Week 24

AP Physics C: Mechanics

Homework bundle for this week

本周作业合集

exercises.lol

Contents 目录

AP Physics C: Mechanics Name: ____________ Score: ______

1. A wheel with $I = 2\ \text{kg} \cdot \text{m}^2$ spins at $\omega = 3$ rad/s. Its rotational kinetic energy (in J)?

   ____________ J

2. A constant torque of $6\ \text{N} \cdot \text{m}$ turns a wheel through $\theta = 4$ rad. The work done (in J)?

   ____________ J

3. A flywheel with $I = 5\ \text{kg} \cdot \text{m}^2$ spins at $\omega = 4$ rad/s. Its angular momentum (in $\text{kg} \cdot \text{m}^2/\text{s}$)?

   ____________ $\text{kg} \cdot \text{m}^2/\text{s}$

4. A skater with $I_i = 8\ \text{kg} \cdot \text{m}^2$ spins at 2 rad/s, then pulls in to $I_f = 4\ \text{kg} \cdot \text{m}^2$. Her new angular speed (in rad/s)?

   ____________ rad/s

5. A wheel of radius 0.3 m rolls without slipping at $\omega = 10$ rad/s. Its forward speed (in m/s)?

   ____________ m/s

6. A satellite moves to a lower circular orbit. Its orbital speed...

   ☐ increases

   ☐ decreases

   ☐ stays the same

7. If you triple a wheel's angular speed but keep its rotational inertia, its rotational kinetic energy becomes...

   ☐ 3 times larger

   ☐ 6 times larger

   ☐ 9 times larger

   ☐ unchanged

8. Angular momentum is conserved exactly when...

   ☐ the net external torque is zero

   ☐ the object is at rest

   ☐ no forces act at all

9. For a wheel rolling without slipping, the point touching the ground is momentarily...

   ☐ at rest

   ☐ moving at v

☐ moving at $2v$

10. What provides the centripetal force that holds a satellite in a circular orbit?

    ☐ gravity

    ☐ air resistance

    ☐ the satellite's engine

AP Physics C: Mechanics

1. **9 J**

   $K = \frac{1}{2}I\omega^2 = \frac{1}{2}(2)(3)^2 = 9$ J.

2. **24 J**

   $W = \tau\theta = 6 \times 4 = 24$ J.

3. **20 $\mathbf{kg \cdot m^2/s}$**

   $L = I\omega = 5 \times 4 = 20\ \mathrm{kg \cdot m^2/s}$.

4. **4 rad/s**

   $I_i\omega_i = I_f\omega_f \Rightarrow 8 \times 2 = 4 \times \omega_f$, so $\omega_f = 4$ rad/s.

5. **3 m/s**

   $v = \omega r = 10 \times 0.3 = 3$ m/s.

6. **increases**

   $v = \sqrt{GM/r}$, so a smaller r gives a larger v.

7. **9 times larger**

   ω is squared, so $3^2 = 9$ times the energy.

8. **the net external torque is zero**

   Zero net external torque means $dL/dt = 0$, so L stays constant.

9. **at rest**

   The contact point has zero velocity at that instant – that is the no-slip condition.

10. **gravity**

    $GMm/r^2 = mv^2/r$ – gravity is the centripetal force.

6.1 Rotational Kinetic Energy

Name: ______________ Class: __________ Date: ____________ **Total: 14 marks**

KEY WORDS rotational kinetic energy 转动动能

Objective

Build the skills to answer exam questions on **rotational kinetic energy**.

You must be able to:

- use **rotational kinetic energy 转动动能**: $K = \frac{1}{2}I\omega^2$
- recognise it as the analogue of $\frac{1}{2}mv^2$
- add translational and rotational kinetic energy for a rolling object
- find K, I, or ω given the others

1 Worked examples

Study these first. Each one shows the method for a question type used later — follow the steps and you can do the Practice and Exam-style questions yourself.

The rotational KE formula

$$K = \tfrac{1}{2}I\omega^2.$$

A disc with $I = 0.5$ kg m^2 spinning at $\omega = 4$ rad s^{-1} stores

$$K = \tfrac{1}{2}(0.5)(4)^2 = 4 \text{ J}.$$

The square of omega

Because ω is squared, doubling the spin rate quadruples the stored energy.

Rolling: two kinds of KE

A rolling object has both, so its total is

$$K = \tfrac{1}{2}mv^2 + \tfrac{1}{2}I\omega^2.$$

2 Practice

Now apply the methods above.

2.1 A wheel with $I = 2$ kg m^2 spins at 3 rad s^{-1}. Find its rotational KE. [2]

2.2 If a spinning object's angular speed triples, by what factor does its rotational KE change? [1]

2.3 State the two terms in the total kinetic energy of a rolling object. [2]

2.4 A flywheel stores 18 J of rotational KE at $I = 4$ kg m^2. Find its angular speed. [2]

3 Exam-style questions

3.1 Rotational kinetic energy is [1]

- **A** $\frac{1}{2}mv^2$
- **B** $\frac{1}{2}I\omega^2$
- **C** $I\omega$
- **D** $\frac{1}{2}kx^2$

3.2 A rolling ball's total kinetic energy is [1]

- **A** only $\frac{1}{2}mv^2$
- **B** only $\frac{1}{2}I\omega^2$
- **C** $\frac{1}{2}mv^2 + \frac{1}{2}I\omega^2$
- **D** zero

3.3 A solid disc has $I = 0.8$ kg m^2 and spins at 5 rad s^{-1}.

(a) Find its rotational kinetic energy. [2]

(b) State its energy if the spin rate doubles. [1]

3.4 A rolling wheel has translational KE = 12 J and rotational KE = 6 J.

(a) Find the total kinetic energy. [1]

(b) State the fraction stored in rotation. [1]

Solutions

2.1 $K = \frac{1}{2}I\omega^2 = \frac{1}{2}(2)(3)^2 = 9$ J.

2.2 Nine times (ω is squared).

2.3 Translational $\frac{1}{2}mv^2$ and rotational $\frac{1}{2}I\omega^2$.

2.4 $\omega = \sqrt{2K/I} = \sqrt{2(18)/4} = \sqrt{9} = 3$ rad s^{-1}.

3.1 **B** — $K = \frac{1}{2}I\omega^2$.

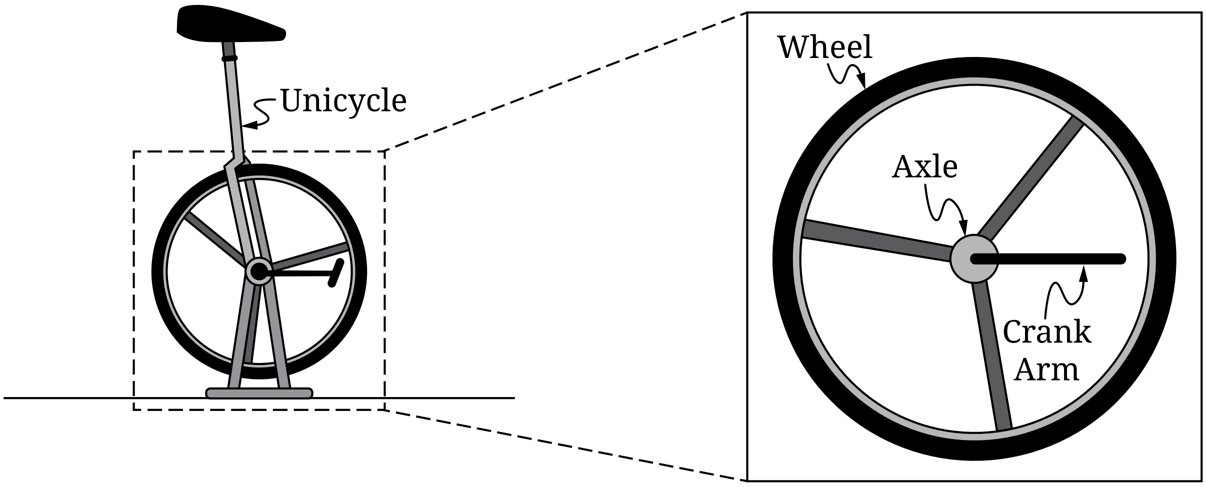
3.2 **C** — a rolling object has both terms.

3.3 (a) $K = \frac{1}{2}(0.8)(5)^2 = 10$ J. (b) Four times, 40 J.

3.4 (a) $12 + 6 = 18$ J. (b) $6/18 = \frac{1}{3}$.

Question 4

4. A unicycle is placed on a stand such that the wheel does not touch the ground. The rotational inertia of the wheel about the axle is I_W. One end of a crank arm can be attached to the wheel, as shown in the side view.



Side View

In Scenarios 1 and 2, crank arms of different lengths are connected to the wheel. In each scenario, the wheel is initially at rest. A force of magnitude F is then exerted at the end of each crank arm. The direction of the force remains perpendicular to each crank arm at all times. The mass of each crank arm is negligible. The scenarios differ as described.

- Scenario 1: The length of the crank arm is ℓ_1. The angular speed of the wheel immediately after the crank arm has been turned through one rotation is ω_1.
- Scenario 2: The length of the crank arm is ℓ_2, where $\ell_2 > \ell_1$. The angular speed of the wheel immediately after the longer crank arm has been turned through one rotation is ω_2.

Part A

Indicate whether ω_2 is greater than, less than, or equal to ω_1 by writing one of the following.

- $\omega_2 > \omega_1$
- $\omega_2 < \omega_1$
- $\omega_2 = \omega_1$

Justify your answer. Your justification may reference equations but must include conceptual reasoning beyond algebraic solutions.

Part B

Derive an expression for the angular speed ω_1 of the wheel in Scenario 1 immediately after the crank arm has been turned through one rotation. Express your answer in terms of I_W, F, ℓ_1, and physical constants, as appropriate. Begin your derivation by writing a fundamental physics principle or an equation from the reference information.

Part C

In Scenario 3, the unicycle is placed on the ground such that the unicycle can move forward without the wheel slipping on the ground. A force of the same magnitude F is exerted at the end of the crank arm of length ℓ_1, as in Scenario 1. The direction of the force remains perpendicular to the crank arm at all times. The resulting angular speed of the wheel immediately after the crank arm has been turned through one rotation is ω_3.

Indicate whether ω_3 is greater than, less than, or equal to ω_1 by writing one of the following.

- $\omega_3 > \omega_1$
- $\omega_3 < \omega_1$
- $\omega_3 = \omega_1$

Briefly **justify** your answer. Your justification may reference equations but must include conceptual reasoning beyond algebraic solutions.

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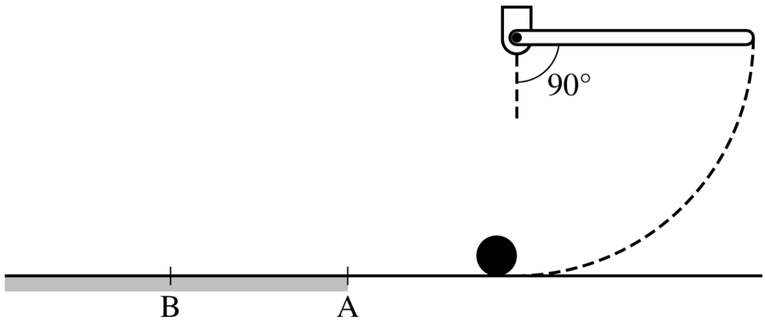

Note: Figure not drawn to scale.

Figure 1

3. A system consists of a small sphere of mass m and radius R at rest on a horizontal surface and a uniform rod of mass $M = 2m$ and length ℓ attached at one end to a pivot with negligible friction, where $R \ll \ell$. There is negligible friction between the surface and the sphere to the right of Point A and nonnegligible friction to the left of Point A. The rod is held horizontally as shown in Figure 1, then is released from rest. The total rotational inertia of the rod about the pivot is $\frac{1}{3}M\ell^2$ and the rotational inertia of the sphere about its center is $\frac{2}{5}mR^2$. After the rod is released, the rod swings down and strikes the sphere head-on. As a result of this collision, the rod is stopped, and the ball initially slides without rotating to the left across the horizontal surface.

(a) Derive an expression for the angular speed of the rod just before striking the sphere in terms of the length ℓ and physical constants as appropriate.

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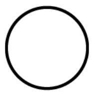
Continue your response to **QUESTION 3** on this page.

(b) Derive an expression for the linear speed v_0 of the sphere immediately after colliding with the rod in terms of the length ℓ and physical constants as appropriate.

After sliding a short distance, at time $t = 0$ the sphere encounters a region of the horizontal surface with a coefficient of kinetic friction μ, beginning at Point A as indicated in Figure 1. The sphere begins rotating while sliding and eventually begins rolling without sliding at Point B, also as indicated.

(c) In the following diagram, which represents the sphere while the sphere is traveling between Points A and B, draw and label the forces (not components) that act on the sphere. Each force must be represented by a distinct arrow starting on, and pointing away from, the point of application on the sphere.



(d) Derive an expression for each of the following as the sphere is rotating and sliding between points A and B in terms of v_0, μ, R, t, and physical constants as appropriate.

i. The linear velocity v of the center of mass of the sphere as a function of time t

ii. The angular velocity ω of the sphere as a function of time t

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Continue your response to **QUESTION 3** on this page.

(e)

i. Derive an expression for the time it takes the sphere to travel from Point A to Point B in terms of v_0, μ, and physical constants as appropriate.

ii. Derive an expression for the linear velocity of the sphere upon reaching Point B in terms of v_0.

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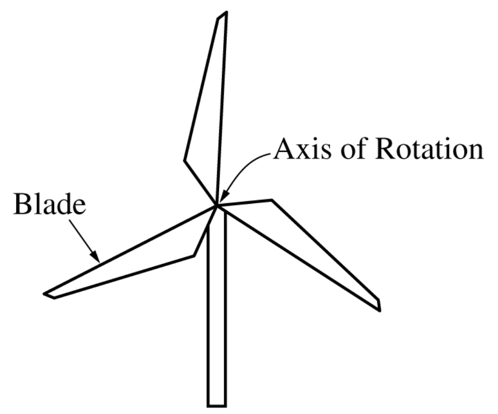

Figure 1

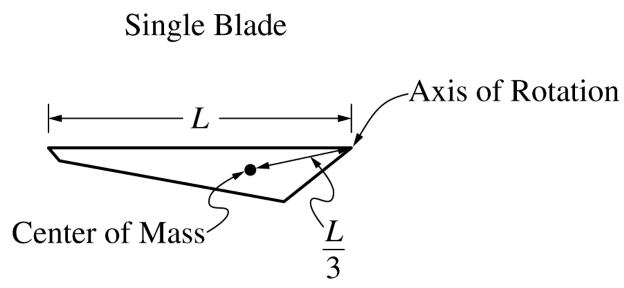

Figure 2

. A wind turbine includes a three-blade system that rotates about an axis through the end of each blade, as shown in Figure 1. Each blade has a length L and mass M, with a center of mass located at a distance $\frac{L}{3}$ from the axis of rotation, as shown in Figure 2.

(a) Derive an expression for the rotational inertia of the three-blade system. Express your answer in terms of M, L, and physical constants, as appropriate. The rotational inertia of each blade about an axis through its center of mass is given by the equation $I_{cm} = \frac{1}{18}ML^2$.

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Continue your response to **QUESTION 3** on this page.

(b) While the wind blows, the three-blade system operates at a constant angular speed $\omega_0 = 2.6\ \text{rad/s}$. The length of one blade is $L = 36\ \text{m}$. The numerical value of the rotational inertia of the system is $I_{sys} = 6.7 \times 10^6\ \text{kg}\cdot\text{m}^2$. Calculate the time T it takes the outer edge of a single blade to complete one revolution.

(c) When the wind stops blowing, the angular speed of the system decreases. The angular speed ω of the system while slowing down is given as a function of time t by the equation $\omega = \omega_0 e^{-\beta_0 t}$, where β_0 is a constant with appropriate units, as shown on the graph in Figure 3.

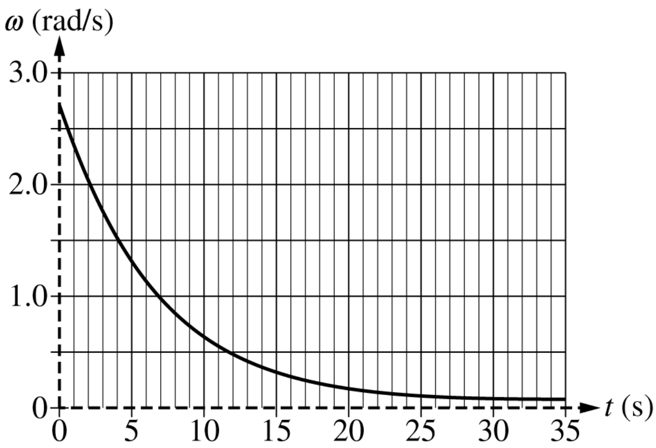

Figure 3

i. Calculate the amount of energy dissipated from $t = 0$, when the wind stops blowing, until the system comes to rest.

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Continue your response to **QUESTION 3** on this page.

ii. Derive an expression for the net torque exerted on the system as a function of t as the system slows down. Express your answers in terms of β_0, ω_0, M, L, I_{sys}, and physical constants, as appropriate.

iii. Derive an expression for the angular displacement of the system $\Delta\theta$ as a function of t. Express your answer in terms of β_0, ω_0, M, L, I_{sys}, and physical constants, as appropriate.

The three-blade system is now replaced with a second three-blade system identical to the first, except that the second three-blade system slows down according to the equation $\omega = \omega_0 e^{-\beta t}$, where $\omega_0 = 2.6\text{ rad/s}$ and $\beta > \beta_0$. The original angular speed function is shown as a dashed line in Figure 4.

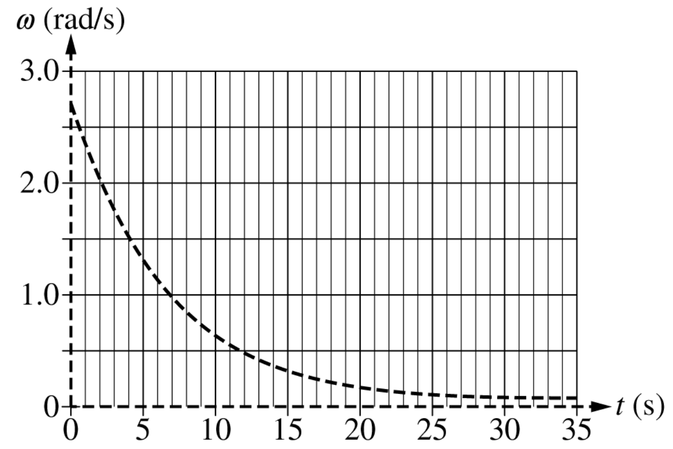

Figure 4

(d) On the graph in Figure 4, sketch the angular speed of the second three-blade system as a function of time t.

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END OF EXAM

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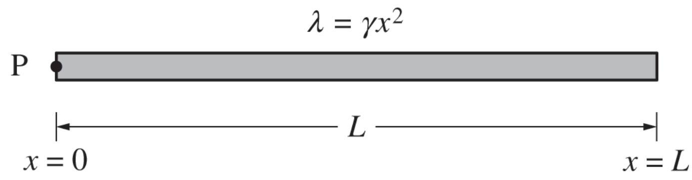

3. A triangular rod of length L and mass M has a nonuniform linear mass density given by the equation $\lambda = \gamma x^2$, where $\gamma = \dfrac{3M}{L^3}$ and x is the distance from point P at the left end of the rod.

(a) Using integral calculus, show that the rotational inertia I of the rod about an axis perpendicular to the page and through point P is $\dfrac{3}{5}ML^2$.

(b) Determine the horizontal location of the center of mass of the rod relative to point P. Express your answer in terms of L.

(c) For an axis perpendicular to the page, is the value of the rotational inertia of the rod around point P greater than, less than, or equal to the value of the rotational inertia of the rod around the rod's center of mass?

_____ Greater than _____ Less than _____ Equal to

Justify your answer.

The rod is released from rest in the position shown, and the rod begins to rotate about a horizontal axis perpendicular to the page and through point P.

(d) On the axes below, sketch graphs of the magnitude of the net torque τ on the rod and the angular speed ω of the rod as functions of time t from the time the rod is released until the time its center of mass reaches its lowest point.

(e) As the rod rotates from the horizontal position down through vertical, is the magnitude of the angular acceleration on the rod increasing, decreasing, or not changing?

_____ Increasing _____ Decreasing _____ Not changing

Justify your answer.

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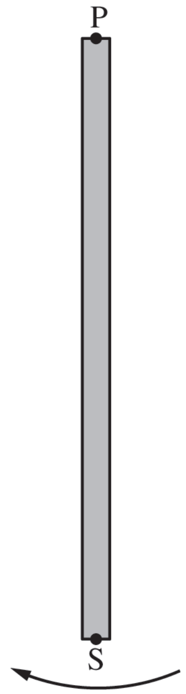

(f) The mass of the rod is 3.0 kg, and the length of the rod is 1.0 m. Calculate the linear speed v of point S as the rod swings through the vertical position shown.

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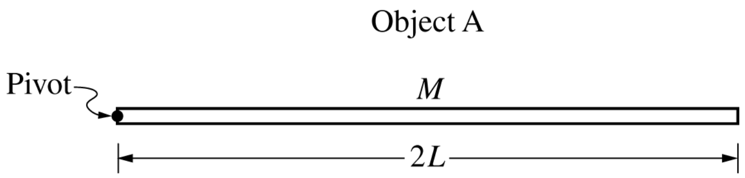
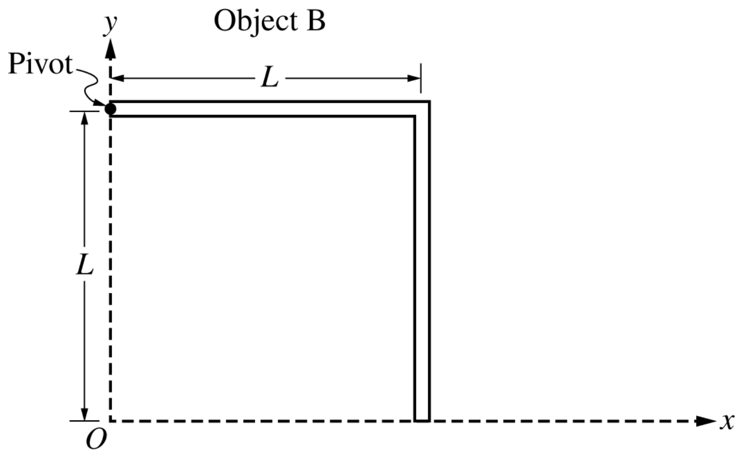


2. Object A is a long, thin, uniform rod of mass M and length $2L$ that is free to rotate about a pivot of negligible friction at its left end, as shown above.

(a) Using integral calculus, derive an expression to show that the rotational inertia I_A of object A about the pivot is given by $\frac{4}{3}ML^2$.



Object B of total mass M is formed by attaching two thin, uniform, identical rods of length L at a right angle to each other. Object B is held in place, as shown above. Express your answers in part (b) in terms of L .

(b) Determine the following for the given coordinate system shown in the figure.

i. The x-coordinate of the center of mass of object B

ii. The y-coordinate of the center of mass of object B

GO ON TO THE NEXT PAGE.

Object B has a rotational inertia of I_B about its pivot.

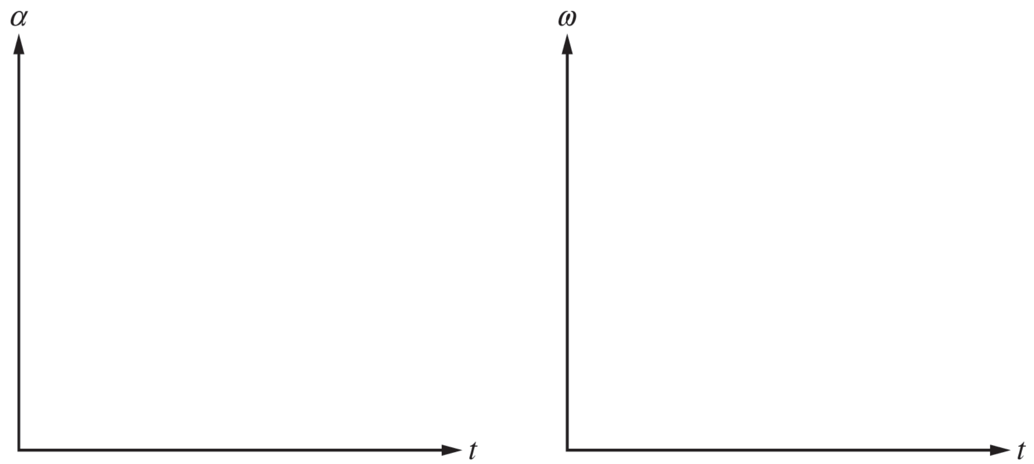
(c) Is the value of I_B greater than, less than, or equal to I_A ?

_____ Greater than _____ Less than _____ Equal to

Justify your answer.

Object B is released from rest and begins to rotate about its pivot.

(d) On the axes below, sketch graphs of the magnitude of the angular acceleration α and the angular speed ω of object B as functions of time t from the time it is released to the time its center of mass reaches its lowest point.



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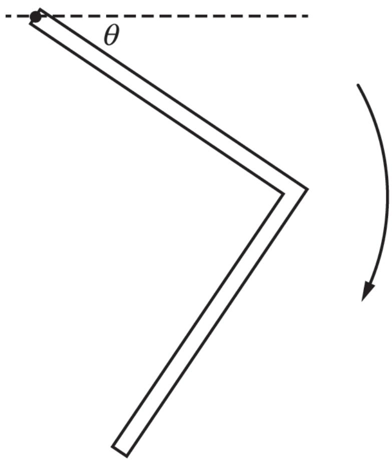
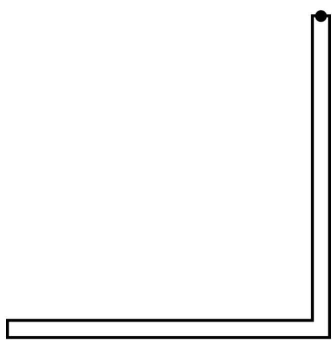


(e) While object B rotates from the horizontal position down through the angle θ shown above, is the magnitude of its angular acceleration increasing, decreasing, or not changing?

_____ Increasing _____ Decreasing _____ Not changing

Justify your answer.



Object B rotates through the position shown above.

(f) Derive an expression for the angular speed of object B when it is in the position shown above. Express your answer in terms of M, L, I_B, and physical constants, as appropriate.

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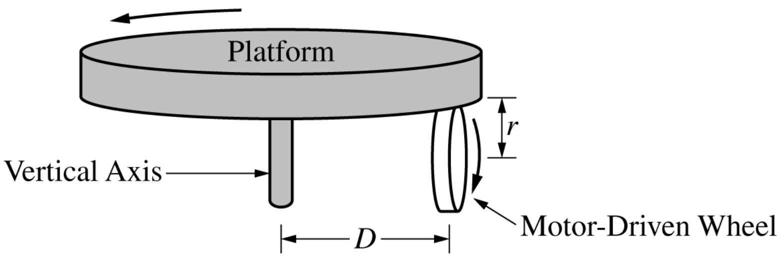
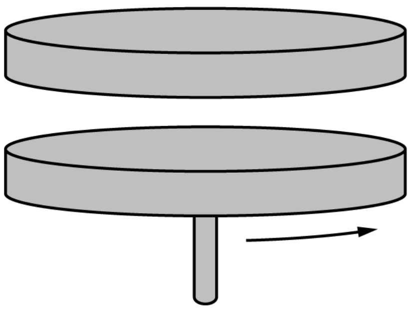


3. A horizontal circular platform with rotational inertia I_P rotates freely without friction on a vertical axis. A small motor-driven wheel that is used to rotate the platform is mounted under the platform and touches it. The wheel has radius r and touches the platform a distance D from the vertical axis of the platform, as shown above. The platform starts at rest, and the wheel exerts a constant horizontal force of magnitude F tangent to the wheel until the platform reaches an angular speed ω_P after time Δt. During time Δt, the wheel stays in contact with the platform without slipping.

(a) Derive an expression for the angular speed ω_P of the platform. Express your answer in terms of I_P, r, D, F, Δt, and physical constants, as appropriate.

(b) Determine an expression for the kinetic energy of the platform at the moment it reaches angular speed ω_P. Express your answer in terms of I_P, r, D, F, Δt, and physical constants, as appropriate.

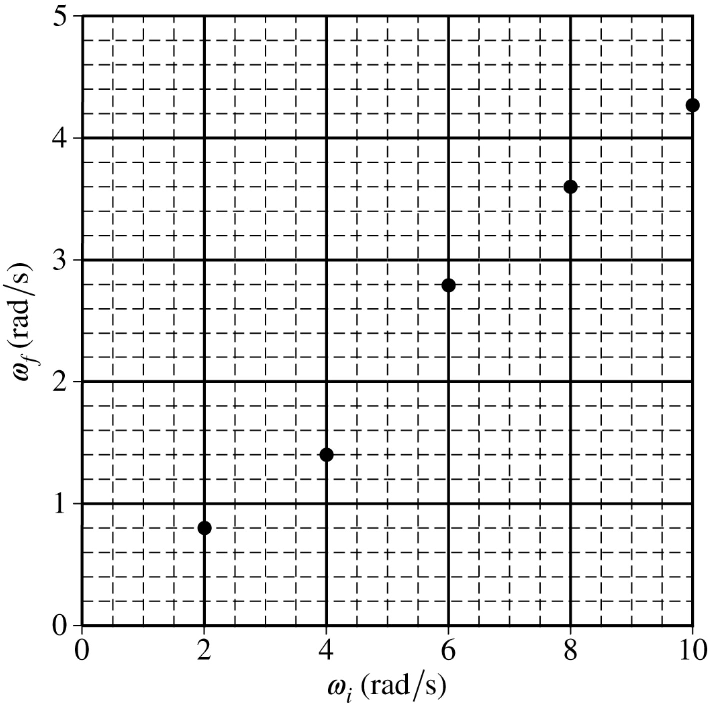
(c) Derive an expression for the angular speed of the wheel ω_W when the platform has reached angular speed ω_P. Express your answer in terms of D, r, ω_P, and physical constants, as appropriate.



When the platform is spinning at angular speed ω_P, the motor-driven wheel is removed. A student holds a disk directly above and concentric with the platform, as shown above. The disk has the same rotational inertia I_P as the platform. The student releases the disk from rest, and the disk falls onto the platform. After a short time, the disk and platform are observed to be rotating together at angular speed ω_f.

(d) Derive an expression for ω_f. Express your answer in terms of ω_P, I_P, and physical constants, as appropriate.

A student now uses the rotating platform $\left(I_P = 3.1\ \text{kg}\bullet\text{m}^2\right)$ to determine the rotational inertia I_U of an unknown object about a vertical axis that passes through the object's center of mass. The platform is rotating at an initial angular speed ω_i when the unknown object is dropped with its center of mass directly above the center of the platform. The platform and object are observed to be rotating together at angular speed ω_f. Trials are repeated for different values of ω_i. A graph of ω_f as a function of ω_i is shown on the axes below.



(e)

i. On the graph on the previous page, draw a best-fit line for the data.

ii. Using the straight line, calculate the rotational inertia of the unknown object I_U about a vertical axis passing through its center of mass.

(f) The kinetic energy of the spinning platform before the object is dropped on it is K_i. The total kinetic energy of the platform-object system when it reaches angular speed ω_f is K_f. Which of the following expressions is true?

____ $K_f < K_i$ ____ $K_f = K_i$ ____ $K_f > K_i$

Justify your answer.

(g) One of the students observes that the center of mass of the object is not actually aligned with the axis of the platform. Is the experimental value of I_U obtained in part (e) greater than, less than, or equal to the actual value of the rotational inertia of the unknown object about a vertical axis that passes through its center of mass?

____ Greater than ____ Less than ____ Equal to

Justify your answer.

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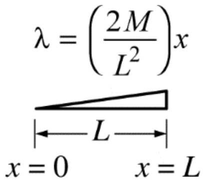


3. A triangular rod, shown above, has length L, mass M, and a nonuniform linear mass density given by the equation $\lambda = \dfrac{2M}{L^2}x$, where x is the distance from one end of the rod.

   (a) Using integral calculus, show that the rotational inertia of the rod about its left end is $ML^2/2$.

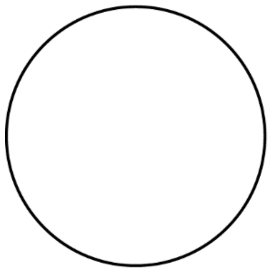


Figure 1

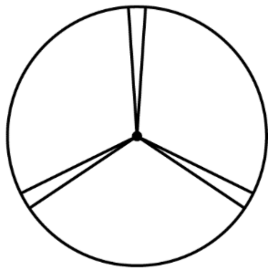


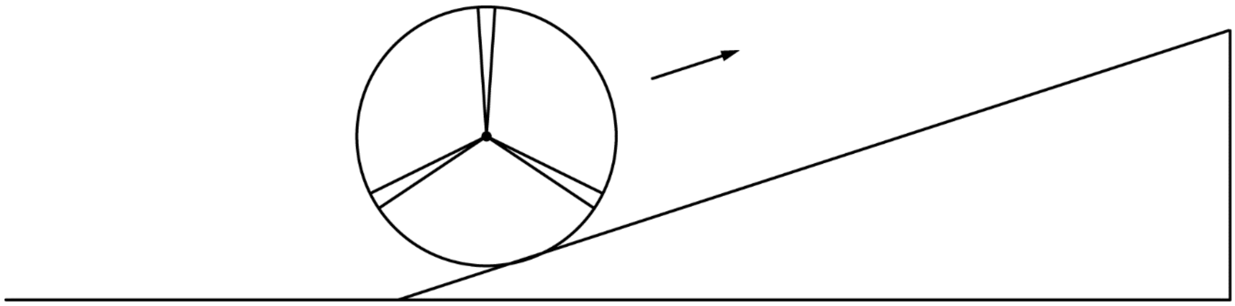
Figure 2

The thin hoop shown above in Figure 1 has a mass M, radius L, and a rotational inertia around its center of ML^2 . Three rods identical to the rod from part (a) are now fastened to the thin hoop, as shown in Figure 2 above.

   (b) Derive an expression for the rotational inertia I_{tot} of the hoop-rods system about the center of the hoop. Express your answer in terms of M, L, and physical constants, as appropriate.

The hoop-rods system is initially at rest and held in place but is free to rotate around its center. A constant force F is exerted tangent to the hoop for a time Δt .

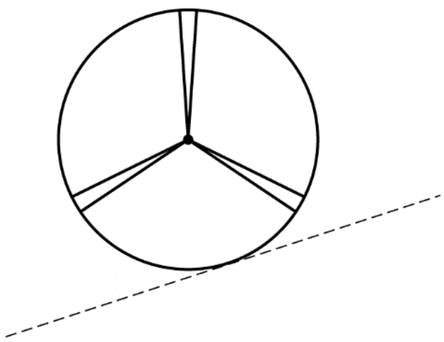
   (c) Derive an expression for the final angular speed ω of the hoop-rods system. Express your answer in terms of M, L, F, Δt , and physical constants, as appropriate.



The hoop-rods system is rolling without slipping along a level horizontal surface with the angular speed ω found in part (c). At time $t = 0$, the system begins rolling without slipping up a ramp, as shown in the figure above.

(d)

i. On the figure of the hoop-rods system below, draw and label the forces (not components) that act on the system. Each force must be represented by a distinct arrow starting at, and pointing away from, the point at which the force is exerted on the system.

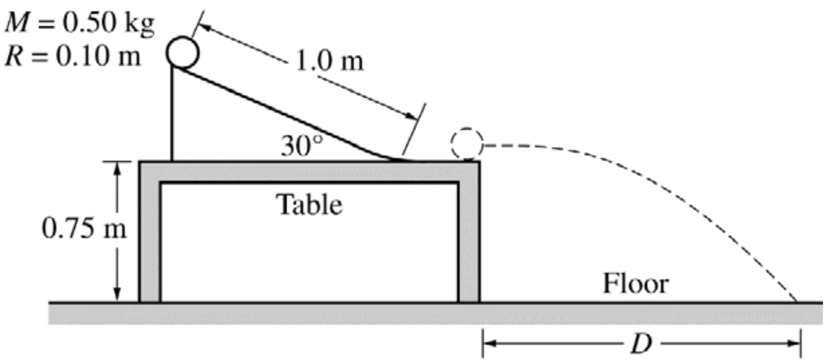


ii. Justify your choice for the direction of each of the forces drawn in part (d)i.

(e) Derive an expression for the change in height of the center of the hoop from the moment it reaches the bottom of the ramp until the moment it reaches its maximum height. Express your answer in terms of M, L, I_{tot}, ω, and physical constants, as appropriate.

STOP

END OF EXAM



3. A uniform solid cylinder of mass $M = 0.50$ kg and radius $R = 0.10$ m is released from rest, rolls without slipping down a 1.0 m long inclined plane, and is launched horizontally from a horizontal table of height 0.75 m. The inclined plane makes an angle of $30°$ with the horizontal. The cylinder lands on the floor a distance D away from the edge of the table, as shown in the figure above. There is a smooth transition from the inclined plane to the horizontal table, and the motion occurs with no frictional energy losses. The rotational inertia of a cylinder around its center is $MR^2/2$.

(a) Calculate the total kinetic energy of the cylinder as it reaches the horizontal table.

(b) Calculate the angular velocity of the cylinder around its axis at the moment it reaches the floor.

(c) Calculate the ratio of the rotational kinetic energy to the total kinetic energy for the cylinder at the moment it reaches the floor.

(d) Calculate the horizontal distance D.

A sphere of the same mass and radius is now rolled down the same inclined plane. The rotational inertia of a sphere around its center is $\frac{2}{5}MR^2$.

(e)

i. Is the total kinetic energy of the sphere at the moment it reaches the floor greater than, less than, or equal to the total kinetic energy of the cylinder at the moment it reaches the floor?

____ Greater than ____ Less than ____ Equal to

Justify your answer.

ii. Is the rotational kinetic energy of the sphere at the moment it reaches the floor greater than, less than, or equal to the rotational kinetic energy of the cylinder at the moment it reaches the floor?

____ Greater than ____ Less than ____ Equal to

Justify your answer.

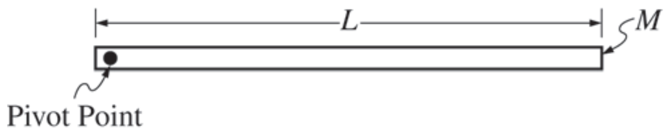
iii. Is the horizontal distance the sphere travels from the table to where it hits the floor greater than, less than, or equal to the horizontal distance the cylinder travels from the table to where it hits the floor?

____ Greater than ____ Less than ____ Equal to

Justify your answer.

STOP

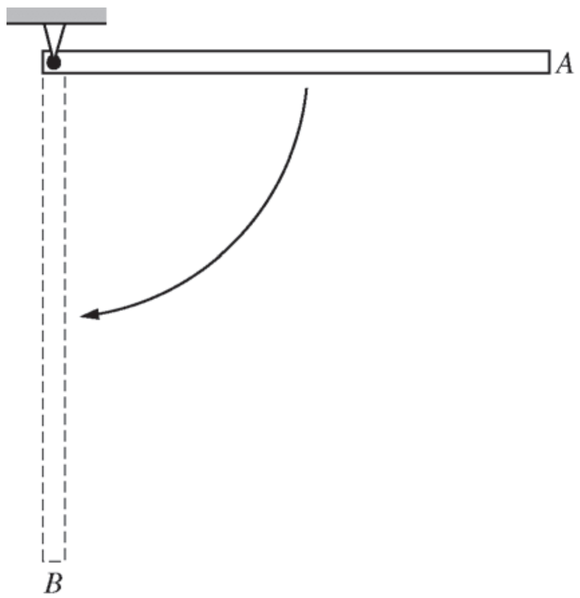
END OF EXAM



Mech.3.

A uniform, thin rod of length L and mass M is allowed to pivot about its end, as shown in the figure above.

(a) Using integral calculus, derive the rotational inertia for the rod around its end to show that it is $ML^2/3$.



The rod is fixed at one end and allowed to fall from the horizontal position A through the vertical position B.

(b) Derive an expression for the velocity of the free end of the rod at position B. Express your answer in terms of M, L, and physical constants, as appropriate.

An experiment is designed to test the validity of the expression found in part (b). A student uses rods of various lengths that all have a uniform mass distribution. The student releases each of the rods from the horizontal position A and uses photogates to measure the velocity of the free end at position B. The data are recorded below.

Length (m)	0.25	0.50	0.75	1.00	1.25	1.50
Velocity (m/s)	2.7	3.8	4.6	5.2	5.8	6.3

(c) Indicate below which quantities should be graphed to yield a straight line whose slope could be used to calculate a numerical value for the acceleration due to gravity g.

Horizontal axis: ____________________

Vertical axis: ____________________

Use the remaining rows in the table above, as needed, to record any quantities that you indicated that are not given. Label each row you use and include units.

(d) Plot the straight line data points on the grid below. Clearly scale and label all axes, including units as appropriate. Draw a straight line that best represents the data.

(e)

i. Using your straight line, determine an experimental value for g.

ii. Describe two ways in which the effects of air resistance could be reduced.

STOP

END OF EXAM

6.2 Torque and Work

Name: ______________ Class: __________ Date: ____________ **Total: 15 marks**

KEY WORDS torque 力矩 • rotational work 转动功 • rotational power 转动功率 • rotational kinetic energy 转动动能

Objective

Build the skills to answer exam questions on **torque and work**.

You must be able to:

- calculate **rotational work 转动功**: $W = \tau\theta$
- use **rotational power 转动功率**: $P = \tau\omega$
- apply the rotational work-energy theorem: $W_{net} = \Delta K_{rot}$
- recognise these as analogues of $W = Fd$ and $P = Fv$

1 Worked examples

Study these first. Each one shows the method for a question type used later — follow the steps and you can do the Practice and Exam-style questions yourself.

■ **Work done by a torque**

$$W = \tau\theta,$$

with θ in radians. A 5 N m torque turning a wheel through 2π rad does $W = 5 \times 2\pi \approx 31.4$ J.

■ **Rotational power**

$$P = \tau\omega.$$

A motor delivering 4 N m to a shaft at 10 rad s^{-1} outputs $P = 40$ W.

■ **Work-energy theorem**

The net rotational work equals the change in rotational kinetic energy:

$$W_{net} = \Delta K_{rot} = \tfrac{1}{2}I\omega_f^2 - \tfrac{1}{2}I\omega_i^2.$$

2 Practice

Now apply the methods above.

2.1 A 6 N m torque turns a wheel through 4 rad. Find the work done. [2]

2.2 A motor supplies 8 N m at 5 rad s^{-1}. Find the power. [2]

2.3 State what the net rotational work done on a body equals. [1]

2.4 In $W = \tau\theta$, in what unit must θ be? [1]

3 Exam-style questions

3.1 The work done by a constant torque is [1]

- **A** τ/θ
- **B** $\tau\theta$
- **C** $\tau\omega$
- **D** $I\alpha$

3.2 Rotational power is the analogue of [1]

- **A** $W = Fd$
- **B** $P = Fv$
- **C** $F = ma$
- **D** $p = mv$

3.3 A motor applies 10 N m while a wheel turns through 3 rad in 2 s.

(a) Find the work done. [2]

(b) Find the average power. [2]

3.4 A net torque does 50 J of work on a wheel with $I = 4$ kg m^2, starting from rest.

(a) State the change in rotational kinetic energy. [1]

(b) Find the final angular speed. [2]

Solutions

2.1 $W = \tau\theta = 6 \times 4 = 24$ J.

2.2 $P = \tau\omega = 8 \times 5 = 40$ W.

2.3 The change in its rotational kinetic energy.

2.4 Radians.

3.1 B — $W = \tau\theta$.

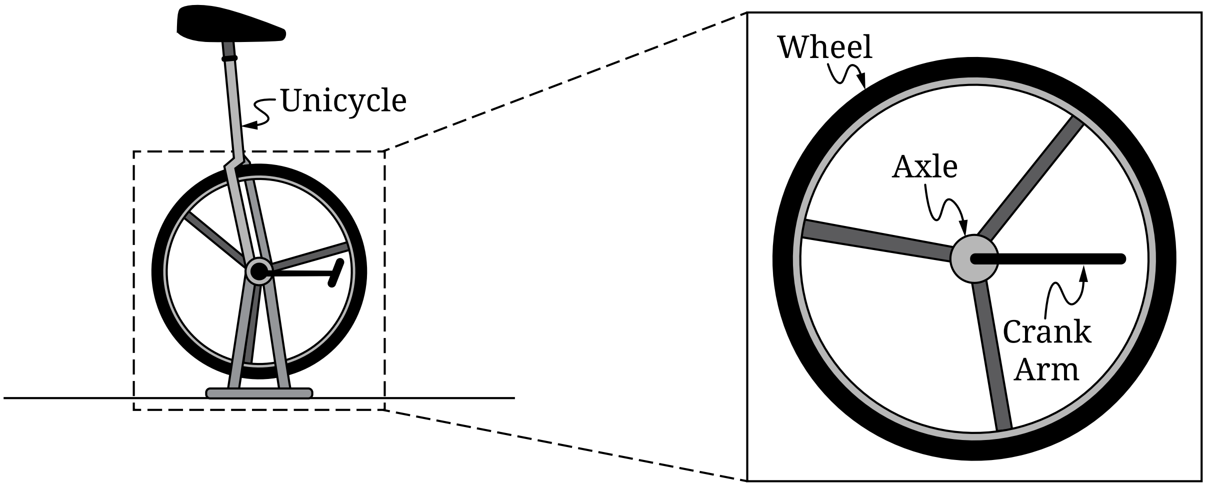
3.2 B — $P = \tau\omega$ mirrors $P = Fv$.

3.3 (a) $W = 10 \times 3 = 30$ J. (b) $P = W/t = 30/2 = 15$ W.

3.4 (a) $\Delta K_{rot} = 50$ J. (b) $\frac{1}{2}I\omega^2 = 50 \Rightarrow \omega = \sqrt{100/4} = 5$ rad s^{-1}.

Question 4

4. A unicycle is placed on a stand such that the wheel does not touch the ground. The rotational inertia of the wheel about the axle is I_W. One end of a crank arm can be attached to the wheel, as shown in the side view.



Side View

In Scenarios 1 and 2, crank arms of different lengths are connected to the wheel. In each scenario, the wheel is initially at rest. A force of magnitude F is then exerted at the end of each crank arm. The direction of the force remains perpendicular to each crank arm at all times. The mass of each crank arm is negligible. The scenarios differ as described.

- Scenario 1: The length of the crank arm is ℓ_1. The angular speed of the wheel immediately after the crank arm has been turned through one rotation is ω_1.
- Scenario 2: The length of the crank arm is ℓ_2, where $\ell_2 > \ell_1$. The angular speed of the wheel immediately after the longer crank arm has been turned through one rotation is ω_2.

Part A

Indicate whether ω_2 is greater than, less than, or equal to ω_1 by writing one of the following.

- $\omega_2 > \omega_1$
- $\omega_2 < \omega_1$
- $\omega_2 = \omega_1$

Justify your answer. Your justification may reference equations but must include conceptual reasoning beyond algebraic solutions.

Part B

Derive an expression for the angular speed ω_1 of the wheel in Scenario 1 immediately after the crank arm has been turned through one rotation. Express your answer in terms of I_W, F, ℓ_1, and physical constants, as appropriate. Begin your derivation by writing a fundamental physics principle or an equation from the reference information.

Part C

In Scenario 3, the unicycle is placed on the ground such that the unicycle can move forward without the wheel slipping on the ground. A force of the same magnitude F is exerted at the end of the crank arm of length ℓ_1, as in Scenario 1. The direction of the force remains perpendicular to the crank arm at all times. The resulting angular speed of the wheel immediately after the crank arm has been turned through one rotation is ω_3.

Indicate whether ω_3 is greater than, less than, or equal to ω_1 by writing one of the following.

- $\omega_3 > \omega_1$
- $\omega_3 < \omega_1$
- $\omega_3 = \omega_1$

Briefly **justify** your answer. Your justification may reference equations but must include conceptual reasoning beyond algebraic solutions.

STOP

END OF EXAM

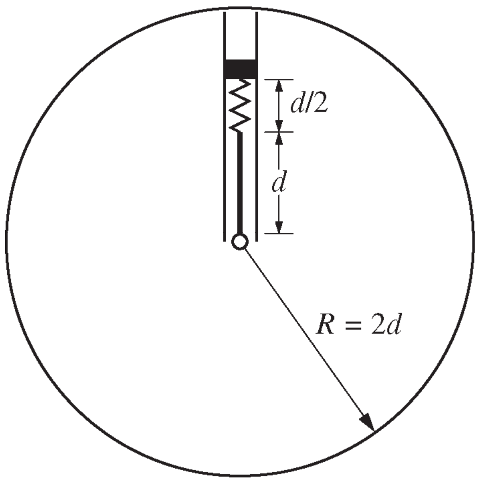


Figure 1

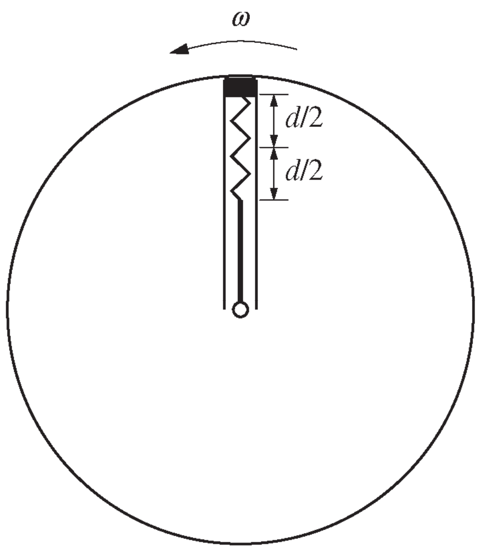


Figure 2

Mech.3.

A uniform rod of length d has one end fixed to the central axis of a horizontal, frictionless circular platform of radius $R = 2d$. Fixed at the other end of the rod is an ideal spring of negligible mass to which a block is attached. The block is set in frictionless grooves so that it can only move along a radius of the platform, as shown in Figure 1 above. The equilibrium length of the spring is $d/2$. Below is a table showing the mass of the block and the masses and rotational inertias of the rod and platform.

	Mass	Rotational Inertia
Block	m	
Rod	$m_R = 3m$	$\frac{m_R d^2}{3}$ (about the end of the rod)
Platform	$m_P = 5m$	$\frac{m_P R^2}{2}$ (about the central axis)

A motor begins to slowly rotate the platform counterclockwise as viewed from above until the platform reaches a constant angular speed ω. Under these conditions, the spring has stretched by an additional length $d/2$, as shown in Figure 2.

Answer the following questions for the platform rotating at constant angular speed ω. Express all algebraic answers in terms of m, d, ω, and physical constants, as appropriate.

(a) Derive an expression for the spring constant of the spring.

(b)

i. Determine an expression for the rotational inertia of the block around the axis of the platform.

ii. Derive an expression for the rotational inertia of the entire system about the axis of the platform.

(c) Determine an expression for the angular momentum of the entire system about the axis of the platform.

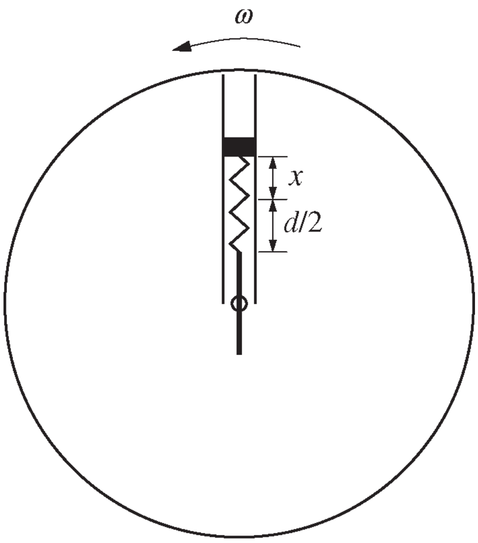


Figure 3

While the system continues to rotate, a small mechanism in the pivot moves the rod slowly until the center of the rod is positioned on the axis, as shown in Figure 3 above. The same constant angular speed ω is maintained by the motor driving the platform.

(d) Derive an expression for the distance x that the spring is stretched when the rod reaches the position shown in Figure 3 above.

For parts (e), (f), and (g), assume the center of the rod is still moving toward the axis of the platform.

(e) Is the angular momentum of the entire system increasing, decreasing, or staying the same?

_____ Increasing _____ Decreasing _____ Staying the same

Justify your answer.

(f) In order to keep the system rotating with constant angular speed ω, is the motor doing positive work, negative work, or no work on the rotating system?

_____ Positive _____ Negative _____ No work

Justify your answer.

(g) On the block in Figure 4 below, draw a single vector representing the direction of the acceleration of the block. Draw the vector so that it is starting on, and pointing away from, the block.

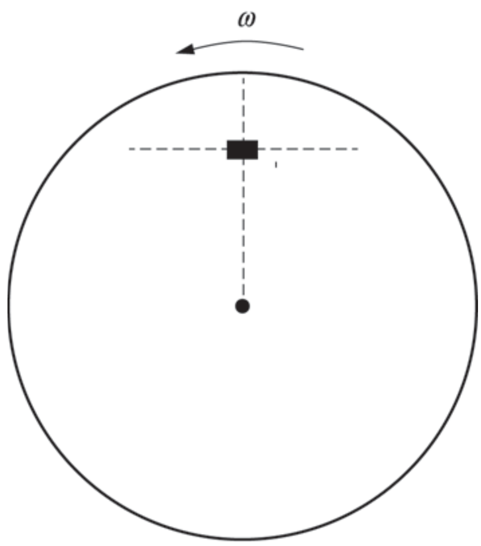


Figure 4

STOP

END OF EXAM

6.3 Angular Momentum and Angular Impulse

Name: ____________ Class: ________ Date: __________ **Total: 14 marks**

KEY WORDS angular momentum 角动量 • angular impulse 角冲量 • torque 力矩

Objective

Build the skills to answer exam questions on **angular momentum and angular impulse**.

You must be able to:

- use **angular momentum 角动量**: $L = I\omega$ (and $L = mvr$ for a point mass)
- use **angular impulse 角冲量**: $\tau\,\Delta t = \Delta L$
- recognise the analogy with $p = mv$ and $F\Delta t = \Delta p$
- state the unit of angular momentum (kg m^2 s^{-1})

1 Worked examples

Study these first. Each one shows the method for a question type used later — follow the steps and you can do the Practice and Exam-style questions yourself.

■ Angular momentum

$$L = I\omega.$$

A wheel with $I = 3$ kg m^2 at $\omega = 2$ rad s^{-1} has $L = 6$ kg m^2 s^{-1}. For a point mass, $L = mvr$.

■ Angular impulse

A torque acting over a time changes the angular momentum:

$$\tau\,\Delta t = \Delta L.$$

A 4 N m torque for 3 s adds $\Delta L = 12$ kg m^2 s^{-1}.

■ The rate form

Equivalently $\tau = \dfrac{dL}{dt}$ – torque is the rate of change of angular momentum.

2 Practice

Now apply the methods above.

2.1 A flywheel with $I = 5$ kg m^2 spins at 4 rad s^{-1}. Find its angular momentum. [2]

2.2 State the unit of angular momentum. [1]

2.3 A 3 N m torque acts for 5 s. Find the change in angular momentum. [2]

2.4 A 2 kg ball moves at 3 m s^{-1} in a circle of radius 4 m. Find its angular momentum. [2]

3 Exam-style questions

3.1 Angular momentum is the rotational analogue of [1]

- **A** force
- **B** momentum
- **C** energy
- **D** acceleration

3.2 For a rigid body, L equals [1]

- **A** $I\alpha$
- **B** $I\omega$
- **C** $\frac{1}{2}I\omega^2$
- **D** $\tau\theta$

3.3 A wheel with $I = 2$ kg m^2 at $\omega = 3$ rad s^{-1} receives a steady 2 N m torque for 4 s.

(a) Find its initial angular momentum. [1]

(b) Find its angular momentum after the torque acts. [2]

3.4 A skater has $I = 4$ kg m^2 spinning at 2 rad s^{-1}.

(a) Find her angular momentum. [1]

(b) State what quantity torque is the rate of change of. [1]

Solutions

2.1 $L = I\omega = 5 \times 4 = 20$ kg m^2 s^{-1}.

2.2 kg m^2 s^{-1}.

2.3 $\Delta L = \tau\,\Delta t = 3 \times 5 = 15$ kg m^2 s^{-1}.

2.4 $L = mvr = 2 \times 3 \times 4 = 24$ kg m^2 s^{-1}.

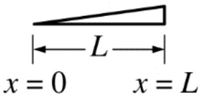
3.1 **B** — angular momentum is the rotational analogue of linear momentum.

3.2 **B** — $L = I\omega$.

3.3 (a) $L_i = 2 \times 3 = 6$ kg m^2 s^{-1}. (b) $\Delta L = 2 \times 4 = 8$; $L_f = 6 + 8 = 14$ kg m^2 s^{-1}.

3.4 (a) $L = 4 \times 2 = 8$ kg m^2 s^{-1}. (b) Angular momentum ($\tau = dL/dt$).

$$\lambda = \left(\frac{2M}{L^2}\right)x$$



3. A triangular rod, shown above, has length L, mass M, and a nonuniform linear mass density given by the equation $\lambda = \dfrac{2M}{L^2}x$, where x is the distance from one end of the rod.

   (a) Using integral calculus, show that the rotational inertia of the rod about its left end is $ML^2/2$.

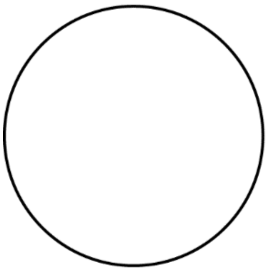


Figure 1

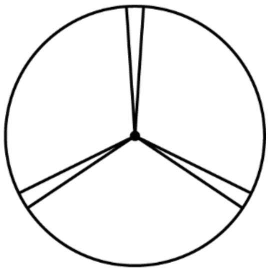


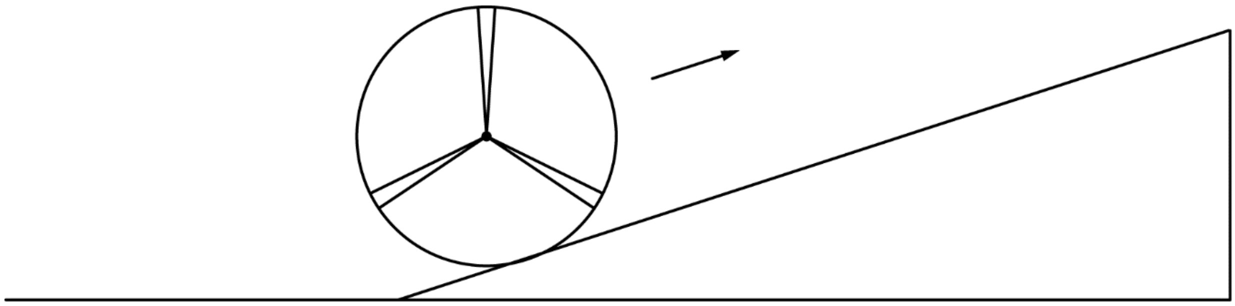
Figure 2

The thin hoop shown above in Figure 1 has a mass M, radius L, and a rotational inertia around its center of ML^2. Three rods identical to the rod from part (a) are now fastened to the thin hoop, as shown in Figure 2 above.

(b) Derive an expression for the rotational inertia I_{tot} of the hoop-rods system about the center of the hoop. Express your answer in terms of M, L, and physical constants, as appropriate.

The hoop-rods system is initially at rest and held in place but is free to rotate around its center. A constant force F is exerted tangent to the hoop for a time Δt.

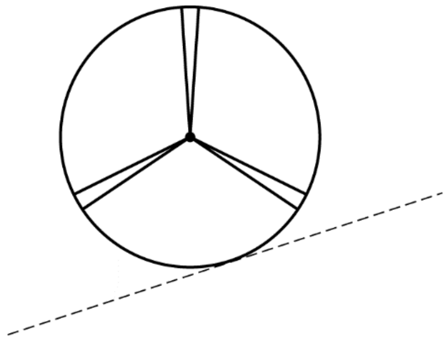
(c) Derive an expression for the final angular speed ω of the hoop-rods system. Express your answer in terms of M, L, F, Δt, and physical constants, as appropriate.



The hoop-rods system is rolling without slipping along a level horizontal surface with the angular speed ω found in part (c). At time $t = 0$, the system begins rolling without slipping up a ramp, as shown in the figure above.

(d)

i. On the figure of the hoop-rods system below, draw and label the forces (not components) that act on the system. Each force must be represented by a distinct arrow starting at, and pointing away from, the point at which the force is exerted on the system.



ii. Justify your choice for the direction of each of the forces drawn in part (d)i.

(e) Derive an expression for the change in height of the center of the hoop from the moment it reaches the bottom of the ramp until the moment it reaches its maximum height. Express your answer in terms of M, L, I_{tot}, ω, and physical constants, as appropriate.

STOP

END OF EXAM

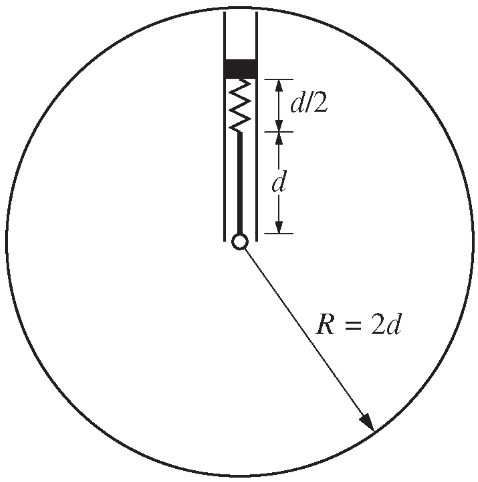


Figure 1

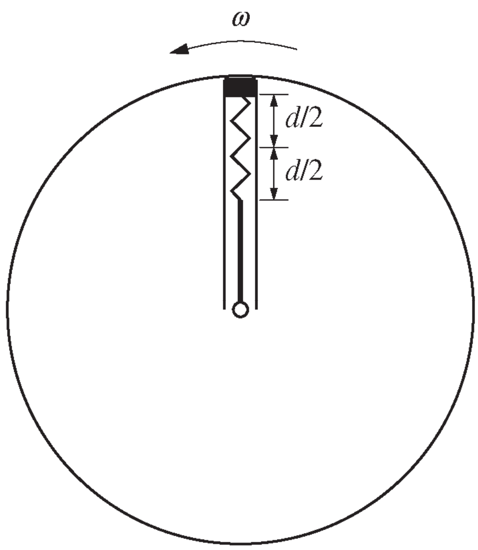


Figure 2

Mech.3.

A uniform rod of length d has one end fixed to the central axis of a horizontal, frictionless circular platform of radius $R = 2d$. Fixed at the other end of the rod is an ideal spring of negligible mass to which a block is attached. The block is set in frictionless grooves so that it can only move along a radius of the platform, as shown in Figure 1 above. The equilibrium length of the spring is $d/2$. Below is a table showing the mass of the block and the masses and rotational inertias of the rod and platform.

	Mass	Rotational Inertia
Block	m	
Rod	$m_R = 3m$	$\frac{m_R d^2}{3}$ (about the end of the rod)
Platform	$m_P = 5m$	$\frac{m_P R^2}{2}$ (about the central axis)

A motor begins to slowly rotate the platform counterclockwise as viewed from above until the platform reaches a constant angular speed ω. Under these conditions, the spring has stretched by an additional length $d/2$, as shown in Figure 2.

Answer the following questions for the platform rotating at constant angular speed ω. Express all algebraic answers in terms of m, d, ω, and physical constants, as appropriate.

(a) Derive an expression for the spring constant of the spring.

(b)

i. Determine an expression for the rotational inertia of the block around the axis of the platform.

ii. Derive an expression for the rotational inertia of the entire system about the axis of the platform.

(c) Determine an expression for the angular momentum of the entire system about the axis of the platform.

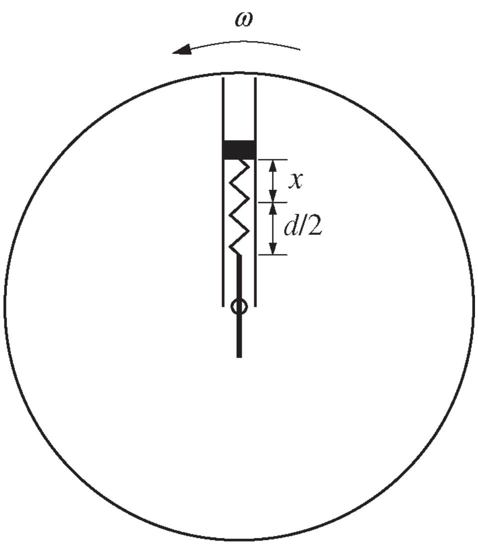


Figure 3

While the system continues to rotate, a small mechanism in the pivot moves the rod slowly until the center of the rod is positioned on the axis, as shown in Figure 3 above. The same constant angular speed ω is maintained by the motor driving the platform.

(d) Derive an expression for the distance x that the spring is stretched when the rod reaches the position shown in Figure 3 above.

For parts (e), (f), and (g), assume the center of the rod is still moving toward the axis of the platform.

(e) Is the angular momentum of the entire system increasing, decreasing, or staying the same?

_____ Increasing   _____ Decreasing   _____ Staying the same

Justify your answer.

(f) In order to keep the system rotating with constant angular speed ω, is the motor doing positive work, negative work, or no work on the rotating system?

_____ Positive   _____ Negative   _____ No work

Justify your answer.

(g) On the block in Figure 4 below, draw a single vector representing the direction of the acceleration of the block. Draw the vector so that it is starting on, and pointing away from, the block.

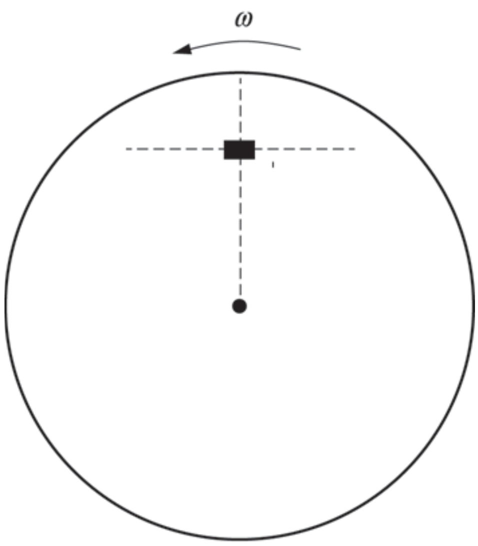


Figure 4

STOP

END OF EXAM

6.4 Conservation of Angular Momentum

Name: ______________ Class: __________ Date: ____________ **Total: 13 marks**

KEY WORDS conserved 守恒 • rotational inertia 转动惯量 • angular momentum 角动量 • torque 力矩 • rotational kinetic energy 转动动能

Objective

Build the skills to answer exam questions on **conservation of angular momentum**.

You must be able to:

- state that **angular momentum is conserved 角动量守恒** when no net external torque acts
- apply $I_i\omega_i = I_f\omega_f$
- explain the skater effect (pulling arms in speeds the spin)
- note that kinetic energy need not be conserved

1 Worked examples

Study these first. Each one shows the method for a question type used later — follow the steps and you can do the Practice and Exam-style questions yourself.

■ Conservation condition

If the net external torque is zero,

$$I_i\omega_i = I_f\omega_f.$$

■ The skater effect

A skater pulls her arms in, lowering I. Since $I\omega$ is fixed, ω rises:

$$I_i\omega_i = I_f\omega_f \Rightarrow \omega_f = \frac{I_i\omega_i}{I_f}.$$

If I halves, ω doubles.

■ Energy is not conserved here

The skater does work pulling her arms in, so the rotational kinetic energy actually **increases** even though L stays constant.

2 Practice

Now apply the methods above.

2.1 State the condition for angular momentum to be conserved. [1]

2.2 A skater at $I_i = 6$ kg m^2 spins at 2 rad s^{-1}, then pulls in to $I_f = 2$ kg m^2. Find the new ω. [2]

2.3 If a spinning body's rotational inertia halves (no external torque), what happens to ω? [1]

2.4 Is rotational kinetic energy always conserved when angular momentum is? Answer yes or no. [1]

3 Exam-style questions

3.1 Angular momentum is conserved when the net external torque is [1]

- **A** maximum
- **B** zero
- **C** increasing
- **D** equal to $I\alpha$

3.2 A star collapses to one third of its rotational inertia. Its spin rate becomes [1]

- **A** one third
- **B** unchanged
- **C** three times faster
- **D** nine times faster

3.3 A turntable ($I = 4$ kg m^2) spins at 3 rad s^{-1}. A blob of clay is dropped on, raising I to 6 kg m^2.

(a) Find the new angular speed. [2]

(b) State whether angular momentum is conserved and why. [1]

3.4 A diver tucks to spin faster during a somersault.

(a) Explain, using $I\omega$, why tucking speeds the spin. [2]

(b) State what provides no external torque here. [1]

Solutions

2.1 When the net external torque is zero.

2.2 $6(2) = 2\omega_f \Rightarrow \omega_f = 6$ rad s^{-1}.

2.3 It doubles.

2.4 No.

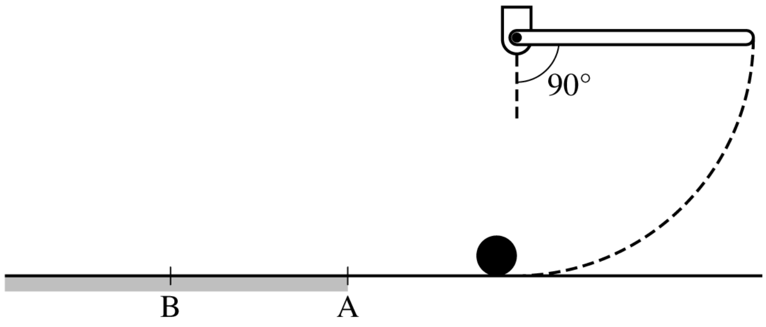
3.1 **B** — zero net external torque.

3.2 **C** — $I \to I/3$ means $\omega \to 3\omega$.

3.3 (a) $4(3) = 6\omega_f \Rightarrow \omega_f = 2$ rad s^{-1}. (b) Yes — no external torque acts on the turntable-plus-clay system.

3.4 (a) Tucking lowers I; since $I\omega$ is fixed, ω rises. (b) Gravity acts through the center of mass, giving no torque about it.

Begin your response to **QUESTION 3** on this page.



Note: Figure not drawn to scale.

Figure 1

3. A system consists of a small sphere of mass m and radius R at rest on a horizontal surface and a uniform rod of mass $M = 2m$ and length ℓ attached at one end to a pivot with negligible friction, where $R \ll \ell$. There is negligible friction between the surface and the sphere to the right of Point A and nonnegligible friction to the left of Point A. The rod is held horizontally as shown in Figure 1, then is released from rest. The total rotational inertia of the rod about the pivot is $\frac{1}{3}M\ell^2$ and the rotational inertia of the sphere about its center is $\frac{2}{5}mR^2$. After the rod is released, the rod swings down and strikes the sphere head-on. As a result of this collision, the rod is stopped, and the ball initially slides without rotating to the left across the horizontal surface.

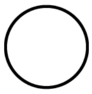
(a) Derive an expression for the angular speed of the rod just before striking the sphere in terms of the length ℓ and physical constants as appropriate.

GO ON TO THE NEXT PAGE.

(b) Derive an expression for the linear speed v_0 of the sphere immediately after colliding with the rod in terms of the length ℓ and physical constants as appropriate.

After sliding a short distance, at time $t = 0$ the sphere encounters a region of the horizontal surface with a coefficient of kinetic friction μ, beginning at Point A as indicated in Figure 1. The sphere begins rotating while sliding and eventually begins rolling without sliding at Point B, also as indicated.

(c) In the following diagram, which represents the sphere while the sphere is traveling between Points A and B, draw and label the forces (not components) that act on the sphere. Each force must be represented by a distinct arrow starting on, and pointing away from, the point of application on the sphere.



(d) Derive an expression for each of the following as the sphere is rotating and sliding between points A and B in terms of v_0, μ, R, t, and physical constants as appropriate.

i. The linear velocity v of the center of mass of the sphere as a function of time t

ii. The angular velocity ω of the sphere as a function of time t

GO ON TO THE NEXT PAGE.

Continue your response to **QUESTION 3** on this page.

(e)

i. Derive an expression for the time it takes the sphere to travel from Point A to Point B in terms of v_0, μ, and physical constants as appropriate.

ii. Derive an expression for the linear velocity of the sphere upon reaching Point B in terms of v_0.

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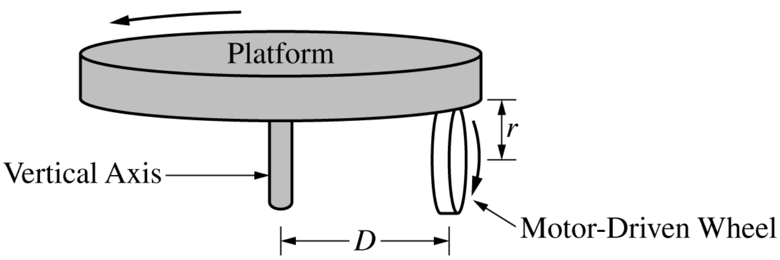
END OF EXAM

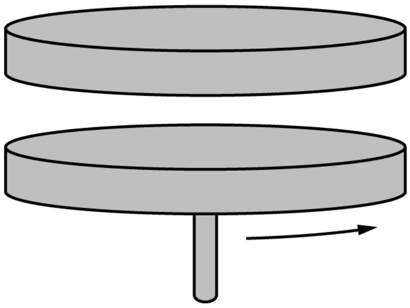


3. A horizontal circular platform with rotational inertia I_P rotates freely without friction on a vertical axis. A small motor-driven wheel that is used to rotate the platform is mounted under the platform and touches it. The wheel has radius r and touches the platform a distance D from the vertical axis of the platform, as shown above. The platform starts at rest, and the wheel exerts a constant horizontal force of magnitude F tangent to the wheel until the platform reaches an angular speed ω_P after time Δt. During time Δt, the wheel stays in contact with the platform without slipping.

(a) Derive an expression for the angular speed ω_P of the platform. Express your answer in terms of I_P, r, D, F, Δt, and physical constants, as appropriate.

(b) Determine an expression for the kinetic energy of the platform at the moment it reaches angular speed ω_P. Express your answer in terms of I_P, r, D, F, Δt, and physical constants, as appropriate.

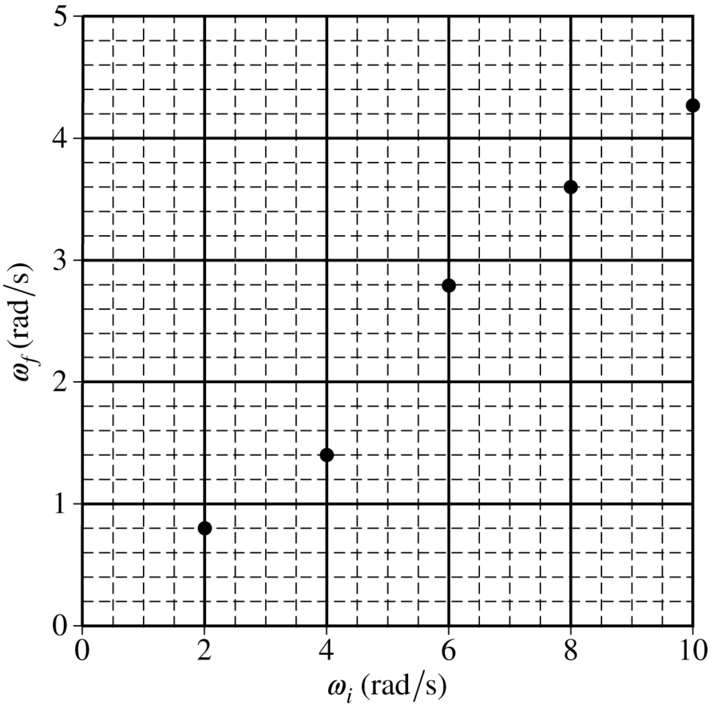
(c) Derive an expression for the angular speed of the wheel ω_W when the platform has reached angular speed ω_P. Express your answer in terms of D, r, ω_P, and physical constants, as appropriate.



When the platform is spinning at angular speed ω_P, the motor-driven wheel is removed. A student holds a disk directly above and concentric with the platform, as shown above. The disk has the same rotational inertia I_P as the platform. The student releases the disk from rest, and the disk falls onto the platform. After a short time, the disk and platform are observed to be rotating together at angular speed ω_f.

(d) Derive an expression for ω_f. Express your answer in terms of ω_P, I_P, and physical constants, as appropriate.

A student now uses the rotating platform $\left(I_P = 3.1\ \text{kg}\bullet\text{m}^2\right)$ to determine the rotational inertia I_U of an unknown object about a vertical axis that passes through the object's center of mass. The platform is rotating at an initial angular speed ω_i when the unknown object is dropped with its center of mass directly above the center of the platform. The platform and object are observed to be rotating together at angular speed ω_f. Trials are repeated for different values of ω_i. A graph of ω_f as a function of ω_i is shown on the axes below.



(e)

i. On the graph on the previous page, draw a best-fit line for the data.

ii. Using the straight line, calculate the rotational inertia of the unknown object I_U about a vertical axis passing through its center of mass.

(f) The kinetic energy of the spinning platform before the object is dropped on it is K_i. The total kinetic energy of the platform-object system when it reaches angular speed ω_f is K_f. Which of the following expressions is true?

____ $K_f < K_i$ ____ $K_f = K_i$ ____ $K_f > K_i$

Justify your answer.

(g) One of the students observes that the center of mass of the object is not actually aligned with the axis of the platform. Is the experimental value of I_U obtained in part (e) greater than, less than, or equal to the actual value of the rotational inertia of the unknown object about a vertical axis that passes through its center of mass?

____ Greater than   ____ Less than   ____ Equal to

Justify your answer.

STOP

END OF EXAM

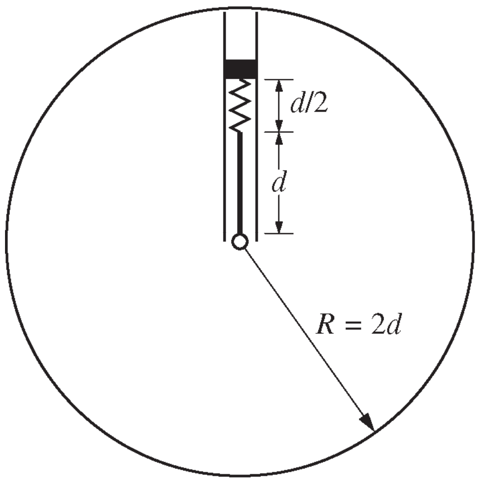


Figure 1

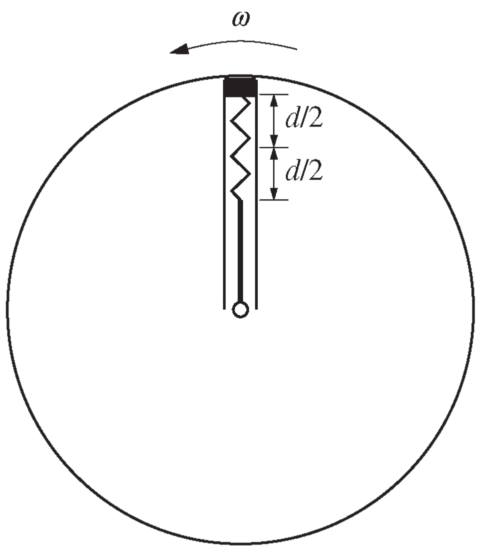


Figure 2

Mech.3.

A uniform rod of length d has one end fixed to the central axis of a horizontal, frictionless circular platform of radius $R = 2d$. Fixed at the other end of the rod is an ideal spring of negligible mass to which a block is attached. The block is set in frictionless grooves so that it can only move along a radius of the platform, as shown in Figure 1 above. The equilibrium length of the spring is $d/2$. Below is a table showing the mass of the block and the masses and rotational inertias of the rod and platform.

	Mass	Rotational Inertia
Block	m	
Rod	$m_R = 3m$	$\frac{m_R d^2}{3}$ (about the end of the rod)
Platform	$m_P = 5m$	$\frac{m_P R^2}{2}$ (about the central axis)

A motor begins to slowly rotate the platform counterclockwise as viewed from above until the platform reaches a constant angular speed ω. Under these conditions, the spring has stretched by an additional length $d/2$, as shown in Figure 2.

Answer the following questions for the platform rotating at constant angular speed ω. Express all algebraic answers in terms of m, d, ω, and physical constants, as appropriate.

(a) Derive an expression for the spring constant of the spring.

(b)

i. Determine an expression for the rotational inertia of the block around the axis of the platform.

ii. Derive an expression for the rotational inertia of the entire system about the axis of the platform.

(c) Determine an expression for the angular momentum of the entire system about the axis of the platform.

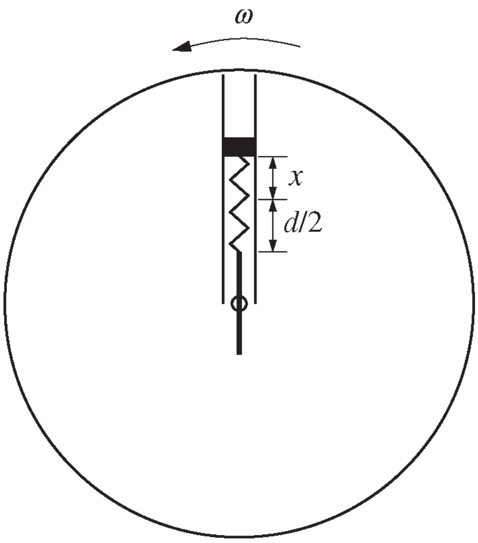


Figure 3

While the system continues to rotate, a small mechanism in the pivot moves the rod slowly until the center of the rod is positioned on the axis, as shown in Figure 3 above. The same constant angular speed ω is maintained by the motor driving the platform.

(d) Derive an expression for the distance x that the spring is stretched when the rod reaches the position shown in Figure 3 above.

For parts (e), (f), and (g), assume the center of the rod is still moving toward the axis of the platform.

(e) Is the angular momentum of the entire system increasing, decreasing, or staying the same?

_____ Increasing _____ Decreasing _____ Staying the same

Justify your answer.

(f) In order to keep the system rotating with constant angular speed ω, is the motor doing positive work, negative work, or no work on the rotating system?

_____ Positive _____ Negative _____ No work

Justify your answer.

(g) On the block in Figure 4 below, draw a single vector representing the direction of the acceleration of the block. Draw the vector so that it is starting on, and pointing away from, the block.

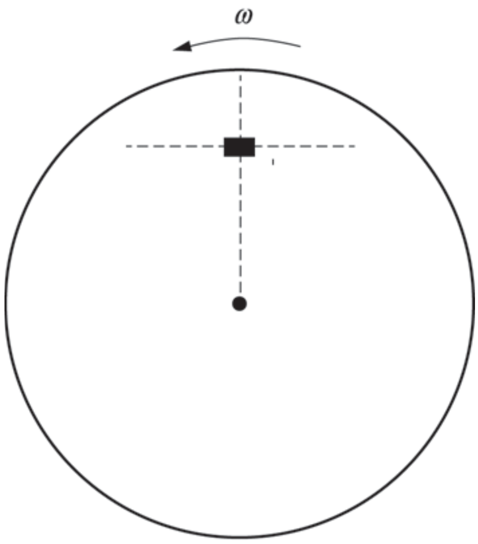
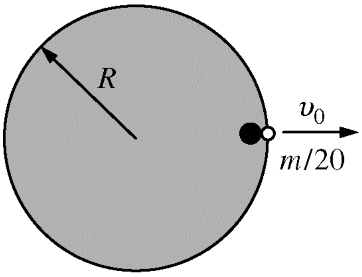


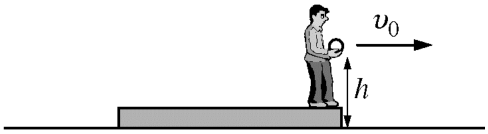
Figure 4

STOP

END OF EXAM



Top View



Side View

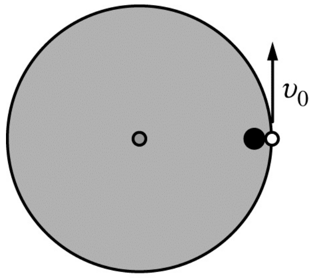
Mech. 3.

A large circular disk of mass m and radius R is initially stationary on a horizontal icy surface. A person of mass $m/2$ stands on the edge of the disk. Without slipping on the disk, the person throws a large stone of mass $m/20$ horizontally at initial speed v_0 from a height h above the ice in a radial direction, as shown in the figures above. The coefficient of friction between the disk and the ice is μ. All velocities are measured relative to the ground. The time it takes to throw the stone is negligible. Express all algebraic answers in terms of m, R, v_0, h, μ, and fundamental constants, as appropriate.

(a) Derive an expression for the length of time it will take the stone to strike the ice.

(b) Assuming that the disk is free to slide on the ice, derive an expression for the speed of the disk and person immediately after the stone is thrown.

(c) Derive an expression for the time it will take the disk to stop sliding.



Top View

The person now stands on a similar disk of mass m and radius R that has a fixed pole through its center so that it can only rotate on the ice. The person throws the same stone horizontally in a tangential direction at initial speed v_0, as shown in the figure above. The rotational inertia of the disk is $mR^2/2$.

(d) Derive an expression for the angular speed ω of the disk immediately after the stone is thrown.

(e) The person now stands on the disk at rest $R/2$ from the center of the disk. The person now throws the stone horizontally with a speed v_0 in the same direction as in part (d). Is the angular speed of the disk immediately after throwing the stone from this new position greater than, less than, or equal to the angular speed found in part (d) ?

____ Greater than   ____ Less than   ____ Equal to

Justify your answer.

STOP

END OF EXAM

6.5 Rolling

Name: ______________ Class: __________ Date: ____________ **Total: 15 marks**

KEY WORDS hoop 圆环 • rolling without slipping 无滑滚动 • rotational inertia 转动惯量

Objective

Build the skills to answer exam questions on **rolling**.

You must be able to:

- use the **rolling without slipping** 无滑滚动 condition: $v = \omega r$ (and $a = \alpha r$)
- state that a rolling object's total kinetic energy is $\frac{1}{2}mv^2 + \frac{1}{2}I\omega^2$
- explain that the contact point is instantaneously at rest
- reason about which shape wins a downhill race

1 Worked examples

Study these first. Each one shows the method for a question type used later — follow the steps and you can do the Practice and Exam-style questions yourself.

■ The no-slip condition

When a wheel rolls without slipping, its speed and spin are locked:

$$v = \omega r.$$

A wheel of radius 0.3 m spinning at 10 rad s^{-1} rolls forward at $v = 3$ m s^{-1}.

■ Total kinetic energy

A rolling object carries both translational and rotational KE:

$$K = \tfrac{1}{2}mv^2 + \tfrac{1}{2}I\omega^2.$$

■ The downhill race

Sharing energy between motion and spin, a shape with a **smaller** rotational inertia (per mr^2) keeps more for speed and reaches the bottom first: solid sphere before cylinder before ring.

2 Practice

Now apply the methods above.

2.1 A wheel of radius 0.4 m rolls without slipping at $\omega = 5$ rad s^{-1}. Find its forward speed. [2]

2.2 State the speed of the point where a rolling wheel touches the ground. [1]

2.3 Write the two terms in the total kinetic energy of a rolling object. [2]

2.4 A rolling ball of radius 0.1 m moves at 2 m s^{-1}. Find its angular velocity. [2]

3 Exam-style questions

3.1 For rolling without slipping, the link between speed and spin is [1]

- **A** $v = \omega/r$
- **B** $v = \omega r$
- **C** $v = r/\omega$
- **D** $v = \omega^2 r$

3.2 Released together down the same ramp, which reaches the bottom first? [1]

- **A** a hollow ring
- **B** a solid cylinder
- **C** a solid sphere
- **D** they tie

3.3 A hoop and a solid disc (same mass and radius) roll from rest down the same ramp.

(a) State which arrives first. [1]

(b) Explain why, in terms of how the energy is shared. [2]

3.4 A ball of radius 0.2 m rolls without slipping at 6 m s^{-1}.

(a) Find its angular velocity. [2]

(b) State what happens to v if it begins to skid (slip). [1]

Solutions

2.1 $v = \omega r = 5 \times 0.4 = 2 \text{ m s}^{-1}$.

2.2 Zero (instantaneously at rest).

2.3 $\frac{1}{2}mv^2$ (translational) and $\frac{1}{2}I\omega^2$ (rotational).

2.4 $\omega = v/r = 2/0.1 = 20 \text{ rad s}^{-1}$.

3.1 **B** — $v = \omega r$.

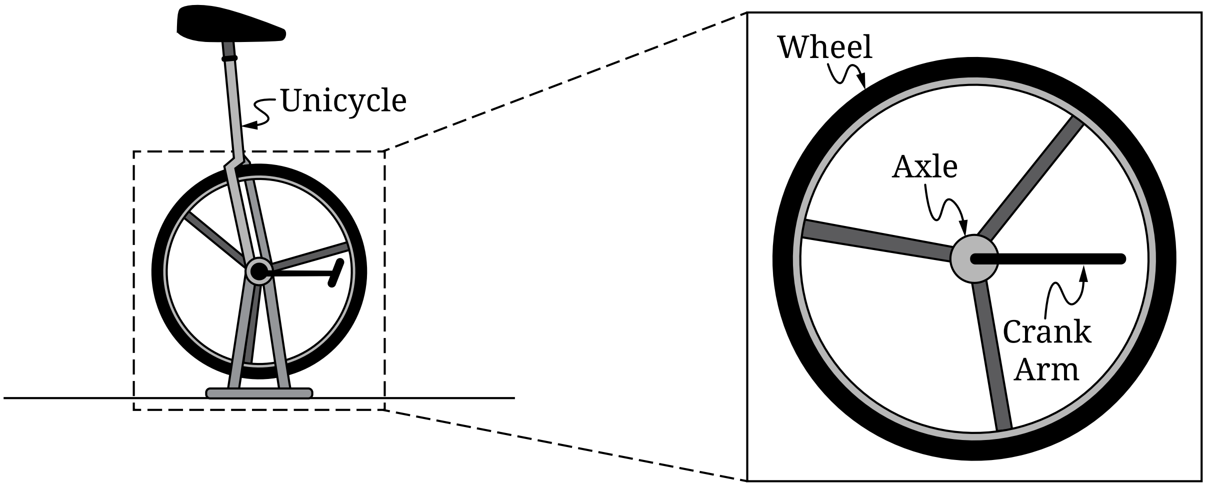
3.2 **C** — the solid sphere (smallest rotational inertia per mr^2) wins.

3.3 (a) The solid disc. (b) The hoop puts more energy into spin (larger I), leaving less for speed, so it is slower.

3.4 (a) $\omega = v/r = 6/0.2 = 30 \text{ rad s}^{-1}$. (b) $v = \omega r$ no longer holds — the speeds decouple.

Question 4

4. A unicycle is placed on a stand such that the wheel does not touch the ground. The rotational inertia of the wheel about the axle is I_W. One end of a crank arm can be attached to the wheel, as shown in the side view.



Side View

In Scenarios 1 and 2, crank arms of different lengths are connected to the wheel. In each scenario, the wheel is initially at rest. A force of magnitude F is then exerted at the end of each crank arm. The direction of the force remains perpendicular to each crank arm at all times. The mass of each crank arm is negligible. The scenarios differ as described.

- Scenario 1: The length of the crank arm is ℓ_1. The angular speed of the wheel immediately after the crank arm has been turned through one rotation is ω_1.
- Scenario 2: The length of the crank arm is ℓ_2, where $\ell_2 > \ell_1$. The angular speed of the wheel immediately after the longer crank arm has been turned through one rotation is ω_2.

Part A

Indicate whether ω_2 is greater than, less than, or equal to ω_1 by writing one of the following.

- $\omega_2 > \omega_1$
- $\omega_2 < \omega_1$
- $\omega_2 = \omega_1$

Justify your answer. Your justification may reference equations but must include conceptual reasoning beyond algebraic solutions.

Part B

Derive an expression for the angular speed ω_1 of the wheel in Scenario 1 immediately after the crank arm has been turned through one rotation. Express your answer in terms of I_W, F, ℓ_1, and physical constants, as appropriate. Begin your derivation by writing a fundamental physics principle or an equation from the reference information.

Part C

In Scenario 3, the unicycle is placed on the ground such that the unicycle can move forward without the wheel slipping on the ground. A force of the same magnitude F is exerted at the end of the crank arm of length ℓ_1, as in Scenario 1. The direction of the force remains perpendicular to the crank arm at all times. The resulting angular speed of the wheel immediately after the crank arm has been turned through one rotation is ω_3.

Indicate whether ω_3 is greater than, less than, or equal to ω_1 by writing one of the following.

- $\omega_3 > \omega_1$
- $\omega_3 < \omega_1$
- $\omega_3 = \omega_1$

Briefly **justify** your answer. Your justification may reference equations but must include conceptual reasoning beyond algebraic solutions.

STOP

END OF EXAM

Question 4

4. A uniform disk and ring, each of mass M and radius R, roll without slipping along a horizontal surface, as shown in Figure 1. The outer edges of the disk and ring are made of the same material. The center of mass of the disk and the center of mass of the ring each initially move with the same constant speed v.

The disk and the ring then smoothly transition to a ramp that is inclined at an angle θ above the horizontal. Both the disk and the ring continue to roll without slipping as they move up the ramp, as shown in Figure 2.

The ring travels a greater distance along the ramp than the disk travels before each momentarily comes to rest.

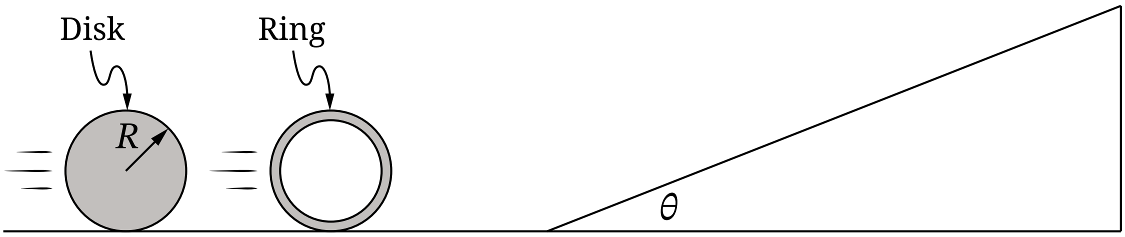


Figure 1

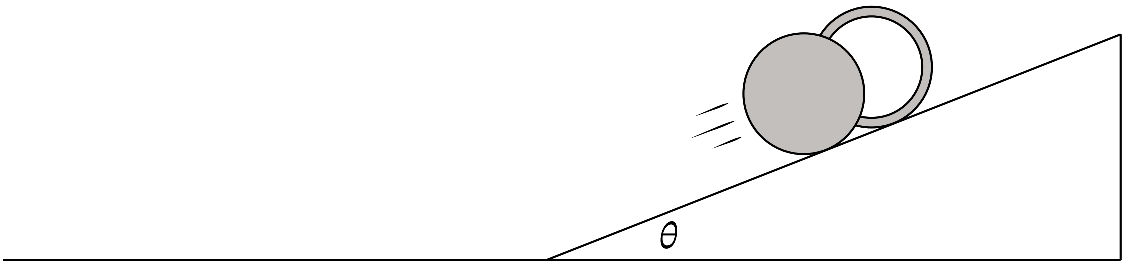


Figure 2

A. While the disk and the ring are rolling on the ramp without slipping, the magnitudes of the static frictional force exerted on the disk and on the ring by the ramp are f_D and f_R, respectively.

Indicate whether f_D is greater than, less than, or equal to f_R by writing one of the following.

- $f_D > f_R$
- $f_D < f_R$
- $f_D = f_R$

Justify your answer using qualitative reasoning beyond referencing equations.

B. A cylinder has mass M, radius R, and rotational inertia I about its central axis. The cylinder rolls without slipping up a ramp that is inclined at an angle θ above the horizontal.

Derive an expression for the magnitude of the static frictional force f exerted on the cylinder by the ramp. Express your answer in terms of M, R, I, θ, and physical constants, as appropriate. Begin your derivation by writing a fundamental physics principle or an equation from the reference information.

C. In a different scenario, the centers of mass of the original disk and ring each have the same initial speed v as they did in the original scenario. The ramp is replaced by a new ramp on which the disk and the ring initially slip as they roll up the new ramp.

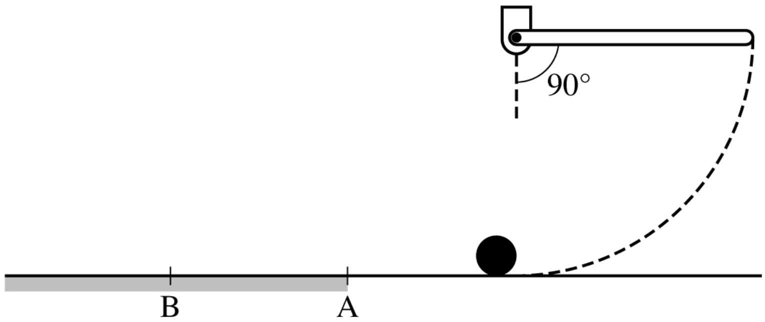
Indicate whether the magnitude of the kinetic frictional force exerted on the disk by the new ramp is greater than, less than, or equal to the magnitude of the kinetic frictional force exerted on the ring by the new ramp while both are slipping.

Briefly **justify** your answer.

STOP

END OF EXAM

Begin your response to **QUESTION 3** on this page.



Note: Figure not drawn to scale.

Figure 1

3. A system consists of a small sphere of mass m and radius R at rest on a horizontal surface and a uniform rod of mass $M = 2m$ and length ℓ attached at one end to a pivot with negligible friction, where $R \ll \ell$. There is negligible friction between the surface and the sphere to the right of Point A and nonnegligible friction to the left of Point A. The rod is held horizontally as shown in Figure 1, then is released from rest. The total rotational inertia of the rod about the pivot is $\frac{1}{3}M\ell^2$ and the rotational inertia of the sphere about its center is $\frac{2}{5}mR^2$. After the rod is released, the rod swings down and strikes the sphere head-on. As a result of this collision, the rod is stopped, and the ball initially slides without rotating to the left across the horizontal surface.

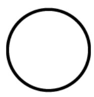
(a) Derive an expression for the angular speed of the rod just before striking the sphere in terms of the length ℓ and physical constants as appropriate.

GO ON TO THE NEXT PAGE.

(b) Derive an expression for the linear speed v_0 of the sphere immediately after colliding with the rod in terms of the length ℓ and physical constants as appropriate.

After sliding a short distance, at time $t = 0$ the sphere encounters a region of the horizontal surface with a coefficient of kinetic friction μ, beginning at Point A as indicated in Figure 1. The sphere begins rotating while sliding and eventually begins rolling without sliding at Point B, also as indicated.

(c) In the following diagram, which represents the sphere while the sphere is traveling between Points A and B, draw and label the forces (not components) that act on the sphere. Each force must be represented by a distinct arrow starting on, and pointing away from, the point of application on the sphere.



(d) Derive an expression for each of the following as the sphere is rotating and sliding between points A and B in terms of v_0, μ, R, t, and physical constants as appropriate.

i. The linear velocity v of the center of mass of the sphere as a function of time t

ii. The angular velocity ω of the sphere as a function of time t

GO ON TO THE NEXT PAGE.

Continue your response to **QUESTION 3** on this page.

(e)

i. Derive an expression for the time it takes the sphere to travel from Point A to Point B in terms of v_0, μ, and physical constants as appropriate.

ii. Derive an expression for the linear velocity of the sphere upon reaching Point B in terms of v_0.

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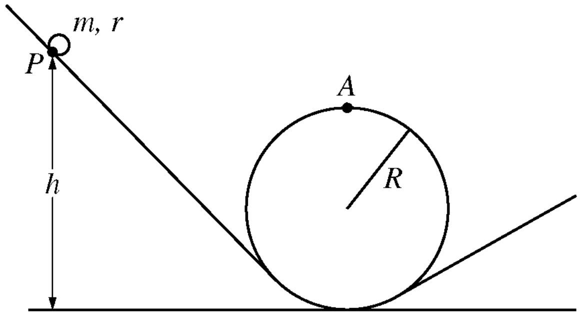
END OF EXAM



Note: Figure not drawn to scale.

3. The rotational inertia of a rolling object may be written in terms of its mass m and radius r as $I = bmr^2$, where b is a numerical value based on the distribution of mass within the rolling object. Students wish to conduct an experiment to determine the value of b for a partially hollowed sphere. The students use a looped track of radius $R >> r$, as shown in the figure above. The sphere is released from rest a height h above the floor and rolls around the loop.

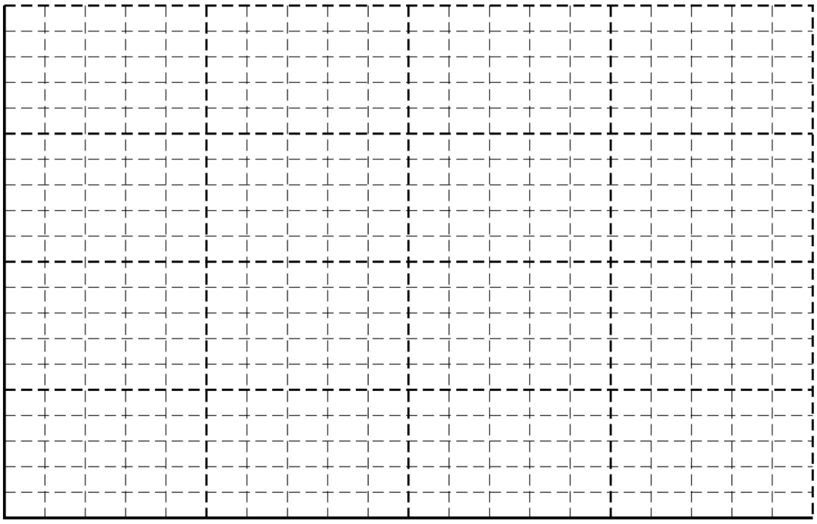
(a) Derive an expression for the minimum speed of the sphere's center of mass that will allow the sphere to just pass point A without losing contact with the track. Express your answer in terms of b, m, R, and fundamental constants, as appropriate.

(b) Suppose the sphere is released from rest at some point P and rolls without slipping. Derive an equation for the minimum release height h that will allow the sphere to pass point A without losing contact with the track. Express your answer in terms of b, m, R, and fundamental constants, as appropriate.

The students perform an experiment by determining the minimum release height h for various other objects of radius r and known values of b. They collect the following data.

Object	b	h (m)
Solid sphere	0.40	1.08
Hollow sphere	0.67	1.13
Solid cylinder	0.50	1.10
Hollow cylinder	1.0	1.20

(c) On the grid below, plot the release height h as a function of b. Clearly scale and label all axes, including units, if appropriate. Draw a straight line that best represents the data.



(d) The students repeat the experiment with the partially hollowed sphere and determine the minimum release height to be 1.16 m. Using the straight line from part (c), determine the value of b for the partially hollowed sphere.

(e) Calculate R, the radius of the loop.

(f) In part (b), the radius r of the rolling sphere was assumed to be much smaller than the radius R of the loop. If the radius r of the rolling sphere was not negligible, would the value of the minimum release height h be greater, less, or the same?

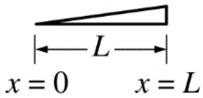
____ Greater ____ Less ____ The same

Justify your answer.

STOP

END OF EXAM

$$\lambda = \left(\frac{2M}{L^2}\right)x$$



3. A triangular rod, shown above, has length L, mass M, and a nonuniform linear mass density given by the equation $\lambda = \dfrac{2M}{L^2}x$, where x is the distance from one end of the rod.

   (a) Using integral calculus, show that the rotational inertia of the rod about its left end is $ML^2/2$.

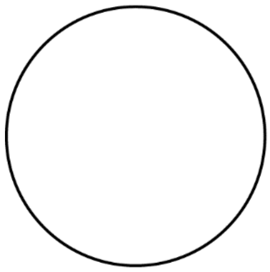


Figure 1

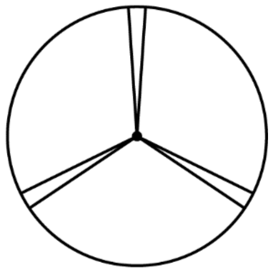


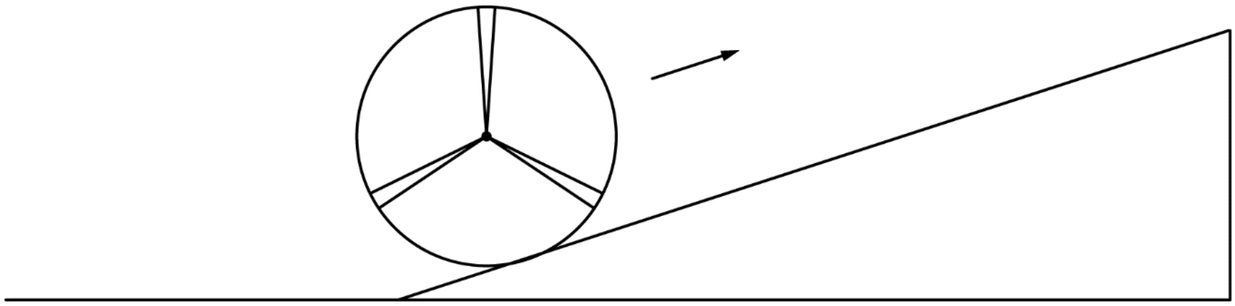
Figure 2

The thin hoop shown above in Figure 1 has a mass M, radius L, and a rotational inertia around its center of ML^2. Three rods identical to the rod from part (a) are now fastened to the thin hoop, as shown in Figure 2 above.

(b) Derive an expression for the rotational inertia I_{tot} of the hoop-rods system about the center of the hoop. Express your answer in terms of M, L, and physical constants, as appropriate.

The hoop-rods system is initially at rest and held in place but is free to rotate around its center. A constant force F is exerted tangent to the hoop for a time Δt.

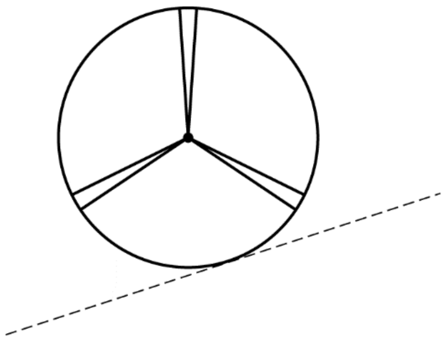
(c) Derive an expression for the final angular speed ω of the hoop-rods system. Express your answer in terms of M, L, F, Δt, and physical constants, as appropriate.



The hoop-rods system is rolling without slipping along a level horizontal surface with the angular speed ω found in part (c). At time $t = 0$, the system begins rolling without slipping up a ramp, as shown in the figure above.

(d)

i. On the figure of the hoop-rods system below, draw and label the forces (not components) that act on the system. Each force must be represented by a distinct arrow starting at, and pointing away from, the point at which the force is exerted on the system.

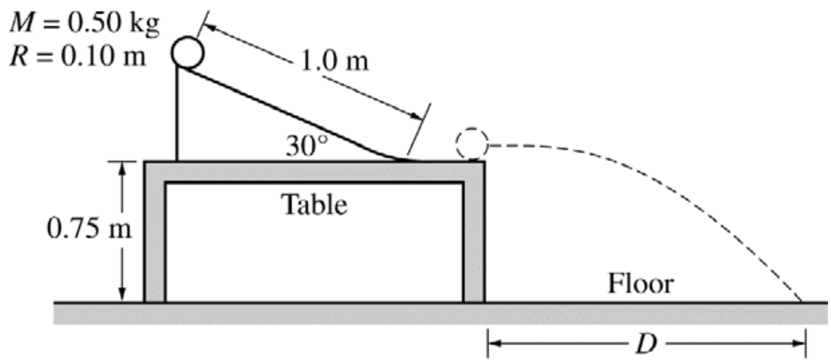


ii. Justify your choice for the direction of each of the forces drawn in part (d)i.

(e) Derive an expression for the change in height of the center of the hoop from the moment it reaches the bottom of the ramp until the moment it reaches its maximum height. Express your answer in terms of M, L, I_{tot}, ω, and physical constants, as appropriate.

STOP

END OF EXAM



3. A uniform solid cylinder of mass $M = 0.50$ kg and radius $R = 0.10$ m is released from rest, rolls without slipping down a 1.0 m long inclined plane, and is launched horizontally from a horizontal table of height 0.75 m. The inclined plane makes an angle of $30°$ with the horizontal. The cylinder lands on the floor a distance D away from the edge of the table, as shown in the figure above. There is a smooth transition from the inclined plane to the horizontal table, and the motion occurs with no frictional energy losses. The rotational inertia of a cylinder around its center is $MR^2/2$.

(a) Calculate the total kinetic energy of the cylinder as it reaches the horizontal table.

(b) Calculate the angular velocity of the cylinder around its axis at the moment it reaches the floor.

(c) Calculate the ratio of the rotational kinetic energy to the total kinetic energy for the cylinder at the moment it reaches the floor.

(d) Calculate the horizontal distance D.

A sphere of the same mass and radius is now rolled down the same inclined plane. The rotational inertia of a sphere around its center is $\frac{2}{5}MR^2$.

(e)

i. Is the total kinetic energy of the sphere at the moment it reaches the floor greater than, less than, or equal to the total kinetic energy of the cylinder at the moment it reaches the floor?

____ Greater than ____ Less than ____ Equal to

Justify your answer.

ii. Is the rotational kinetic energy of the sphere at the moment it reaches the floor greater than, less than, or equal to the rotational kinetic energy of the cylinder at the moment it reaches the floor?

____ Greater than ____ Less than ____ Equal to

Justify your answer.

iii. Is the horizontal distance the sphere travels from the table to where it hits the floor greater than, less than, or equal to the horizontal distance the cylinder travels from the table to where it hits the floor?

____ Greater than ____ Less than ____ Equal to

Justify your answer.

STOP

END OF EXAM

6.6 Motion of Orbiting Satellites

Name: ______________ Class: __________ Date: ____________ **Total: 13 marks**

KEY WORDS satellite 卫星 • kepler's third law 开普勒第三定律 • centripetal force 向心力 • gravity 引力

Objective

Build the skills to answer exam questions on **the motion of orbiting satellites**.

You must be able to:

- explain that **gravity 引力** provides the centripetal force for a circular orbit
- derive the orbital speed: $v = \sqrt{\dfrac{GM}{r}}$
- use **Kepler's third law 开普勒第三定律**: $T^2 \propto r^3$
- state that a satellite's own mass cancels out

1 Worked examples

Study these first. Each one shows the method for a question type used later — follow the steps and you can do the Practice and Exam-style questions yourself.

■ Gravity is the centripetal force

For a circular orbit,

$$\frac{GMm}{r^2} = \frac{mv^2}{r} \Rightarrow v = \sqrt{\frac{GM}{r}}.$$

Lower orbits are faster; higher orbits are slower.

■ Kepler's third law

The period and radius are linked by

$$T^2 = \frac{4\pi^2}{GM} r^3,$$

so $T^2 \propto r^3$ – a bigger orbit takes much longer.

■ The mass cancels

The satellite's mass m cancels, so orbital speed and period do not depend on it.

2 Practice

Now apply the methods above.

2.1 What provides the centripetal force that holds a satellite in orbit? [1]

2.2 A satellite moves to a lower orbit. State whether its speed increases or decreases. [1]

2.3 Compute the orbital speed where $GM/r = 4.0 \times 10^7\ \mathrm{m^2\ s^{-2}}$. [2]

2.4 State whether a heavier satellite at the same radius orbits faster, slower, or the same. [1]

3 Exam-style questions

3.1 The orbital speed of a satellite depends on [1]

- **A** its own mass
- **B** the central mass and the orbital radius
- **C** its colour
- **D** the launch angle

3.2 According to Kepler's third law, a larger orbit has a [1]

- **A** shorter period
- **B** longer period
- **C** the same period
- **D** zero period

3.3 Satellite B orbits at twice the radius of satellite A (same planet).

(a) State how B's orbital speed compares with A's (qualitatively). [1]

(b) Using $T^2 \propto r^3$, state how B's period compares with A's. [2]

3.4 A satellite orbits where $GM/r = 2.5 \times 10^7\ \mathrm{m^2\ s^{-2}}$.

(a) Find its orbital speed. [2]

(b) State whether doubling the satellite's mass changes this speed. [1]

Solutions

2.1 Gravity.

2.2 It increases (lower orbits are faster).

2.3 $v = \sqrt{4.0 \times 10^7} \approx 6300\ \text{m s}^{-1}$.

2.4 The same (mass cancels).

3.1 **B** — the central mass and the orbital radius.

3.2 **B** — a larger orbit has a longer period.

3.3 (a) B is slower (larger r gives smaller v). (b) $T^2 \propto r^3$, so $T_B/T_A = 2^{3/2} \approx 2.8$ times longer.

3.4 (a) $v = \sqrt{2.5 \times 10^7} \approx 5000\ \text{m s}^{-1}$. (b) No — mass cancels, so the speed is unchanged.