your phone all follow the same equation: the equation of simple harmonic motion.

Simple harmonic motion 简谐运动 (SHM) is a special case of **periodic motion** 周期运动. It appears whenever two things are true: there is an **equilibrium** 平衡位置 position where the net force is zero, and displacing the object produces a **restoring force** 回复力 – a force pointing back toward equilibrium – whose magnitude is *proportional* to the displacement:

$$F = -k\,\Delta x.$$

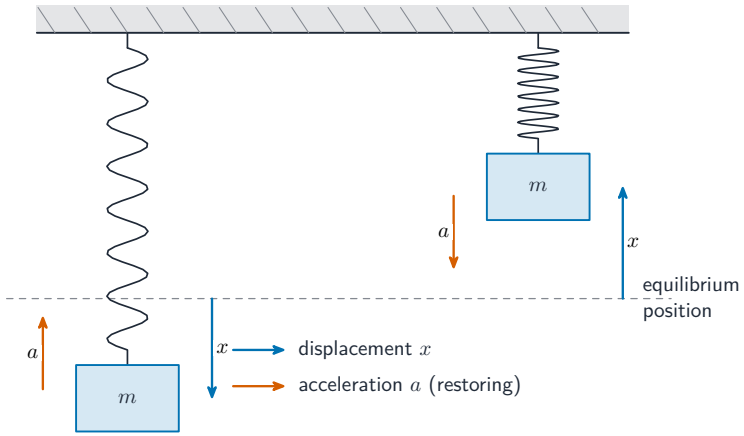
Newton's second law then gives the defining **differential equation** 微分方程:

$$\frac{d^2x}{dt^2} = -\frac{k}{m}\,x = -\omega^2 x, \qquad \omega = \sqrt{\frac{k}{m}}.$$

Its solution is sinusoidal, $x(t) = A\cos(\omega t + \phi)$. You do not need to prove this solution – you need to *recognise* the equation: any system whose motion obeys "$\ddot{x} = -\omega^2 x$" is a simple harmonic oscillator, whatever it is made of. (A mass hanging on a vertical spring works the same way: gravity only shifts the equilibrium point; the oscillation about it is unchanged.)

Is it SHM? Write the net force (or torque) as a function of displacement from equilibrium and check that it is **proportional to the displacement and opposite to it**:

- $F = -50x$: yes, $k = 50$ N/m.
- $F = -4x^3$: no – the restoring force is not proportional to x, so the period depends on amplitude.
- A ball bouncing on the floor: periodic, but not SHM – the force is constant (mg) while it is in the air.
- A pendulum at large angles: the restoring torque is proportional to $\sin\theta$, not θ, so it is only approximately SHM for small swings.



In SHM the acceleration always points back towards equilibrium, opposite the displacement

Frequency and Period of SHM

The **angular frequency** 角频率 ω sets the **period** 周期 and **frequency** 频率:

$$T = \frac{2\pi}{\omega} = \frac{1}{f}.$$

For the two standard systems:

$$T_{\text{spring}} = 2\pi\sqrt{\frac{m}{k}}, \qquad T_{\text{pendulum}} = 2\pi\sqrt{\frac{\ell}{g}}.$$

Two classic conceptual traps: the period of SHM never depends on the **amplitude** 振幅, and each system ignores one obvious variable – the pendulum's period does not involve its mass, and the spring's period does not involve g (a spring–block oscillator keeps perfect time in orbit; a pendulum clock does not).

Representing and Analyzing SHM

Differentiate $x(t) = A\cos(\omega t + \phi)$ twice:

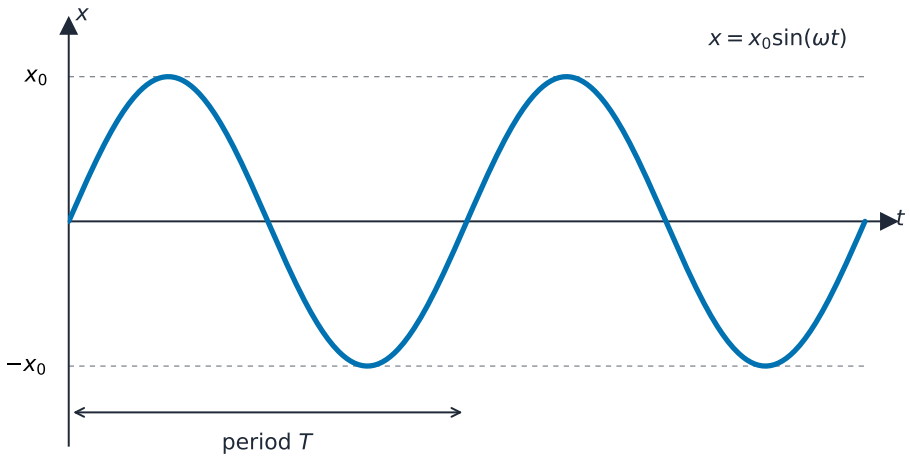
$$v = -A\omega\sin(\omega t + \phi), \qquad a = -A\omega^2\cos(\omega t + \phi) = -\omega^2 x.$$

So at the ends of the swing ($x = \pm A$) the speed is zero and the acceleration is largest; passing through equilibrium ($x = 0$) the acceleration is zero and the speed is largest, $v_{\max} = A\omega$. The amplitude A and the **phase constant** 相位常数 ϕ come from the

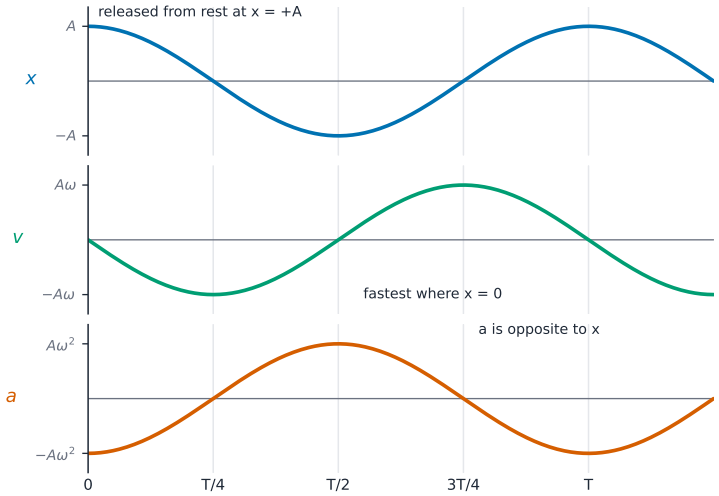
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initial conditions: where the object starts and how fast it is moving. Eliminating t (or using energy, below) gives signed velocity as a function of position:

$$v = \pm\omega\sqrt{A^2 - x^2}.$$



Displacement varies sinusoidally with time in simple harmonic motion



Position, velocity and acceleration against time for SHM released from rest at $x = +A$: velocity is a quarter cycle out of step with position, and acceleration is always opposite to position

Read these graphs the same way as in Unit 1: the slope of the x–t graph is v, and the slope of the v–t graph is a. Where x is at a peak, its slope is zero, so $v = 0$; where x crosses zero most steeply, $|v|$ is largest. The a–t graph is the x–t graph flipped and scaled by ω^2.

> **Worked example.** A 0.50 kg mass on a $k = 200$ N/m spring: $\omega = \sqrt{k/m} = 20$ rad/s, $T = 2\pi/\omega = 0.31$ s. With $A = 0.10$ m: $v_{\max} = A\omega = 2.0$ m/s at equilibrium, and at $x = 0.050$ m the speed is $v = \omega\sqrt{A^2 - x^2} = 20\sqrt{0.10^2 - 0.050^2} = 1.7$ m/s.

> **Worked example (phase from initial conditions).** The same oscillator ($\omega = 20$ rad/s) is at equilibrium at $t = 0$ and moving in the $+x$ direction at 2.0 m/s. Since $x(0) = A\cos\phi = 0$, $\phi = \pm\pi/2$. Since $v(0) = -A\omega\sin\phi > 0$, $\sin\phi < 0$, so $\phi = -\pi/2$, and $A = v_{\max}/\omega = 0.10$ m. So $x = 0.10\cos(20t - \pi/2) = 0.10\sin(20t)$ m. Starting at equilibrium gives a sine; starting at an end gives a cosine.

> **Worked example (vertical spring).** A 0.20 kg mass hangs at rest on a spring and stretches it by 0.050 m. It is then pulled down a further 0.030 m and released. At equilibrium $kx_0 = mg$, so $k = \dfrac{0.20(10)}{0.050} = 40$ N/m. $\omega = \sqrt{40/0.20} = 14$ rad/s and $T = 2\pi/\omega = 0.44$ s. The mass oscillates with amplitude 0.030 m **about the new equilibrium**, 0.050 m below the spring's natural length; gravity changed where the centre of the motion is, not the period.

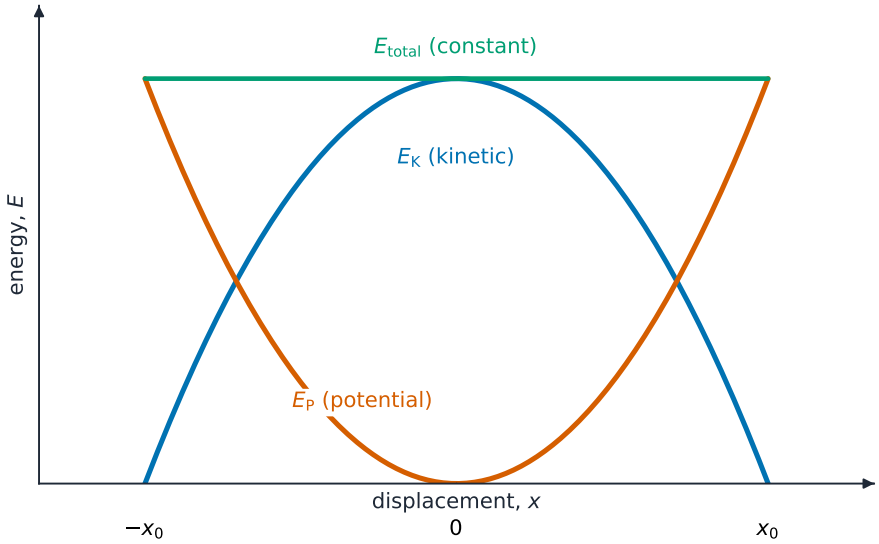
Energy of Simple Harmonic Oscillators

The total energy of the oscillator is the sum $E = U + K$, and with no friction it is constant – energy just trades back and forth between the spring's potential energy and the mass's kinetic energy:

$$E = \tfrac{1}{2}kA^2 = \tfrac{1}{2}kx^2 + \tfrac{1}{2}mv^2.$$

All potential at the ends (where $K = 0$), all kinetic at equilibrium (where U is minimum). Because $E \propto A^2$, doubling the amplitude quadruples the energy.

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Kinetic and potential energy swap over a cycle while the total energy stays constant

> **Worked example.** Where is the energy split evenly? Set $\frac{1}{2}kx^2 = \frac{1}{2}E = \frac{1}{4}kA^2$, so $x = \pm A/\sqrt{2} \approx \pm 0.71A$ – much closer to the end than to the middle. For the system above ($A = 0.10$ m): $x = \pm 0.071$ m.

Left alone, a system oscillates at its own **natural frequency** 固有频率 (set by k and m: $f_0 = \frac{1}{2\pi}\sqrt{k/m}$). With weak damping, a periodic driving force near this frequency can produce a large amplitude – **resonance** 共振. Well-timed pushes add energy to the motion, while damping removes it. The precise peak frequency depends on damping; for weak damping it is close to f_0 (as when pumping a swing in time). Resonance can be useful (tuning a radio, MRI scanners, musical instruments) or dangerous (soldiers break step on bridges). The Tacoma Narrows Bridge collapse in 1940 involved wind-driven aeroelastic flutter, rather than simple forced resonance. A **tuned mass damper** like the one in Taipei 101 is an oscillator tuned to the building's natural frequency so that it absorbs energy at exactly the frequency where the building would otherwise resonate.

Simple and Physical Pendulums

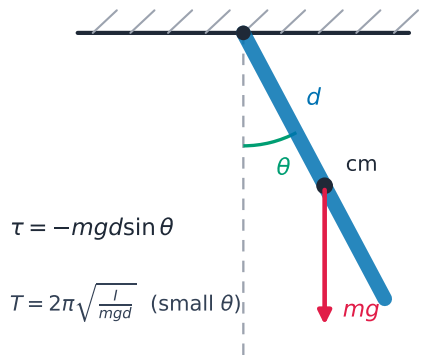
A **physical pendulum** 物理摆 is any rigid body swinging about a fixed pivot. Displace it by an angle θ and gravity, acting at the center of mass a distance d from the pivot, supplies a restoring torque

$$\tau = -mgd\sin\theta.$$

For small angles, apply the **small-angle approximation** 小角度近似 $\sin\theta \approx \theta$ and Newton's second law in rotational form ($\tau = I\alpha$):

$$\frac{d^2\theta}{dt^2} = -\frac{mgd}{I}\,\theta = -\omega^2\theta \quad \Rightarrow \quad T_{\text{phys}} = 2\pi\sqrt{\frac{I}{mgd}}.$$

This is the same "$\ddot\theta = -\omega^2\theta$" pattern as before – recognising it *is* the derivation. A **simple pendulum** 单摆 is the special case of a point mass on a light string: $I = m\ell^2$ and $d = \ell$ give $T = 2\pi\sqrt{\ell/g}$.



physical pendulum: $T = 2\pi\sqrt{I/mgd}$

Gravity acting at the center of mass provides a physical pendulum's restoring torque

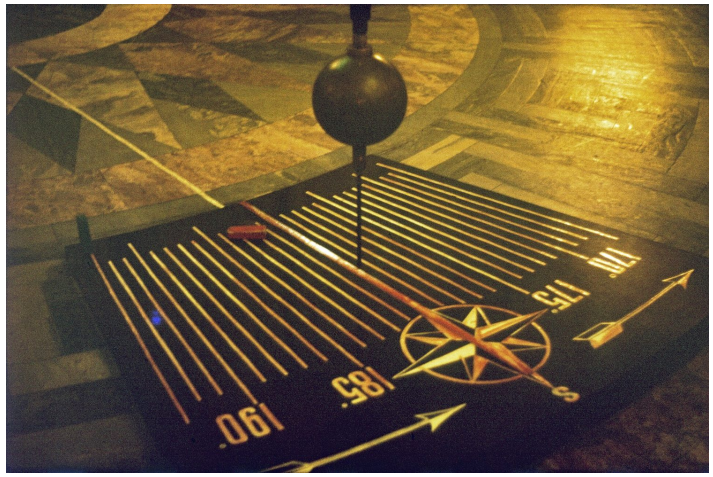
Worked example. A uniform rod (mass M, length L) swings from one end: $I = \frac{1}{3}ML^2$, $d = \frac{L}{2}$, so

$$T = 2\pi\sqrt{\frac{\frac{1}{3}ML^2}{Mg\frac{L}{2}}} = 2\pi\sqrt{\frac{2L}{3g}}$$

– shorter than a simple pendulum of length L, because the rod's mass sits nearer the pivot.

A **torsion pendulum** 扭摆 – a disk hanging from a wire that twists – is SHM one more time: the wire's restoring torque is proportional to the twist angle, $I\alpha = -\kappa\,\Delta\theta$, giving $T = 2\pi\sqrt{I/\kappa}$.

Exam skill. To derive a small-angle gravitational pendulum's period: write the restoring torque about the pivot, apply the small-angle approximation, and match the result to $\ddot\theta = -\omega^2\theta$ to read off ω. State the small-angle step explicitly – it justifies the SHM model and explains why large-amplitude swings are *not* simple harmonic.



A Foucault pendulum: a long, heavy pendulum whose swing plane slowly turns as Earth rotates beneath it

Worked example (springs in series). A block of mass m hangs from N identical springs joined end to end, each of constant k.

1. In series the springs share the stretch, so $k_{\text{eq}} = \dfrac{k}{N}$.
2. Period: $T = 2\pi\sqrt{\dfrac{m}{k_{\text{eq}}}} = 2\pi\sqrt{\dfrac{mN}{k}}$, so $T \propto \sqrt{N}$.
3. To test this, plot T^2 against N: a straight line through the origin with slope $\dfrac{4\pi^2 m}{k}$. So $k = \dfrac{4\pi^2 m}{\text{slope}}$. With $m = 1.5$ kg and a slope of 0.20 s^2, $k \approx 300$ N/m.

If a manufacturer's k is larger than the measured value, a likely cause is that the springs' own mass, ignored in the formula, adds to the mass that oscillates and makes T longer.

Torsional pendulum. A disk hung on a wire that twists back has restoring torque $\tau = -\kappa\,\Delta\theta$, so $\alpha = -\dfrac{\kappa}{I}\Delta\theta$: SHM with $T = 2\pi\sqrt{\dfrac{I}{\kappa}}$. It has the same form as a block on a spring, with I in place of m and κ in place of k. Its energy is $U = \frac{1}{2}\kappa(\Delta\theta)^2$.

Collision, then oscillation. A block on a spring is hit by a second block that sticks to it. Momentum (not kinetic energy) is conserved in the collision; afterwards mechanical energy is conserved. The oscillating mass is now larger, so the **period**

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increases: $T = 2\pi\sqrt{\dfrac{m_1+m_2}{k}}$. The amplitude comes from the energy just after the collision: $\frac{1}{2}kA^2 = \frac{1}{2}(m_1+m_2)v^2$, if the collision happens at the equilibrium position.

Practice

Answer each before reading the answers below.

1. A mass moves with $x(t) = 0.20\cos(5.0t)$ (m, s). Find the amplitude, period, maximum speed and maximum acceleration.
2. Find the period of a 0.50 kg mass on a spring with $k = 200$ N/m.
3. A simple pendulum has a period of 2.0 s on Earth. Find its period on the Moon, where g is one-sixth as large.
4. A spring with $k = 200$ N/m oscillates with amplitude 0.10 m. Find the total energy and the kinetic energy at $x = 0.05$ m.
5. Show that a uniform rod pivoted at one end has period $T = 2\pi\sqrt{\dfrac{2L}{3g}}$ for small swings.
6. Show that a mass on a spring obeys $\dfrac{d^2x}{dt^2} = -\omega^2 x$, and identify ω.

Answers

1. $A = 0.20$ m; $T = 2\pi/5.0 = 1.26$ s; $v_{\max} = A\omega = 1.0$ m/s; $a_{\max} = A\omega^2 = 5.0$ m/s^2.
2. $T = 2\pi\sqrt{0.50/200} = 0.31$ s.
3. $T \propto 1/\sqrt{g}$: $2.0\sqrt{6} = 4.9$ s.
4. $E = \frac{1}{2}(200)(0.10)^2 = 1.0$ J; $U = \frac{1}{2}(200)(0.05)^2 = 0.25$ J, so $K = 0.75$ J.
5. $T = 2\pi\sqrt{\dfrac{I}{mgd}}$ with $I = \frac{1}{3}mL^2$ and $d = L/2$: $T = 2\pi\sqrt{\dfrac{mL^2/3}{mgL/2}} = 2\pi\sqrt{\dfrac{2L}{3g}}$.
6. Newton's second law: $m\dfrac{d^2x}{dt^2} = -kx$, so $\dfrac{d^2x}{dt^2} = -\dfrac{k}{m}x$ with $\omega = \sqrt{k/m}$.

Exam tips

- Identify SHM from a **linear restoring force** $F = -kx$, which gives $\frac{d^2x}{dt^2} = -\omega^2 x$ with $\omega = \sqrt{k/m}$.
- Write the solution $x = A\cos(\omega t + \phi)$ and get v, a by differentiating; period $T = \frac{2\pi}{\omega}$ is amplitude-independent.
- Energy trades between $\frac{1}{2}kx^2$ and $\frac{1}{2}mv^2$, with total $\frac{1}{2}kA^2$.
- For a pendulum, use the small-angle approximation $\sin\theta \approx \theta$ to reach SHM.
- Match phase to the start: released from rest at maximum displacement uses cosine.

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Common mistakes

- Writing $\omega = f$. Angular frequency is $\omega = 2\pi f = 2\pi/T$.
- Saying a larger amplitude gives a longer period. For SHM the period depends only on m and k (or L and g).
- Using the full length of a physical pendulum for d. In $T = 2\pi\sqrt{I/(mgd)}$, d is the distance from the pivot to the centre of mass.
- Using $T = 2\pi\sqrt{L/g}$ at large angles. It relies on the small-angle approximation $\sin\theta \approx \theta$.
- Starting from sin when the mass is released from rest at $x = A$. That start needs $x = A\cos(\omega t)$.