Oscillations

AP Physics C: Mechanics

Defining Simple Harmonic Motion

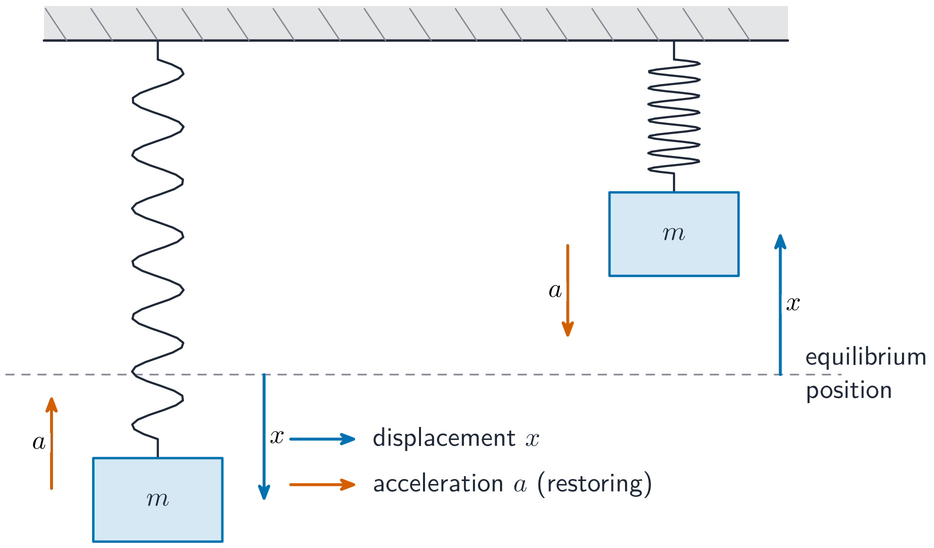
Simple harmonic motion 简谐运动 (SHM) is a special case of **periodic motion** 周期运动. It appears whenever two things are true: there is an **equilibrium** 平衡位置 position where the net force is zero, and displacing the object produces a **restoring force** 回复力–a force pointing back toward equilibrium –whose magnitude is *proportional* to the displacement:

$$F = -k\,\Delta x.$$

Newton's second law then gives the defining **differential equation** 微分方程:

$$\frac{d^2x}{dt^2} = -\frac{k}{m}\,x = -\omega^2 x, \qquad \omega = \sqrt{\frac{k}{m}}.$$

Its solution is sinusoidal, $x(t) = A\cos(\omega t + \phi)$. You do not need to prove this solution –you need to *recognise* the equation: any system whose motion obeys ”$\ddot{x} = -\omega^2 x$” is a simple harmonic oscillator, whatever it is made of. (A mass hanging on a vertical spring works the same way: gravity only shifts the equilibrium point; the oscillation about it is unchanged.)



In SHM the acceleration always points back towards equilibrium, opposite the displacement

Frequency and Period of SHM

The **angular frequency** 角频率 ω sets the **period** 周期 and **frequency** 频率:

$$T = \frac{2\pi}{\omega} = \frac{1}{f}.$$

For the two standard systems:

$$T_{\text{spring}} = 2\pi\sqrt{\frac{m}{k}}, \qquad T_{\text{pendulum}} = 2\pi\sqrt{\frac{\ell}{g}}.$$

Two classic conceptual traps: the period of SHM never depends on the **amplitude** 振幅, and each system ignores one obvious variable –the pendulum's period does not involve its mass, and the spring's period does not involve g (a spring–block oscillator keeps perfect time in orbit; a pendulum clock does not).

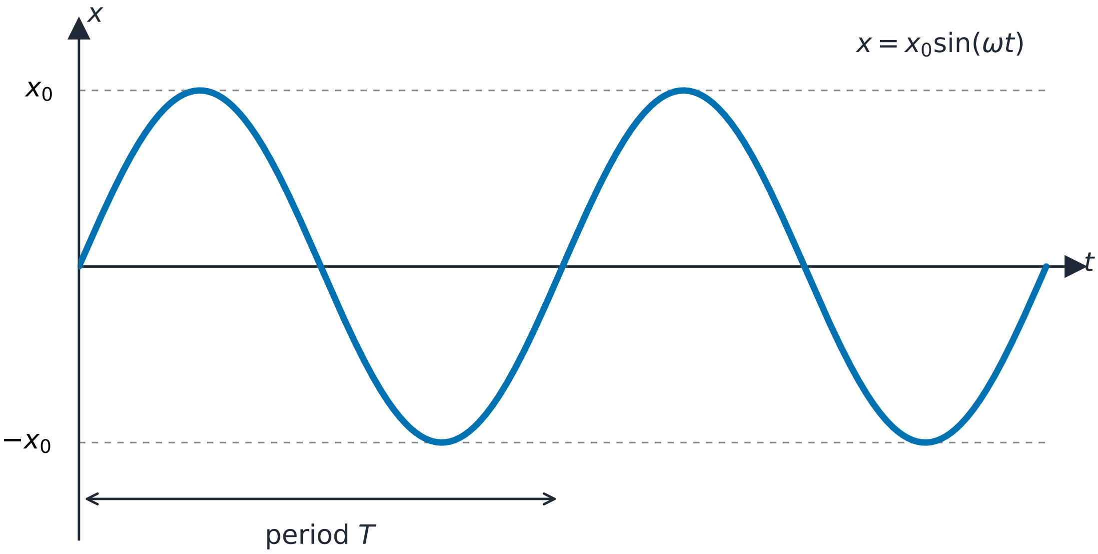
Representing and Analyzing SHM

Differentiate $x(t) = A\cos(\omega t + \phi)$ twice:

$$v = -A\omega\sin(\omega t + \phi), \qquad a = -A\omega^2\cos(\omega t + \phi) = -\omega^2 x.$$

So at the ends of the swing ($x = \pm A$) the speed is zero and the acceleration is largest; passing through equilibrium ($x = 0$) the acceleration is zero and the speed is largest, $v_{\max} = A\omega$. The amplitude A and the **phase constant** 相位常数 ϕ come from the initial conditions: where the object starts and how fast it is moving. Eliminating t (or using energy, below) gives speed as a function of position:

$$v = \pm\,\omega\sqrt{A^2 - x^2}.$$



Displacement varies sinusoidally with time in simple harmonic motion

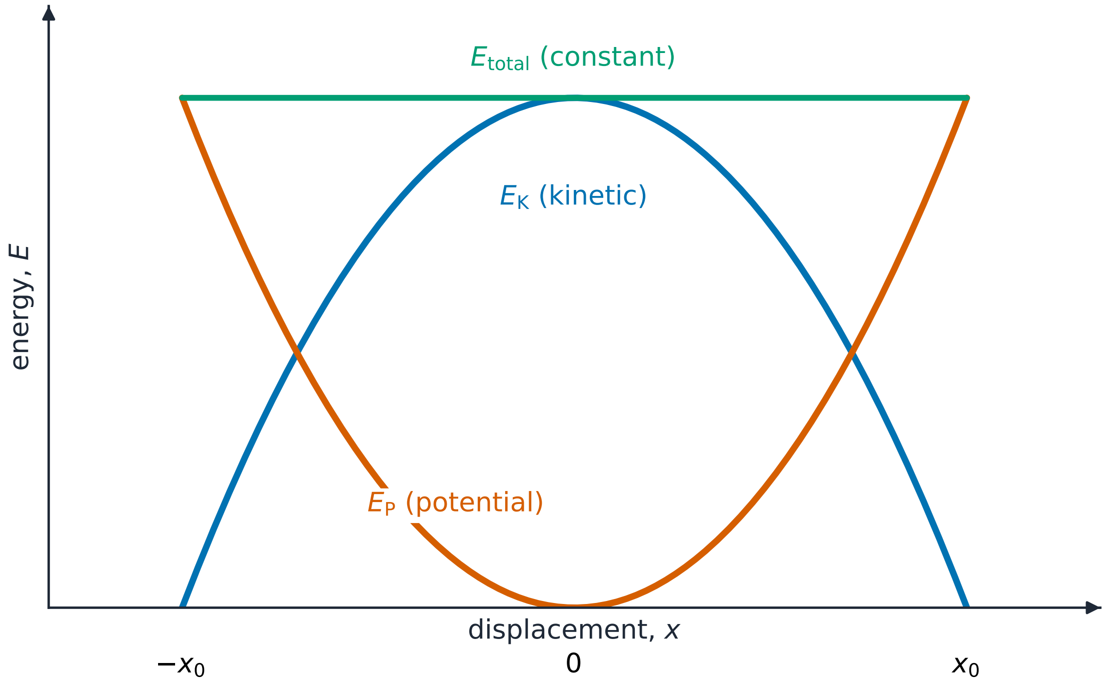
Worked example. A 0.50 kg mass on a $k = 200$ N/m spring: $\omega = \sqrt{k/m} = 20$ rad/s, $T = 2\pi/\omega = 0.31$ s. With $A = 0.10$ m: $v_{\max} = A\omega = 2.0$ m/s at equilibrium, and at $x = 0.050$ m the speed is $v = \omega\sqrt{A^2 - x^2} = 20\sqrt{0.10^2 - 0.050^2} = 1.7$ m/s.

Energy of Simple Harmonic Oscillators

The total energy of the oscillator is the sum $E = U + K$, and with no friction it is constant –energy just trades back and forth between the spring's potential energy and the mass's kinetic energy:

$$E = \tfrac{1}{2}kA^2 = \tfrac{1}{2}kx^2 + \tfrac{1}{2}mv^2.$$

All potential at the ends (where $K = 0$), all kinetic at equilibrium (where U is minimum). Because $E \propto A^2$, doubling the amplitude quadruples the energy.



Kinetic and potential energy swap over a cycle while the total energy stays constant

Worked example. Where is the energy split evenly? Set $\frac{1}{2}kx^2 = \frac{1}{2}E = \frac{1}{4}kA^2$, so $x = A/\sqrt{2} \approx 0.71A$ –much closer to the end than to the middle. For the system above ($A = 0.10$ m): $x = 0.071$ m.

Simple and Physical Pendulums

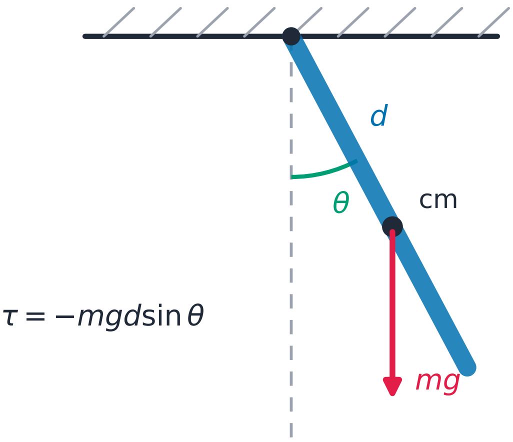
A **physical pendulum** 物理摆 is any rigid body swinging about a fixed pivot. Displace it by an angle θ and gravity, acting at the center of mass a distance d from the pivot, supplies a restoring torque

$$\tau = -mgd\sin\theta.$$

For small angles, apply the **small-angle approximation** 小角度近似 $\sin\theta \approx \theta$ and Newton's second law in rotational form ($\tau = I\alpha$):

$$\frac{d^2\theta}{dt^2} = -\frac{mgd}{I}\,\theta = -\omega^2\theta \quad \Rightarrow \quad T_{\text{phys}} = 2\pi\sqrt{\frac{I}{mgd}}.$$

This is the same "$\ddot{\theta} = -\omega^2\theta$" pattern as before –recognising it *is* the derivation. A **simple pendulum** 单摆 is the special case of a point mass on a light string: $I = m\ell^2$ and $d = \ell$ give $T = 2\pi\sqrt{\ell/g}$.



Gravity acting at the center of mass provides a physical pendulum's restoring torque

Worked example. A uniform rod (mass M, length L) swings from one end: $I = \frac{1}{3}ML^2$, $d = \frac{L}{2}$, so

$$T = 2\pi\sqrt{\frac{\frac{1}{3}ML^2}{Mg\,\frac{L}{2}}} = 2\pi\sqrt{\frac{2L}{3g}}$$

–shorter than a simple pendulum of length L, because the rod's mass sits nearer the pivot.

A **torsion pendulum** 扭摆–a disk hanging from a wire that twists –is SHM one more time: the wire's restoring torque is proportional to the twist angle, $I\alpha = -\kappa\,\Delta\theta$, giving $T = 2\pi\sqrt{I/\kappa}$.

Exam skill. Every pendulum FRQ wants the same three steps: write the restoring torque about the pivot, apply the small-angle approximation, and match the result to $\ddot{\theta} = -\omega^2\theta$ to read off ω. State the small-angle step explicitly –it is a scored point, and it is why large-amplitude swings are *not* simple harmonic.

Exam tips

- Identify SHM from a **linear restoring force** $F = -kx$, which gives $\frac{d^2x}{dt^2} = -\omega^2 x$ with $\omega = \sqrt{k/m}$.
- Write the solution $x = A\cos(\omega t + \phi)$ and get v, a by differentiating; period $T = \frac{2\pi}{\omega}$ is amplitude-independent.
- Energy trades between $\frac{1}{2}kx^2$ and $\frac{1}{2}mv^2$, with total $\frac{1}{2}kA^2$.
- For a pendulum, use the small-angle approximation $\sin\theta \approx \theta$ to reach SHM.
- Match phase to the start: released from rest at maximum displacement uses cosine.