Week 20

AP Physics C: Mechanics

Homework bundle for this week

本周作业合集

exercises.lol

Contents 目录

5. Torque and Rotational Dynamics

Starter Quiz · 课前小测

AP Physics C: Mechanics   Name: ____________   Score: ______

1. A disk spins at $f = 4$ turns per second. What is its angular velocity (in rad/s)?

   ______________ rad/s

2. A wheel of radius 0.5 m spins at $\omega = 8\ \frac{\text{rad}}{\text{s}}$. The linear speed of a rim point (in m/s)?

   ______________ m/s

3. A 20 N force pushes perpendicular to a wrench 0.25 m from the bolt. What torque results (in $\text{N} \cdot \text{m}$)?

   ______________ $\text{N} \cdot \text{m}$

4. Two 3 kg masses sit 2 m from an axis on a light rod. Total rotational inertia (in $\text{kg} \cdot \text{m\^{}2}$)?

   ______________ $\text{kg} \cdot \text{m\^{}2}$

5. A 20 kg child sits 3 m left of a pivot. A 30 kg child balances it on the right. How far from the pivot must they sit (in m)?

   ______________ m

6. Angular acceleration α is the rotational twin of which linear quantity?

   ☐ position

   ☐ velocity

   ☐ acceleration

   ☐ mass

7. Two points on the same rigid spinning disk, at different radii, share the same...

   ☐ linear speed

   ☐ angular velocity

   ☐ distance travelled

   ☐ centripetal acceleration

8. Moving a mass twice as far from the axis changes its contribution to I by a factor of...

   ☐ 2

   ☐ 4

   ☐ 1 (no change)

   ☐ 0.5

9. For an object to be in full static equilibrium, which must be true?

- ☐ net force is zero only
- ☐ net torque is zero only
- ☐ both net force and net torque are zero
- ☐ the object must be at rest and never moving

10. In $\tau = I\alpha$, rotational inertia I plays the role of which quantity in $F = ma$?

- ☐ force
- ☐ mass
- ☐ acceleration
- ☐ velocity

AP Physics C: Mechanics

1. **25.13 rad/s**

   $\omega = 2\pi f = 2\pi(4) \approx 25.13\ \frac{\text{rad}}{\text{s}}$.

2. **4 m/s**

   $v = \omega r = 8 \times 0.5 = 4\ \frac{\text{m}}{\text{s}}$.

3. **5 N · m**

   $\tau = rF\sin 90° = 0.25 \times 20 \times 1 = 5\ \text{N} \cdot \text{m}$.

4. **24 kg · m^2**

   $I = \sum mr^2 = 3(2)^2 + 3(2)^2 = 12 + 12 = 24\ \text{kg} \cdot \text{m}^2$.

5. **2 m**

   Balance torques, $20 \times 3 = 30 \times x$, so $x = 60/30 = 2$ m.

6. **acceleration**

   $\alpha = d\omega/dt$ mirrors $a = dv/dt$ – angular acceleration is the twin of linear acceleration.

7. **angular velocity**

   A rigid body turns as one, so every point has the same ω. Linear speed differs with radius.

8. **4**

   $I \propto r^2$, so doubling r multiplies the contribution by $2^2 = 4$.

9. **both net force and net torque are zero**

   Static equilibrium requires zero net force *and* zero net torque.

10. **mass**

    Torque ~ force, I ~ mass, α ~ acceleration. So I is the rotational mass.

5.1 Rotational Kinematics

Name: ____________ Class: ________ Date: __________ **Total: 15 marks**

KEY WORDS angular acceleration 角加速度 • angular velocity 角速度 • radians 弧度 • torque 力矩

Objective

Build the skills to answer exam questions on **rotational kinematics**.

You must be able to:

- use **angular velocity 角速度** ω and **angular acceleration 角加速度** α
- apply the rotational equations of motion (analogues of the linear ones)
- convert between revolutions, degrees, and radians
- relate constant α to ω and θ over time

1 Worked examples

Study these first. Each one shows the method for a question type used later — follow the steps and you can do the Practice and Exam-style questions yourself.

The rotational analogues

Position, velocity, and acceleration become angle θ, angular velocity ω, and angular acceleration α. For constant α:

$$\omega = \omega_0 + \alpha t, \qquad \theta = \omega_0 t + \tfrac{1}{2}\alpha t^2.$$

Using the equations

A wheel starts from rest ($\omega_0 = 0$) with $\alpha = 3$ rad s^{-2}. After 4 s:

$$\omega = 0 + 3(4) = 12 \text{ rad s}^{-1}.$$

Radians

One full turn is 2π rad. Angular quantities in the equations must be in radians.

2 Practice

Now apply the methods above.

2.1 A disc spins at a constant 6 rad s^{-1}. State its angular acceleration. [1]

2.2 A wheel starts from rest with $\alpha = 2$ rad s^{-2}. Find ω after 5 s. [2]

2.3 How many radians are in one full revolution? [1]

2.4 A flywheel at 10 rad s^{-1} decelerates at 2 rad s^{-2}. Find ω after 3 s. [2]

3 Exam-style questions

3.1 The rotational analogue of linear acceleration is [1]

- **A** angular velocity
- **B** angular acceleration
- **C** torque
- **D** angular momentum

3.2 A wheel starts from rest and reaches 20 rad s^{-1} in 4 s. Its angular acceleration is [1]

- **A** 5 rad s^{-2}
- **B** 80 rad s^{-2}
- **C** 16 rad s^{-2}
- **D** 0.2 rad s^{-2}

3.3 A turntable starts from rest with $\alpha = 4$ rad s^{-2}.

(a) Find its angular velocity after 3 s. [2]

(b) Find the angle turned in that time. [2]

3.4 A fan blade slows from 30 rad s^{-1} to rest in 6 s at constant deceleration.

(a) Find the angular acceleration. [2]

(b) State its sign and what it means. [1]

Solutions

2.1 Zero (constant ω).

2.2 $\omega = \omega_0 + \alpha t = 0 + 2(5) = 10 \text{ rad s}^{-1}$.

2.3 2π rad.

2.4 $\omega = 10 + (-2)(3) = 4 \text{ rad s}^{-1}$.

3.1 B — angular acceleration.

3.2 A — $\alpha = 20/4 = 5 \text{ rad s}^{-2}$.

3.3 (a) $\omega = 4(3) = 12 \text{ rad s}^{-1}$. (b) $\theta = \frac{1}{2}(4)(3)^2 = 18$ rad.

3.4 (a) $\alpha = (0 - 30)/6 = -5 \text{ rad s}^{-2}$. (b) Negative — it is decelerating (slowing the spin).

Three-Blade System

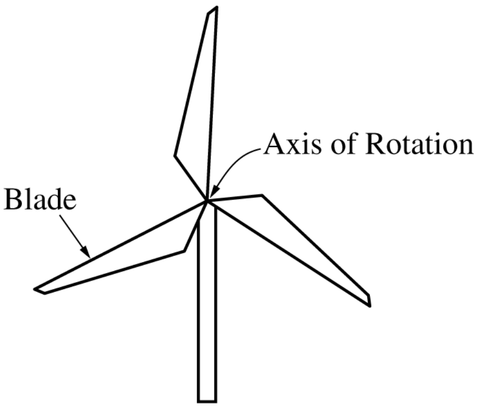

Figure 1

Single Blade

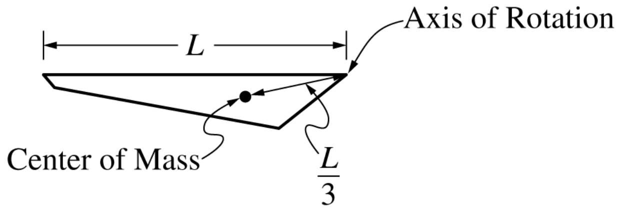

Figure 2

. A wind turbine includes a three-blade system that rotates about an axis through the end of each blade, as shown in Figure 1. Each blade has a length L and mass M, with a center of mass located at a distance $\frac{L}{3}$ from the axis of rotation, as shown in Figure 2.

(a) Derive an expression for the rotational inertia of the three-blade system. Express your answer in terms of M, L, and physical constants, as appropriate. The rotational inertia of each blade about an axis through its center of mass is given by the equation $I_{cm} = \frac{1}{18}ML^2$.

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Continue your response to **QUESTION 3** on this page.

(b) While the wind blows, the three-blade system operates at a constant angular speed $\omega_0 = 2.6$ rad/s. The length of one blade is $L = 36$ m. The numerical value of the rotational inertia of the system is $I_{sys} = 6.7 \times 10^6\ \text{kg}\cdot\text{m}^2$. Calculate the time T it takes the outer edge of a single blade to complete one revolution.

(c) When the wind stops blowing, the angular speed of the system decreases. The angular speed ω of the system while slowing down is given as a function of time t by the equation $\omega = \omega_0 e^{-\beta_0 t}$, where β_0 is a constant with appropriate units, as shown on the graph in Figure 3.

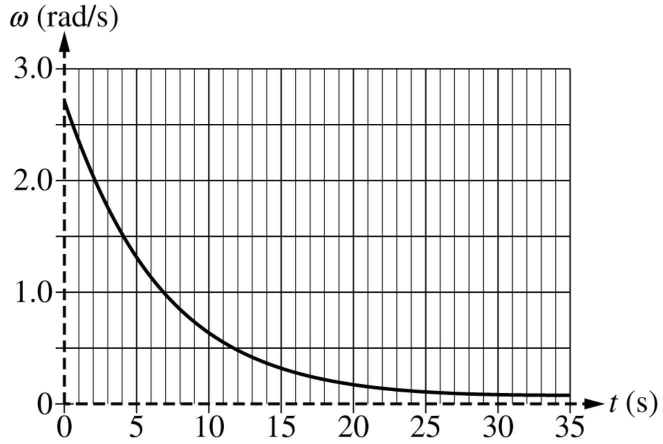

Figure 3

i. Calculate the amount of energy dissipated from $t = 0$, when the wind stops blowing, until the system comes to rest.

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ii. Derive an expression for the net torque exerted on the system as a function of t as the system slows down. Express your answers in terms of β_0, ω_0, M, L, I_{sys}, and physical constants, as appropriate.

iii. Derive an expression for the angular displacement of the system $\Delta\theta$ as a function of t. Express your answer in terms of β_0, ω_0, M, L, I_{sys}, and physical constants, as appropriate.

The three-blade system is now replaced with a second three-blade system identical to the first, except that the second three-blade system slows down according to the equation $\omega = \omega_0 e^{-\beta t}$, where $\omega_0 = 2.6$ rad/s and $\beta > \beta_0$. The original angular speed function is shown as a dashed line in Figure 4.

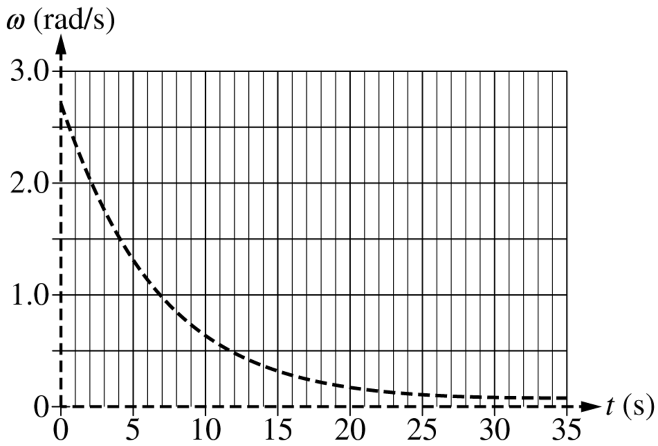

Figure 4

(d) On the graph in Figure 4, sketch the angular speed of the second three-blade system as a function of time t.

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END OF EXAM

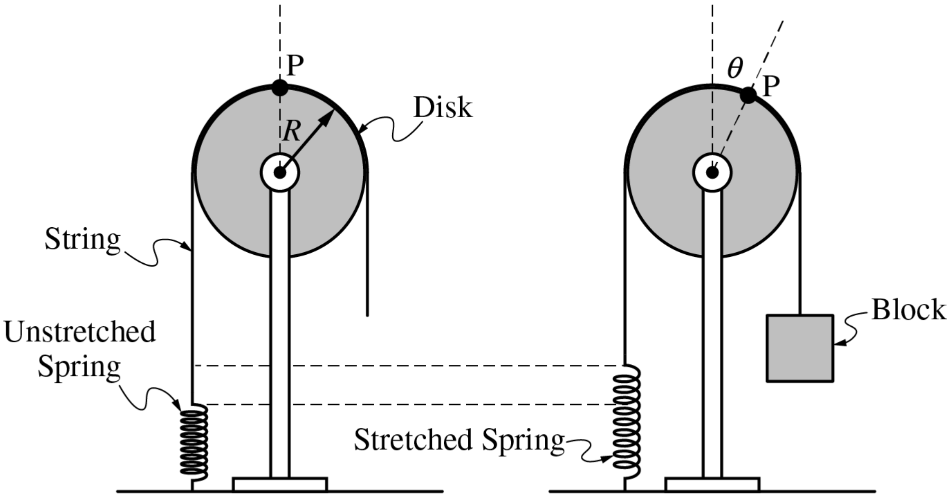

Figure 1 Figure 2

Note: Figures not drawn to scale.

. A solid uniform disk is supported by a vertical stand. The disk is able to rotate with negligible friction about an axle that passes through the center of the disk. The mass and radius of the disk are given by M_d and R, respectively. The rotational inertia of the disk is $I_d = \frac{1}{2}M_dR^2$. A string of negligible mass is draped over the disk and attached to the top of the disk at point P. One end of the string is connected to an unstretched ideal spring of spring constant k, which is fixed to the ground as shown in Figure 1.

A block of mass m_B is then attached to the string on the right side of the disk. The block is slowly lowered until the spring-disk-block system reaches equilibrium, as shown in Figure 2. In this equilibrium position, the disk has rotated clockwise through a small angle θ.

Give all algebraic answers in terms of M_d, R, k, θ, and physical constants, as appropriate.

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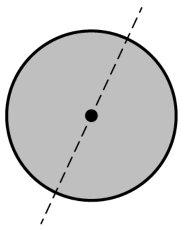
Continue your response to **QUESTION 3** on this page.

(a) Derive an expression for the mass m_B of the block.

(b) At time $t = 0$, the string on the right side of the disk is cut and the block falls to the ground. On the circle below, which represents the disk, draw and label the forces (not components) that act on the disk immediately after the string is cut and the block is falling to the ground. Each force should be represented by an arrow that starts on and is directed away from the point of application.



(c) Derive an expression for the angular acceleration α of the disk immediately after the string is cut.

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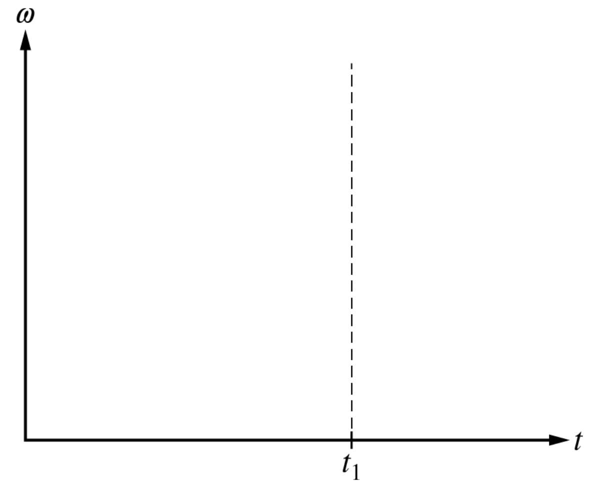
(d) At $t = t_1$, the disk has rotated and point P is again directly above the axle. Sketch a graph of the magnitude of the angular velocity ω of the disk as a function of time t from $t = 0$ to $t = t_1$.



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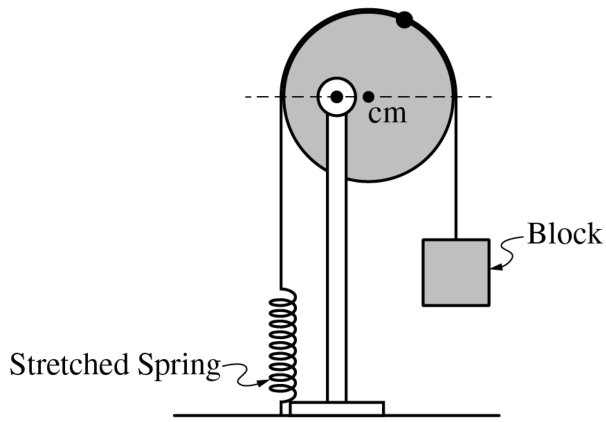

Figure 3

Note: Figure not drawn to scale.

(e) The disk is adjusted on the support so that the axle does not pass through the center of mass of the disk. The block is again hung on the right side of the disk and the spring-disk-block system comes to equilibrium, as shown in Figure 3. The axle does not exert a torque on the disk. For each force on the disk, indicate whether the magnitude of the torque about the axle caused by that force increases, decreases, or stays the same relative to part (b).

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END OF EXAM

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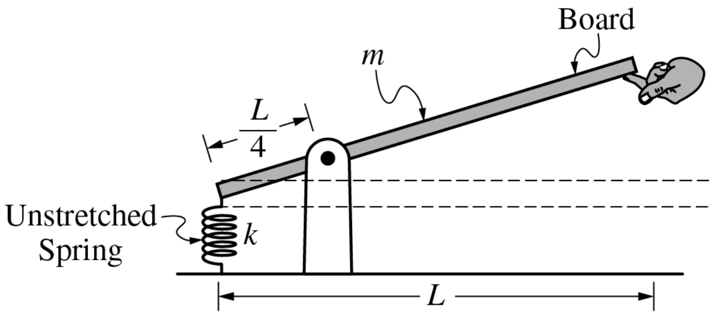


Figure 1

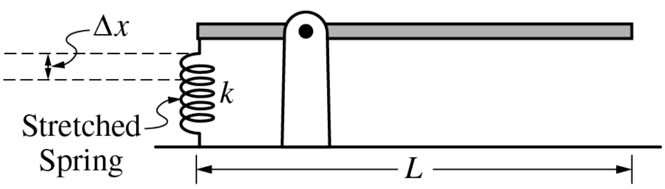
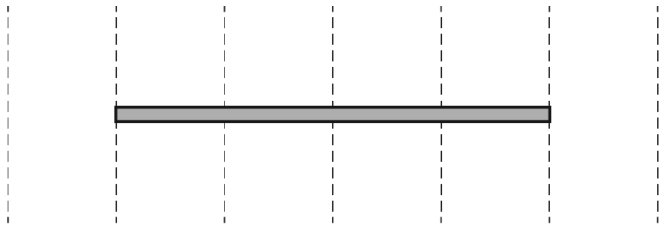


Figure 2

Note: Figures not drawn to scale.

. A uniform board of length L and mass m is attached to a pivot $\frac{L}{4}$ from the left end of the board. The left end of the board is attached to an ideal spring of spring constant k that is attached to the ground. The right end of the board is initially held by a student so that the spring is unstretched, as shown in Figure 1. The student slowly lowers and then releases the board. The board remains at rest in the horizontal position, with the spring stretched, as shown in Figure 2. The rotational inertia of the board about the pivot is I.

(a) On the rectangle below, which represents the board, draw and label the forces (not components) that act on the board while the board-spring system is in equilibrium. Each force should be represented by an arrow that starts on, and points away from, the board, and should represent the location at which that force acts.



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(b) Derive an expression for the distance the spring stretches, Δx, when the board is in equilibrium. Express your answer in terms of k, L, m, and physical constants, as appropriate.

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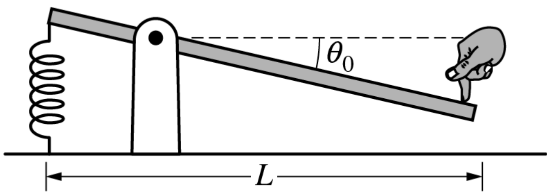
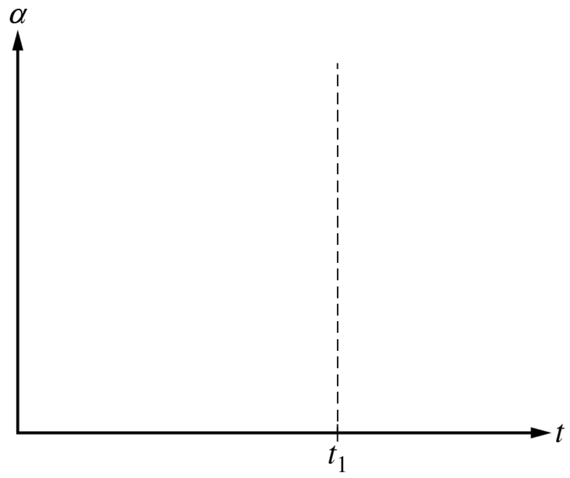


Note: Figure not drawn to scale.

(c) A student pushes the board down on the right side, stretching the spring a new distance Δx_2 from the unstretched position. The board is held at a small angle θ_0 with the horizontal, as shown. The student then releases the board from rest.

i. At time $t = 0$, the board is released. At $t = t_1$, the board first crosses the horizontal. Sketch a graph of the magnitude of the angular acceleration α of the board as a function of time t from $t = 0$ to $t = t_1$.



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ii. Derive an expression for the angular acceleration α_0 of the board immediately after the board is released. Express your answer in terms of k, L, m, I, Δx_2, θ_0, and physical constants, as appropriate.

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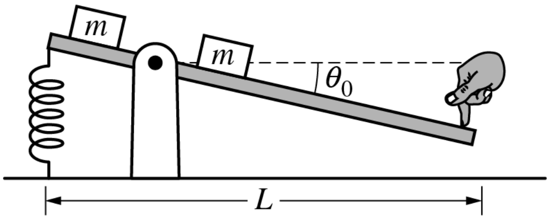

Note: Figure not drawn to scale.

(d) Two blocks of equal mass m are attached to the board equal distances from the pivot point, as shown. The board is again pushed down on the right side so that the spring stretches the same distance Δx_2 as in part (c). The board is then released. How does the new angular acceleration α' when the blocks are attached compare to the angular acceleration α_0 from part (c) ?

___$\alpha' > \alpha_0$ ___$\alpha' < \alpha_0$ ___ $\alpha' = \alpha_0$

Justify your answer.

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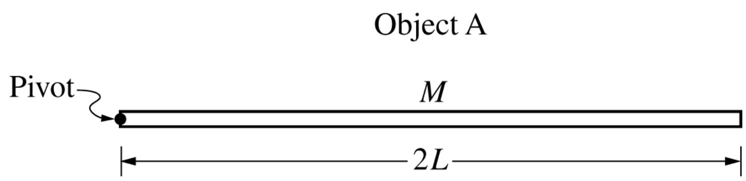
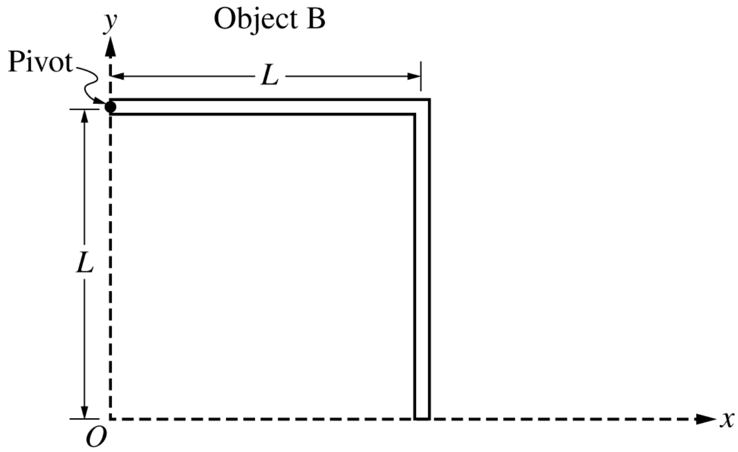


2. Object A is a long, thin, uniform rod of mass M and length $2L$ that is free to rotate about a pivot of negligible friction at its left end, as shown above.

(a) Using integral calculus, derive an expression to show that the rotational inertia I_A of object A about the pivot is given by $\frac{4}{3}ML^2$.



Object B of total mass M is formed by attaching two thin, uniform, identical rods of length L at a right angle to each other. Object B is held in place, as shown above. Express your answers in part (b) in terms of L .

(b) Determine the following for the given coordinate system shown in the figure.

i. The x-coordinate of the center of mass of object B

ii. The y-coordinate of the center of mass of object B

Object B has a rotational inertia of I_B about its pivot.

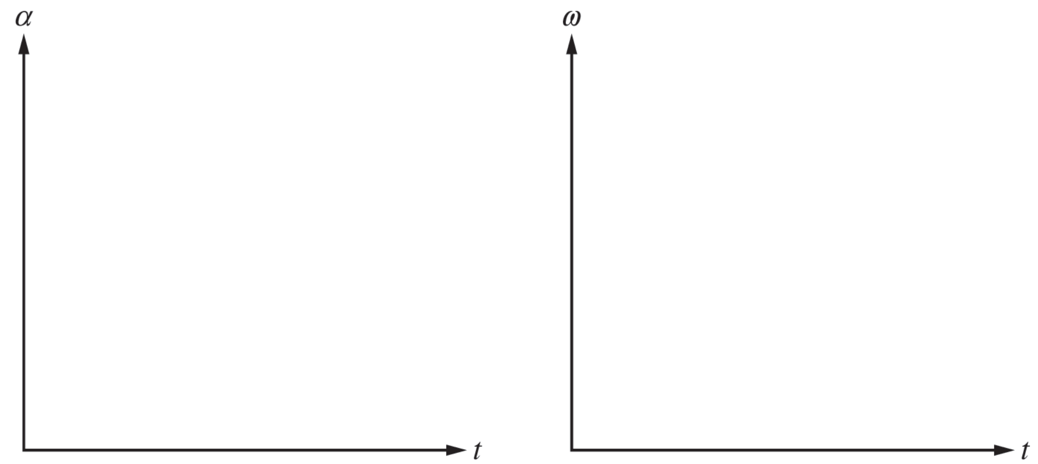
(c) Is the value of I_B greater than, less than, or equal to I_A ?

_____ Greater than _____ Less than _____ Equal to

Justify your answer.

Object B is released from rest and begins to rotate about its pivot.

(d) On the axes below, sketch graphs of the magnitude of the angular acceleration α and the angular speed ω of object B as functions of time t from the time it is released to the time its center of mass reaches its lowest point.



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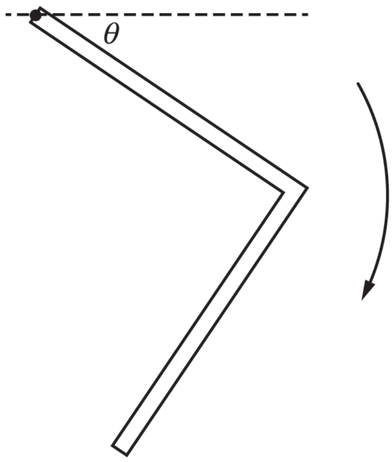
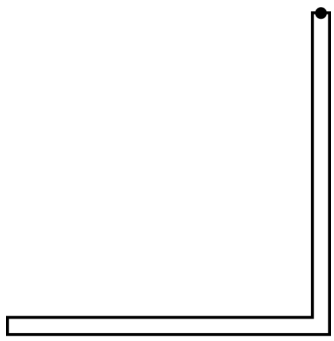


(e) While object B rotates from the horizontal position down through the angle θ shown above, is the magnitude of its angular acceleration increasing, decreasing, or not changing?

_____ Increasing _____ Decreasing _____ Not changing

Justify your answer.



Object B rotates through the position shown above.

(f) Derive an expression for the angular speed of object B when it is in the position shown above. Express your answer in terms of M, L, I_B, and physical constants, as appropriate.

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5.2 Connecting Linear and Rotational Motion

Name: ______________ Class: __________ Date: ____________ **Total: 15 marks**

KEY WORDS tangential acceleration 切向加速度 • angular velocity 角速度 • angular acceleration 角加速度 • pulley 滑轮

Objective

Build the skills to answer exam questions on **connecting linear and rotational motion**.

You must be able to:

- relate linear and angular speed: $v = \omega r$
- relate linear and angular acceleration: $a = \alpha r$
- explain that points farther from the axis move faster
- apply these to wheels, gears, and pulleys

1 Worked examples

Study these first. Each one shows the method for a question type used later — follow the steps and you can do the Practice and Exam-style questions yourself.

■ Speed from angular speed

A point at radius r on a body spinning at ω moves at

$$v = \omega r.$$

The rim of a wheel of radius 0.3 m spinning at 10 rad s^{-1} moves at $v = 10 \times 0.3 = 3$ m s^{-1}.

■ Acceleration link

Likewise the tangential acceleration is

$$a = \alpha r.$$

■ Farther out is faster

For the same ω, a bigger r gives a bigger v. The edge of a disc outruns points near the centre.

2 Practice

Now apply the methods above.

2.1 A wheel of radius 0.5 m spins at 8 rad s^{-1}. Find the rim speed. [2]

2.2 State how the speed of a point on a spinning disc depends on its distance from the axis. [1]

2.3 A point at $r = 2$ m has angular acceleration 3 rad s^{-2}. Find its tangential acceleration. [2]

2.4 A rim moves at 6 m s^{-1} on a wheel of radius 0.2 m. Find the angular velocity. [2]

3 Exam-style questions

3.1 The relationship between rim speed and angular speed is [1]

- **A** $v = \omega/r$
- **B** $v = \omega r$
- **C** $v = r/\omega$
- **D** $v = \omega r^2$

3.2 Two points on a spinning disc: A is near the centre, B is at the rim. Compared with A, point B moves [1]

- **A** slower
- **B** faster
- **C** at the same speed
- **D** in the opposite direction

3.3 A bicycle wheel of radius 0.35 m rolls so its rim moves at 7 m s^{-1}.

(a) Find its angular velocity. [2]

(b) State the speed of the point where it touches the ground. [1]

3.4 A grinding wheel of radius 0.1 m has angular acceleration 20 rad s^{-2}.

(a) Find the tangential acceleration of a point on the rim. [2]

(b) State whether a point halfway to the centre has more or less tangential acceleration. [1]

Solutions

2.1 $v = \omega r = 8 \times 0.5 = 4$ m s^{-1}.

2.2 It is proportional to the distance from the axis (farther is faster).

2.3 $a = \alpha r = 3 \times 2 = 6$ m s^{-2}.

2.4 $\omega = v/r = 6/0.2 = 30$ rad s^{-1}.

3.1 B — $v = \omega r$.

3.2 B — the rim (larger r) moves faster.

3.3 (a) $\omega = v/r = 7/0.35 = 20$ rad s^{-1}. (b) Zero — the contact point is instantaneously at rest.

3.4 (a) $a = \alpha r = 20 \times 0.1 = 2$ m s^{-2}. (b) Less (smaller r).

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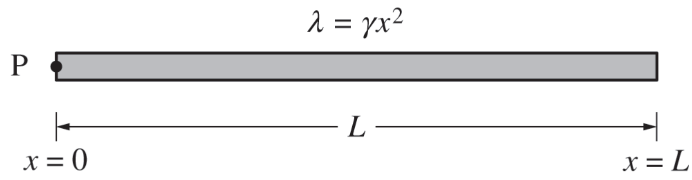

3. A triangular rod of length L and mass M has a nonuniform linear mass density given by the equation $\lambda = \gamma x^2$, where $\gamma = \dfrac{3M}{L^3}$ and x is the distance from point P at the left end of the rod.

(a) Using integral calculus, show that the rotational inertia I of the rod about an axis perpendicular to the page and through point P is $\dfrac{3}{5}ML^2$.

(b) Determine the horizontal location of the center of mass of the rod relative to point P. Express your answer in terms of L.

(c) For an axis perpendicular to the page, is the value of the rotational inertia of the rod around point P greater than, less than, or equal to the value of the rotational inertia of the rod around the rod's center of mass?

_____ Greater than   _____ Less than   _____ Equal to

Justify your answer.

GO ON TO THE NEXT PAGE.

The rod is released from rest in the position shown, and the rod begins to rotate about a horizontal axis perpendicular to the page and through point P.

(d) On the axes below, sketch graphs of the magnitude of the net torque τ on the rod and the angular speed ω of the rod as functions of time t from the time the rod is released until the time its center of mass reaches its lowest point.

(e) As the rod rotates from the horizontal position down through vertical, is the magnitude of the angular acceleration on the rod increasing, decreasing, or not changing?

_____ Increasing _____ Decreasing _____ Not changing

Justify your answer.

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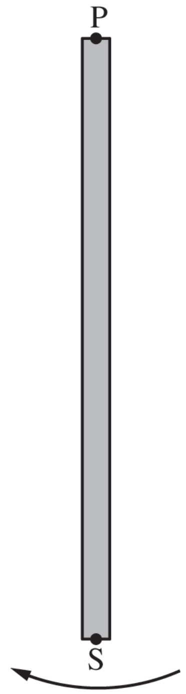

(f) The mass of the rod is 3.0 kg, and the length of the rod is 1.0 m. Calculate the linear speed v of point S as the rod swings through the vertical position shown.

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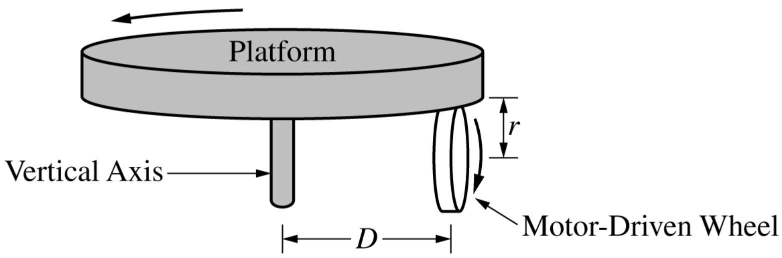
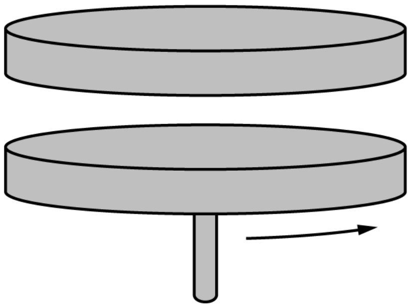


3. A horizontal circular platform with rotational inertia I_P rotates freely without friction on a vertical axis. A small motor-driven wheel that is used to rotate the platform is mounted under the platform and touches it. The wheel has radius r and touches the platform a distance D from the vertical axis of the platform, as shown above. The platform starts at rest, and the wheel exerts a constant horizontal force of magnitude F tangent to the wheel until the platform reaches an angular speed ω_P after time Δt . During time Δt , the wheel stays in contact with the platform without slipping.

   (a) Derive an expression for the angular speed ω_P of the platform. Express your answer in terms of I_P , r, D, F, Δt , and physical constants, as appropriate.

   (b) Determine an expression for the kinetic energy of the platform at the moment it reaches angular speed ω_P . Express your answer in terms of I_P , r, D, F, Δt , and physical constants, as appropriate.

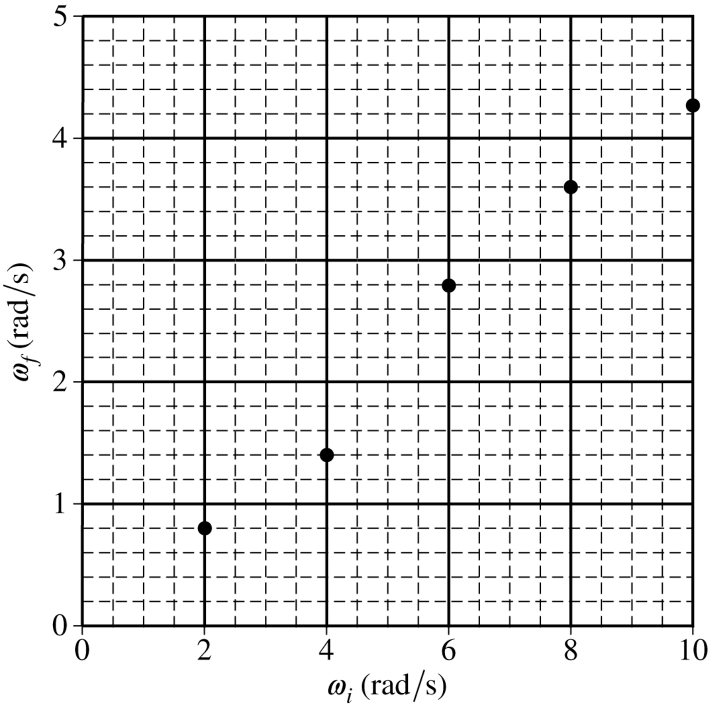
   (c) Derive an expression for the angular speed of the wheel ω_W when the platform has reached angular speed ω_P . Express your answer in terms of D, r, ω_P , and physical constants, as appropriate.



When the platform is spinning at angular speed ω_P , the motor-driven wheel is removed. A student holds a disk directly above and concentric with the platform, as shown above. The disk has the same rotational inertia I_P as the platform. The student releases the disk from rest, and the disk falls onto the platform. After a short time, the disk and platform are observed to be rotating together at angular speed ω_f .

   (d) Derive an expression for ω_f . Express your answer in terms of ω_P , I_P , and physical constants, as appropriate.

A student now uses the rotating platform $\left(I_P = 3.1\ \text{kg}\cdot\text{m}^2\right)$ to determine the rotational inertia I_U of an unknown object about a vertical axis that passes through the object's center of mass. The platform is rotating at an initial angular speed ω_i when the unknown object is dropped with its center of mass directly above the center of the platform. The platform and object are observed to be rotating together at angular speed ω_f. Trials are repeated for different values of ω_i. A graph of ω_f as a function of ω_i is shown on the axes below.



(e)

i. On the graph on the previous page, draw a best-fit line for the data.

ii. Using the straight line, calculate the rotational inertia of the unknown object I_U about a vertical axis passing through its center of mass.

(f) The kinetic energy of the spinning platform before the object is dropped on it is K_i. The total kinetic energy of the platform-object system when it reaches angular speed ω_f is K_f. Which of the following expressions is true?

____ $K_f < K_i$ ____ $K_f = K_i$ ____ $K_f > K_i$

Justify your answer.

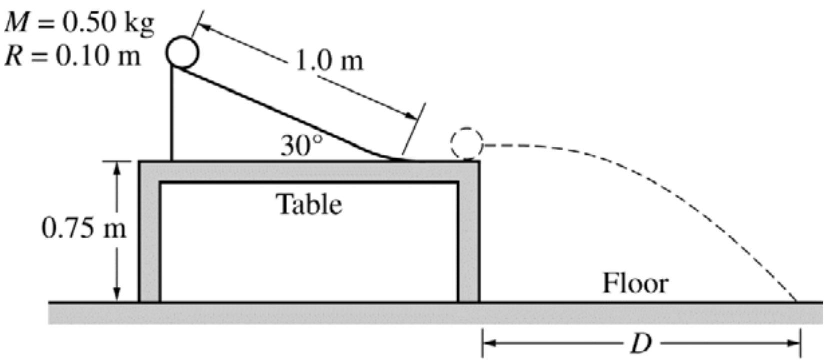
(g) One of the students observes that the center of mass of the object is not actually aligned with the axis of the platform. Is the experimental value of I_U obtained in part (e) greater than, less than, or equal to the actual value of the rotational inertia of the unknown object about a vertical axis that passes through its center of mass?

____ Greater than ____ Less than ____ Equal to

Justify your answer.

STOP

END OF EXAM



3. A uniform solid cylinder of mass $M = 0.50$ kg and radius $R = 0.10$ m is released from rest, rolls without slipping down a 1.0 m long inclined plane, and is launched horizontally from a horizontal table of height 0.75 m. The inclined plane makes an angle of $30°$ with the horizontal. The cylinder lands on the floor a distance D away from the edge of the table, as shown in the figure above. There is a smooth transition from the inclined plane to the horizontal table, and the motion occurs with no frictional energy losses. The rotational inertia of a cylinder around its center is $MR^2/2$.

(a) Calculate the total kinetic energy of the cylinder as it reaches the horizontal table.

(b) Calculate the angular velocity of the cylinder around its axis at the moment it reaches the floor.

(c) Calculate the ratio of the rotational kinetic energy to the total kinetic energy for the cylinder at the moment it reaches the floor.

(d) Calculate the horizontal distance D.

A sphere of the same mass and radius is now rolled down the same inclined plane. The rotational inertia of a sphere around its center is $\frac{2}{5}MR^2$.

(e)

i. Is the total kinetic energy of the sphere at the moment it reaches the floor greater than, less than, or equal to the total kinetic energy of the cylinder at the moment it reaches the floor?

____ Greater than ____ Less than ____ Equal to

Justify your answer.

ii. Is the rotational kinetic energy of the sphere at the moment it reaches the floor greater than, less than, or equal to the rotational kinetic energy of the cylinder at the moment it reaches the floor?

____ Greater than ____ Less than ____ Equal to

Justify your answer.

iii. Is the horizontal distance the sphere travels from the table to where it hits the floor greater than, less than, or equal to the horizontal distance the cylinder travels from the table to where it hits the floor?

____ Greater than ____ Less than ____ Equal to

Justify your answer.

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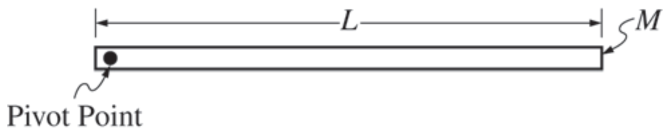
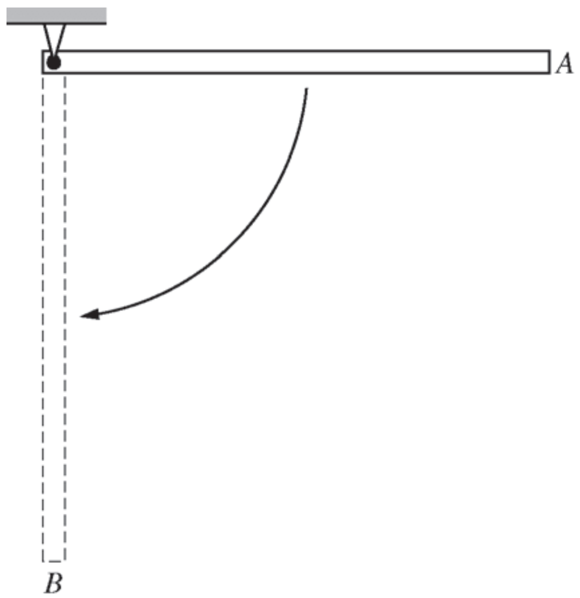
END OF EXAM



Mech.3.

A uniform, thin rod of length L and mass M is allowed to pivot about its end, as shown in the figure above.

(a) Using integral calculus, derive the rotational inertia for the rod around its end to show that it is $ML^2/3$.



The rod is fixed at one end and allowed to fall from the horizontal position A through the vertical position B.

(b) Derive an expression for the velocity of the free end of the rod at position B. Express your answer in terms of M, L, and physical constants, as appropriate.

An experiment is designed to test the validity of the expression found in part (b). A student uses rods of various lengths that all have a uniform mass distribution. The student releases each of the rods from the horizontal position A and uses photogates to measure the velocity of the free end at position B. The data are recorded below.

Length (m)	0.25	0.50	0.75	1.00	1.25	1.50
Velocity (m/s)	2.7	3.8	4.6	5.2	5.8	6.3

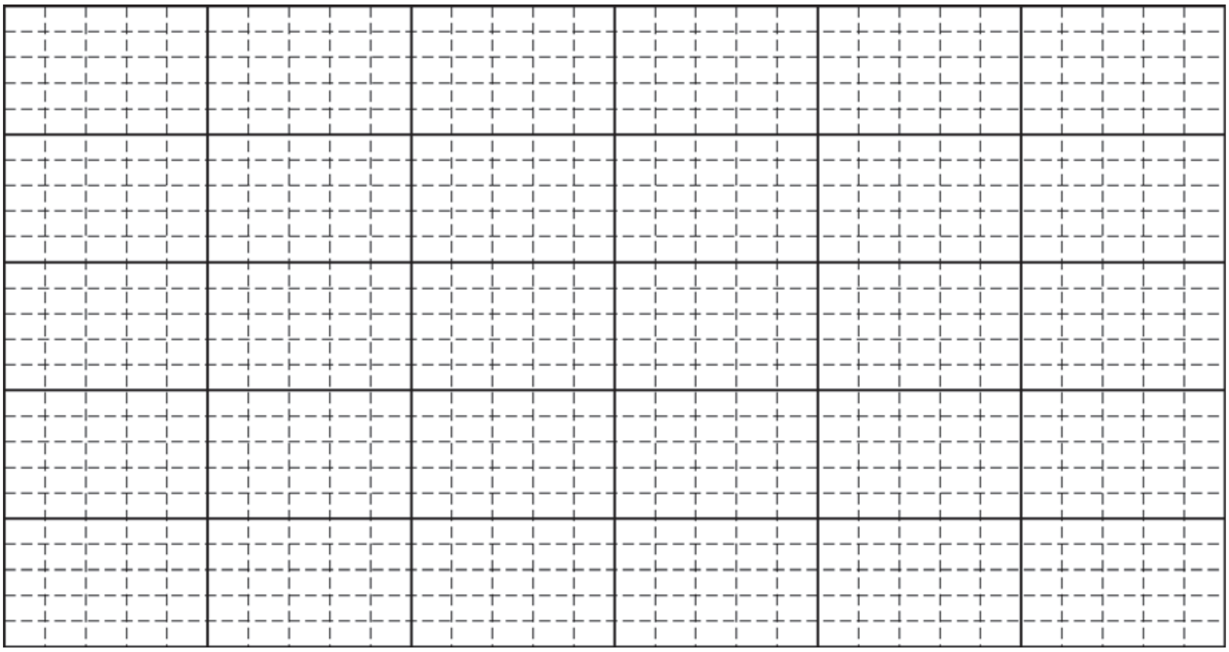
(c) Indicate below which quantities should be graphed to yield a straight line whose slope could be used to calculate a numerical value for the acceleration due to gravity g.

Horizontal axis: ____________________

Vertical axis: ____________________

Use the remaining rows in the table above, as needed, to record any quantities that you indicated that are not given. Label each row you use and include units.

(d) Plot the straight line data points on the grid below. Clearly scale and label all axes, including units as appropriate. Draw a straight line that best represents the data.



(e)

i. Using your straight line, determine an experimental value for g.

ii. Describe two ways in which the effects of air resistance could be reduced.

STOP

END OF EXAM

5.3 Torque

Name: ______________ Class: __________ Date: ____________ **Total: 15 marks**

KEY WORDS torque 力矩 • lever arm 力臂

Objective

Build the skills to answer exam questions on **torque**.

You must be able to:

- calculate **torque 力矩**: $\tau = rF\sin\theta$
- identify the **lever arm 力臂** (perpendicular distance from the axis to the line of the force)
- explain how the angle and distance affect the turning effect
- add torques with signs (clockwise vs anticlockwise)

1 Worked examples

Study these first. Each one shows the method for a question type used later — follow the steps and you can do the Practice and Exam-style questions yourself.

■ The torque formula

$$\tau = rF\sin\theta,$$

where r is the distance from the axis to where the force acts and θ is the angle between $\vec{r}$ and $\vec{F}$. A 10 N force at 0.5 m, perpendicular ($\theta = 90°$): $\tau = 0.5 \times 10 \times 1 = 5$ N m.

■ The lever arm

The perpendicular distance $r\sin\theta$ is the lever arm. A force pushing straight toward or away from the axis ($\theta = 0$) has zero lever arm and produces no torque.

■ Balancing torques

Anticlockwise torques are usually taken positive, clockwise negative. Sum them to find the net turning effect.

2 Practice

Now apply the methods above.

2.1 A 20 N force acts perpendicular to a spanner 0.3 m from the bolt. Find the torque. [2]

2.2 State the torque when a force acts straight along the line toward the axis. [1]

2.3 Define the lever arm. [1]

2.4 A 50 N force acts at 30° to a 0.4 m arm. Find the torque. [2]

3 Exam-style questions

3.1 The torque of a force is greatest when the force acts [1]

- **A** along the arm toward the axis
- **B** perpendicular to the arm
- **C** at 45° to the arm
- **D** at the axis

3.2 A force of 8 N acts at 0.25 m, perpendicular to the arm. The torque is [1]

- **A** 2 N m
- **B** 32 N m
- **C** 0.03 N m
- **D** 8 N m

3.3 Two children sit on a seesaw. A 300 N child sits 1.5 m from the pivot.

(a) Find the torque this child produces. [2]

(b) State where a 450 N child must sit on the other side to balance it. [2]

3.4 A 40 N force is applied to a door 0.8 m from the hinge, at 60° to the door.

(a) Find the torque about the hinge. [2]

(b) State how to increase the torque without changing the force. [1]

Solutions

2.1 $\tau = rF = 0.3 \times 20 = 6$ N m.

2.2 Zero (the lever arm is zero).

2.3 The perpendicular distance from the axis to the line of the force.

2.4 $\tau = rF \sin\theta = 0.4 \times 50 \times \sin 30° = 10$ N m.

3.1 **B** — perpendicular gives the maximum torque.

3.2 **A** — $\tau = 0.25 \times 8 = 2$ N m.

3.3 (a) $\tau = 1.5 \times 300 = 450$ N m. (b) $450 = 450 \times d \Rightarrow d = 1.0$ m from the pivot.

3.4 (a) $\tau = 0.8 \times 40 \times \sin 60° = 27.7$ N m. (b) Apply it perpendicular to the door (or farther from the hinge).

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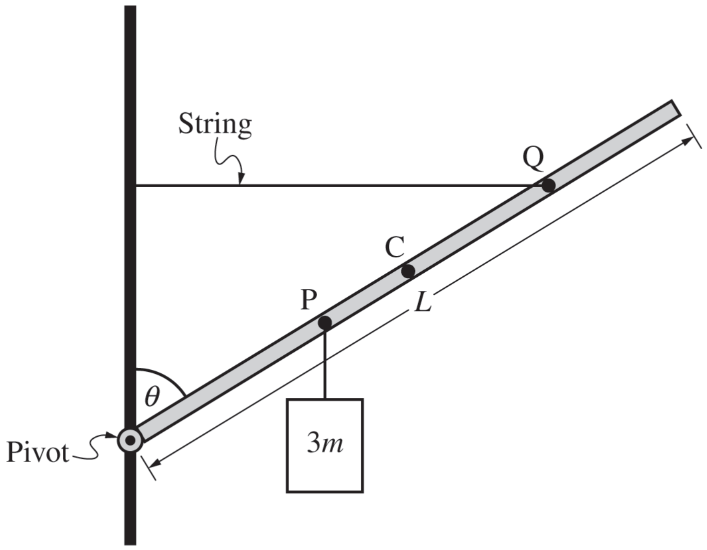
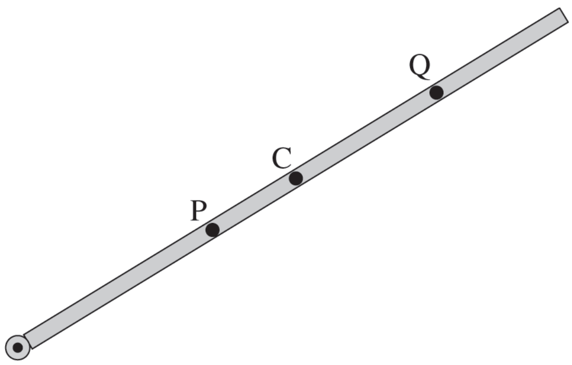


Figure 1

Note: Figure not drawn to scale.

3. A uniform rod of length L and mass m is attached to a pivot on a vertical pole, as shown in Figure 1. There is negligible friction between the rod and the pivot. A horizontal string connects Point Q on the rod to the pole. The rod makes an angle θ with the pole. A block of mass $3m$ hangs from the rod at Point P. The center of mass of the rod is located at Point C.

(a) On the following representation of the rod, **draw** and **label** the forces (not components) that are exerted on the rod. Each force must be represented by a distinct arrow that starts on and points away from the point at which the force is exerted on the rod.



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Continue your response to **QUESTION 3** on this page.

(b) In Figure 1, Point P is located $\frac{3}{8}L$ from the pivot and Point Q is located $\frac{6}{8}L$ from the pivot. **Derive** an equation for the tension F_T in the horizontal string in terms of L, m, θ, and physical constants, as appropriate.

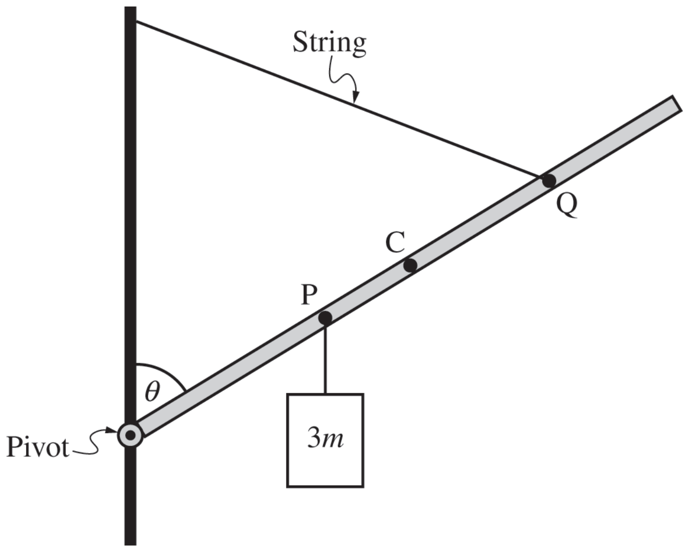

Figure 2

<u>Note:</u> Figure not drawn to scale.

(c) The original string is replaced with a longer string that connects Point Q to a higher location on the vertical pole, as shown in Figure 2. The angle θ remains the same. How does the new tension $F_{T,\text{ new}}$ compare with the original tension F_T from part (b) ? **Justify** your reasoning.

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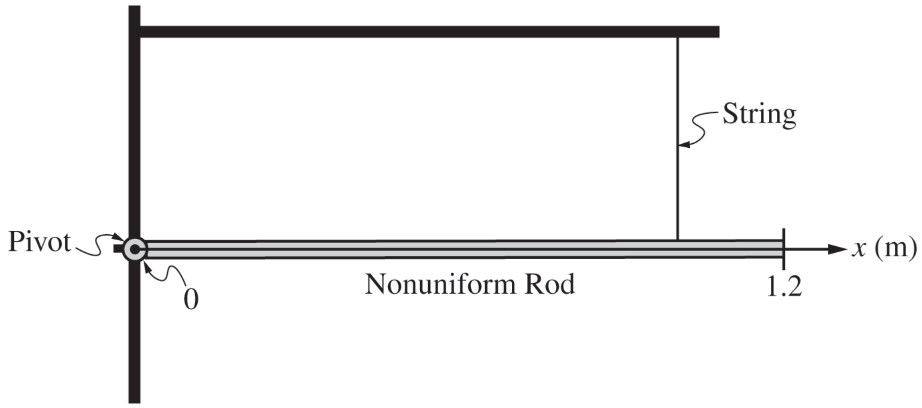

Figure 3

Note: Figure not drawn to scale.

(d) A nonuniform rod is now attached to the pivot, as shown in Figure 3. There is negligible friction between the nonuniform rod and the pivot. The rod has a length of 1.2 m and a linear mass density $\lambda(x) = A + Bx$, where x is the distance from the pivot, $A = 6.0$ kg/m, and $B = 10.0$ kg/m^2.

i. **Calculate** the mass of the rod.

ii. **Calculate** the rotational inertia of the rod about the pivot.

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END OF EXAM

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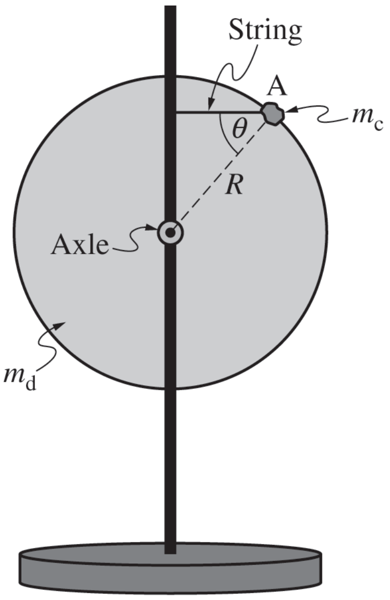
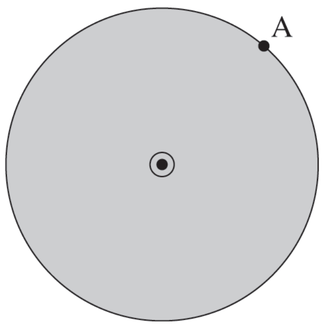


Figure 1

<u>Note:</u> Figure not drawn to scale.

3. A uniform disk of radius R and mass m_d is attached to a vertical pole by a horizontal axle that passes through the center of the disk. Friction between the axle and the disk is negligible. A lump of clay of mass m_c is attached to the edge of the disk at Point A. The size of the lump of clay is small compared with the radius of the disk. A horizontal string is connected from the pole to the edge of the disk at Point A. The string makes an angle θ with the line between Point A and the axle, as shown in Figure 1.

(a) On the following representation of the clay-disk system, **draw** and **label** the external forces (not components) exerted on the system. Each force must be represented by a distinct arrow that starts on, and points away from, the point at which the force is exerted on the system.



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Continue your response to **QUESTION 3** on this page.

(b) **Derive** an expression for the tension F_T in the string when the clay is at Point A, as shown in Figure 1, in terms of R, m_d, m_c, θ, and physical constants, as appropriate.

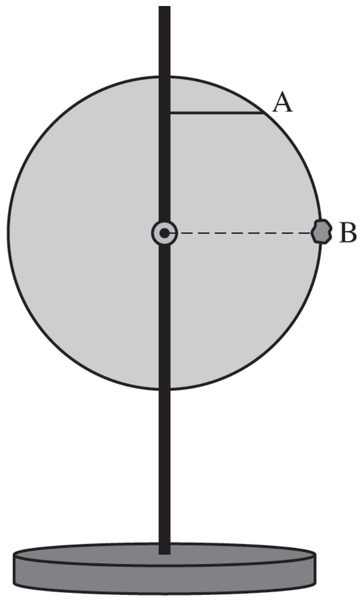

Figure 2

Note: Figure not drawn to scale.

(c) The string remains connected to the edge of the disk at Point A. The clay is moved to Point B, which is horizontally in line with the axle, as shown in Figure 2. How does the new tension $F_{T,\,new}$ compare with tension F_T from part (b)? **Justify** your reasoning.

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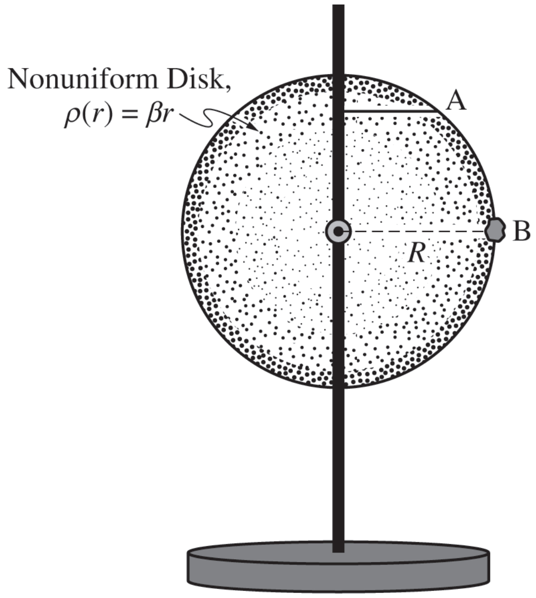

Figure 3

Note: Figure not drawn to scale.

(d) A nonuniform disk is now attached to the axle. The lump of clay is attached to the disk at Point B, as shown in Figure 3. The clay has mass $m_c = 0.60\,\text{kg}$ and the disk has a radius $R = 0.30\,\text{m}$. The mass density of the disk varies radially and can be modeled by $\rho(r) = \beta r$, where r is the radial distance from the axle and $\beta = 4.0\,\text{kg/m}^3$.

i. **Calculate** the rotational inertia of the disk about the axle.

ii. The string connecting the disk to the pole is cut. **Calculate** the magnitude of the initial angular acceleration of the clay-disk system.

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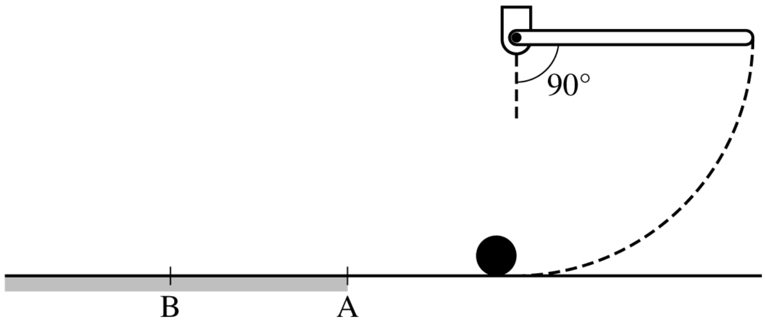
END OF EXAM



<u>Note</u>: Figure not drawn to scale.

Figure 1

3. A system consists of a small sphere of mass m and radius R at rest on a horizontal surface and a uniform rod of mass $M = 2m$ and length ℓ attached at one end to a pivot with negligible friction, where $R \ll \ell$. There is negligible friction between the surface and the sphere to the right of Point A and nonnegligible friction to the left of Point A. The rod is held horizontally as shown in Figure 1, then is released from rest. The total rotational inertia of the rod about the pivot is $\frac{1}{3}M\ell^2$ and the rotational inertia of the sphere about its center is $\frac{2}{5}mR^2$. After the rod is released, the rod swings down and strikes the sphere head-on. As a result of this collision, the rod is stopped, and the ball initially slides without rotating to the left across the horizontal surface.

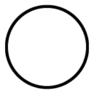
(a) Derive an expression for the angular speed of the rod just before striking the sphere in terms of the length ℓ and physical constants as appropriate.

GO ON TO THE NEXT PAGE.

(b) Derive an expression for the linear speed v_0 of the sphere immediately after colliding with the rod in terms of the length ℓ and physical constants as appropriate.

After sliding a short distance, at time $t = 0$ the sphere encounters a region of the horizontal surface with a coefficient of kinetic friction μ, beginning at Point A as indicated in Figure 1. The sphere begins rotating while sliding and eventually begins rolling without sliding at Point B, also as indicated.

(c) In the following diagram, which represents the sphere while the sphere is traveling between Points A and B, draw and label the forces (not components) that act on the sphere. Each force must be represented by a distinct arrow starting on, and pointing away from, the point of application on the sphere.



(d) Derive an expression for each of the following as the sphere is rotating and sliding between points A and B in terms of v_0, μ, R, t, and physical constants as appropriate.

i. The linear velocity v of the center of mass of the sphere as a function of time t

ii. The angular velocity ω of the sphere as a function of time t

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Continue your response to **QUESTION 3** on this page.

(e)

i. Derive an expression for the time it takes the sphere to travel from Point A to Point B in terms of v_0, μ, and physical constants as appropriate.

ii. Derive an expression for the linear velocity of the sphere upon reaching Point B in terms of v_0.

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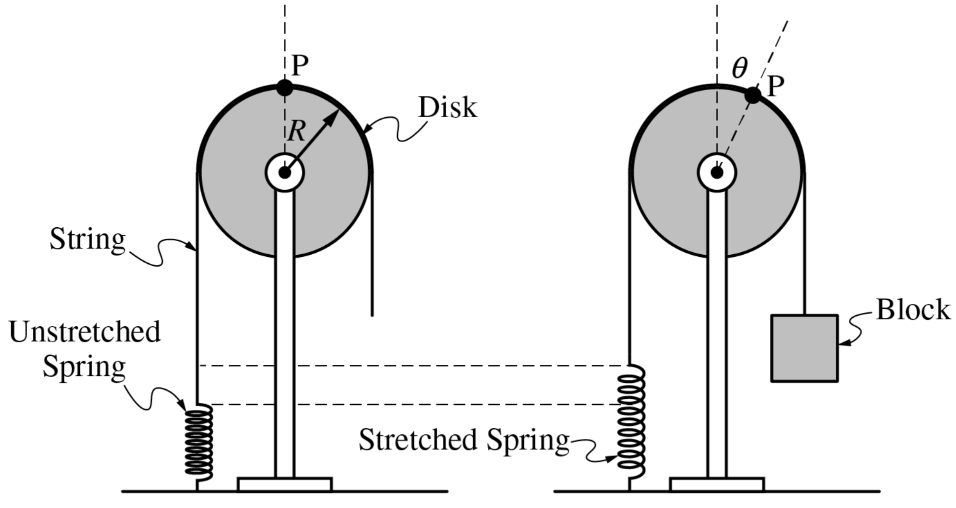

Figure 1 Figure 2

Note: Figures not drawn to scale.

. A solid uniform disk is supported by a vertical stand. The disk is able to rotate with negligible friction about an axle that passes through the center of the disk. The mass and radius of the disk are given by M_d and R, respectively. The rotational inertia of the disk is $I_d = \frac{1}{2} M_d R^2$. A string of negligible mass is draped over the disk and attached to the top of the disk at point P. One end of the string is connected to an unstretched ideal spring of spring constant k, which is fixed to the ground as shown in Figure 1.

A block of mass m_B is then attached to the string on the right side of the disk. The block is slowly lowered until the spring-disk-block system reaches equilibrium, as shown in Figure 2. In this equilibrium position, the disk has rotated clockwise through a small angle θ.

Give all algebraic answers in terms of M_d, R, k, θ, and physical constants, as appropriate.

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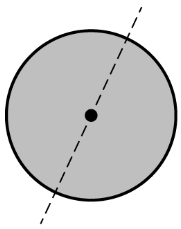
Continue your response to **QUESTION 3** on this page.

(a) Derive an expression for the mass m_B of the block.

(b) At time $t = 0$, the string on the right side of the disk is cut and the block falls to the ground. On the circle below, which represents the disk, draw and label the forces (not components) that act on the disk immediately after the string is cut and the block is falling to the ground. Each force should be represented by an arrow that starts on and is directed away from the point of application.



(c) Derive an expression for the angular acceleration α of the disk immediately after the string is cut.

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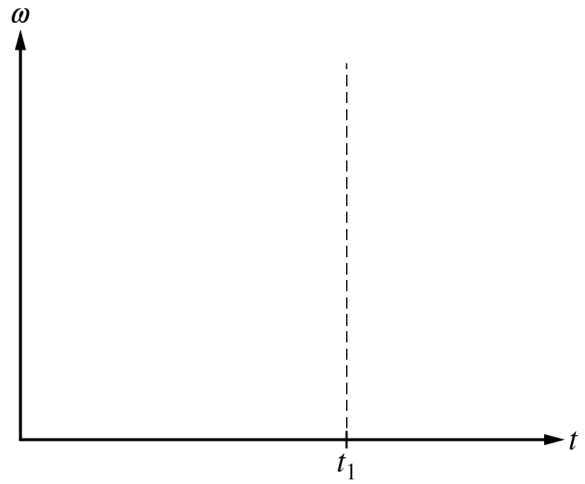
Continue your response to **QUESTION 3** on this page.

(d) At $t = t_1$, the disk has rotated and point P is again directly above the axle. Sketch a graph of the magnitude of the angular velocity ω of the disk as a function of time t from $t = 0$ to $t = t_1$.



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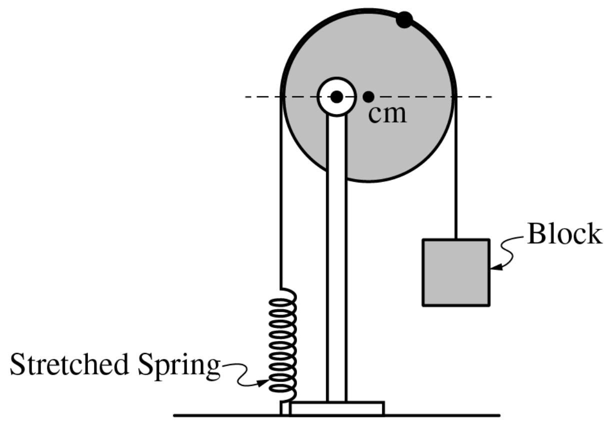

Figure 3

Note: Figure not drawn to scale.

(e) The disk is adjusted on the support so that the axle does not pass through the center of mass of the disk. The block is again hung on the right side of the disk and the spring-disk-block system comes to equilibrium, as shown in Figure 3. The axle does not exert a torque on the disk. For each force on the disk, indicate whether the magnitude of the torque about the axle caused by that force increases, decreases, or stays the same relative to part (b).

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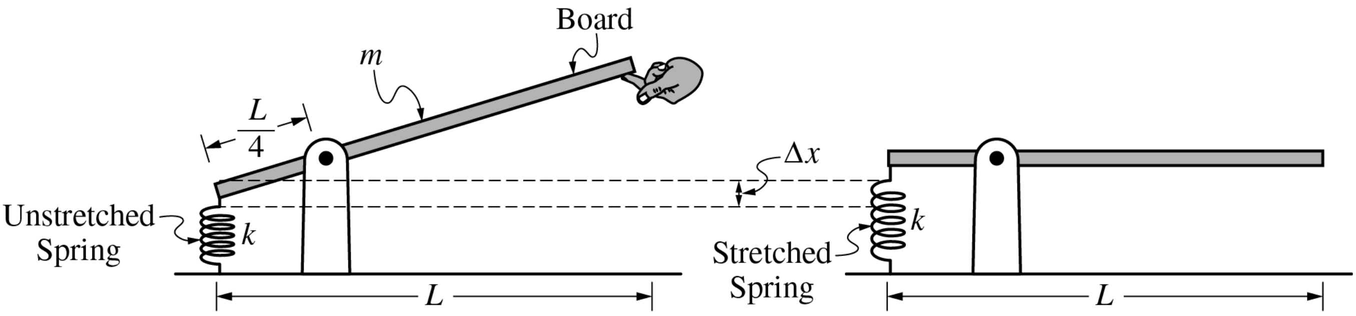
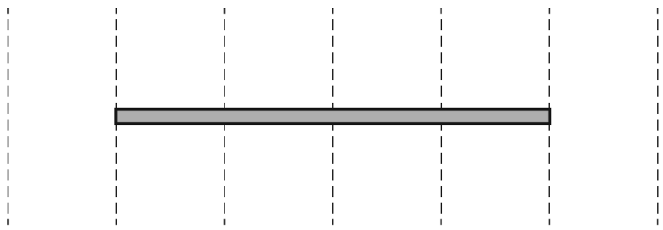


Figure 1 Figure 2

Note: Figures not drawn to scale.

. A uniform board of length L and mass m is attached to a pivot $\frac{L}{4}$ from the left end of the board. The left end of the board is attached to an ideal spring of spring constant k that is attached to the ground. The right end of the board is initially held by a student so that the spring is unstretched, as shown in Figure 1. The student slowly lowers and then releases the board. The board remains at rest in the horizontal position, with the spring stretched, as shown in Figure 2. The rotational inertia of the board about the pivot is I.

(a) On the rectangle below, which represents the board, draw and label the forces (not components) that act on the board while the board-spring system is in equilibrium. Each force should be represented by an arrow that starts on, and points away from, the board, and should represent the location at which that force acts.



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(b) Derive an expression for the distance the spring stretches, Δx, when the board is in equilibrium. Express your answer in terms of k, L, m, and physical constants, as appropriate.

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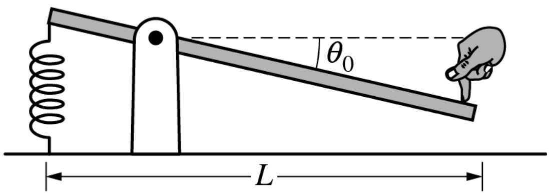
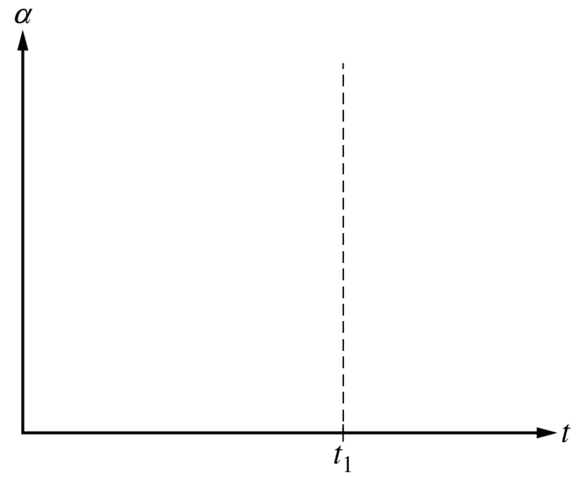


Note: Figure not drawn to scale.

(c) A student pushes the board down on the right side, stretching the spring a new distance Δx_2 from the unstretched position. The board is held at a small angle θ_0 with the horizontal, as shown. The student then releases the board from rest.

i. At time $t = 0$, the board is released. At $t = t_1$, the board first crosses the horizontal. Sketch a graph of the magnitude of the angular acceleration α of the board as a function of time t from $t = 0$ to $t = t_1$.



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ii. Derive an expression for the angular acceleration α_0 of the board immediately after the board is released. Express your answer in terms of k, L, m, I, Δx_2, θ_0, and physical constants, as appropriate.

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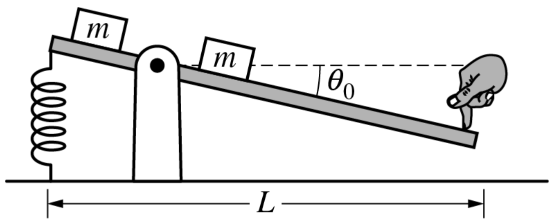
Continue your response to **QUESTION 3** on this page.



Note: Figure not drawn to scale.

(d) Two blocks of equal mass m are attached to the board equal distances from the pivot point, as shown. The board is again pushed down on the right side so that the spring stretches the same distance Δx_2 as in part (c). The board is then released. How does the new angular acceleration α' when the blocks are attached compare to the angular acceleration α_0 from part (c) ?

___$\alpha' > \alpha_0$ ___$\alpha' < \alpha_0$ ___ $\alpha' = \alpha_0$

Justify your answer.

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END OF EXAM

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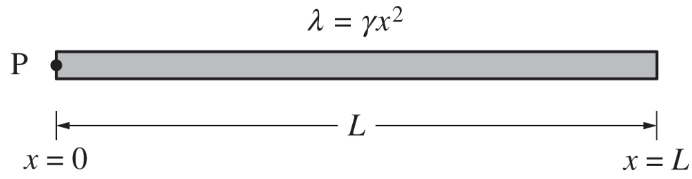

3. A triangular rod of length L and mass M has a nonuniform linear mass density given by the equation $\lambda = \gamma x^2$, where $\gamma = \frac{3M}{L^3}$ and x is the distance from point P at the left end of the rod.

(a) Using integral calculus, show that the rotational inertia I of the rod about an axis perpendicular to the page and through point P is $\frac{3}{5}ML^2$.

(b) Determine the horizontal location of the center of mass of the rod relative to point P. Express your answer in terms of L.

(c) For an axis perpendicular to the page, is the value of the rotational inertia of the rod around point P greater than, less than, or equal to the value of the rotational inertia of the rod around the rod's center of mass?

_____ Greater than _____ Less than _____ Equal to

Justify your answer.

GO ON TO THE NEXT PAGE.

The rod is released from rest in the position shown, and the rod begins to rotate about a horizontal axis perpendicular to the page and through point P.

(d) On the axes below, sketch graphs of the magnitude of the net torque τ on the rod and the angular speed ω of the rod as functions of time t from the time the rod is released until the time its center of mass reaches its lowest point.

(e) As the rod rotates from the horizontal position down through vertical, is the magnitude of the angular acceleration on the rod increasing, decreasing, or not changing?

_____ Increasing   _____ Decreasing   _____ Not changing

Justify your answer.

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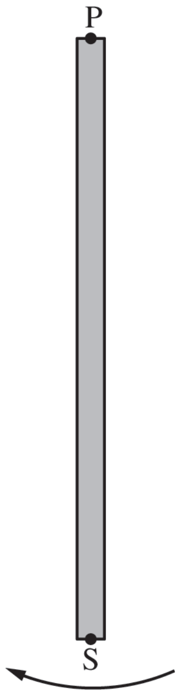

(f) The mass of the rod is 3.0 kg, and the length of the rod is 1.0 m. Calculate the linear speed v of point S as the rod swings through the vertical position shown.

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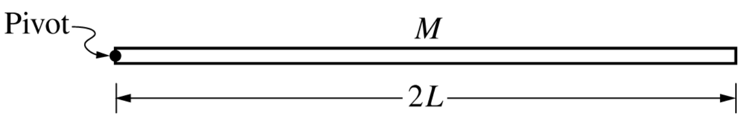
END OF EXAM

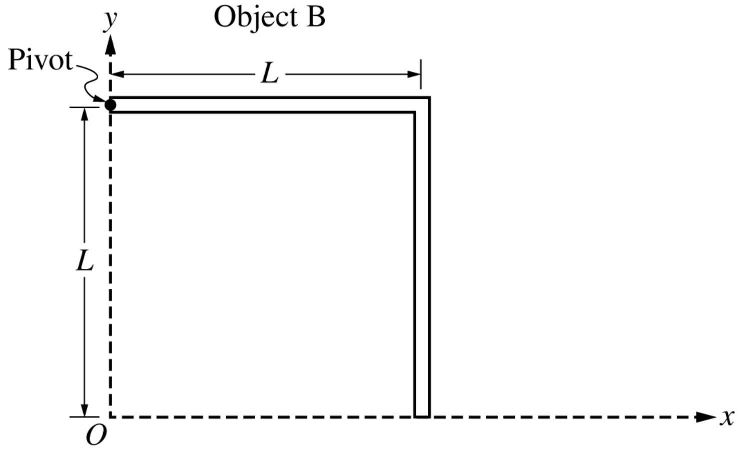
Object A



2. Object A is a long, thin, uniform rod of mass M and length $2L$ that is free to rotate about a pivot of negligible friction at its left end, as shown above.

(a) Using integral calculus, derive an expression to show that the rotational inertia I_A of object A about the pivot is given by $\frac{4}{3}ML^2$.

Object B



Object B of total mass M is formed by attaching two thin, uniform, identical rods of length L at a right angle to each other. Object B is held in place, as shown above. Express your answers in part (b) in terms of L .

(b) Determine the following for the given coordinate system shown in the figure.

i. The x-coordinate of the center of mass of object B

ii. The y-coordinate of the center of mass of object B

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Object B has a rotational inertia of I_B about its pivot.

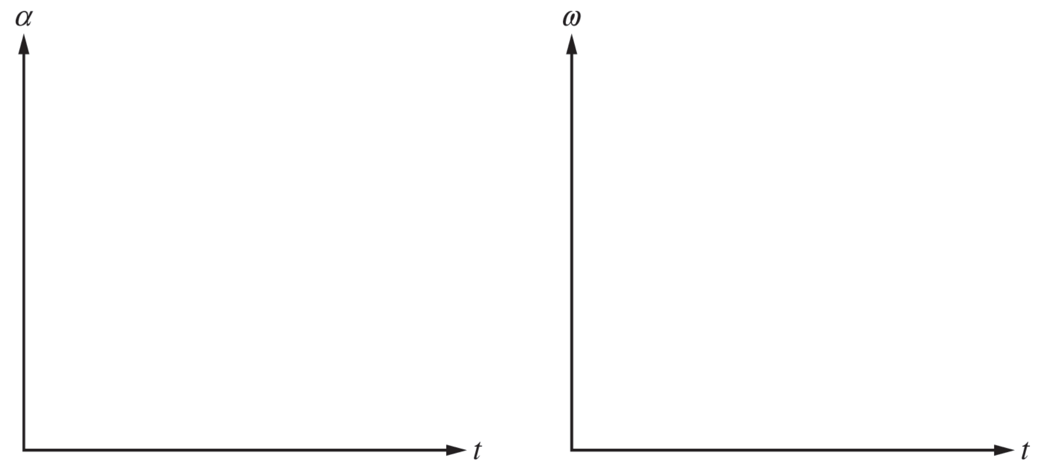
(c) Is the value of I_B greater than, less than, or equal to I_A ?

_____ Greater than _____ Less than _____ Equal to

Justify your answer.

Object B is released from rest and begins to rotate about its pivot.

(d) On the axes below, sketch graphs of the magnitude of the angular acceleration α and the angular speed ω of object B as functions of time t from the time it is released to the time its center of mass reaches its lowest point.



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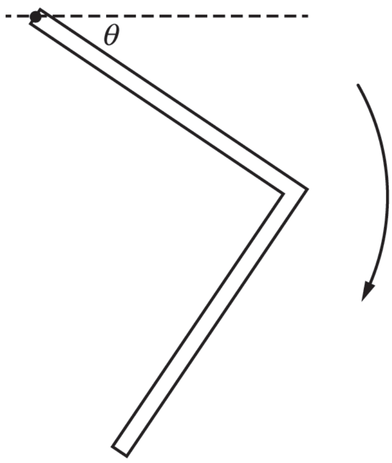
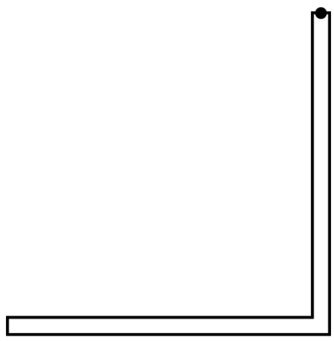


(e) While object B rotates from the horizontal position down through the angle θ shown above, is the magnitude of its angular acceleration increasing, decreasing, or not changing?

_____ Increasing _____ Decreasing _____ Not changing

Justify your answer.



Object B rotates through the position shown above.

(f) Derive an expression for the angular speed of object B when it is in the position shown above. Express your answer in terms of M, L, I_B, and physical constants, as appropriate.

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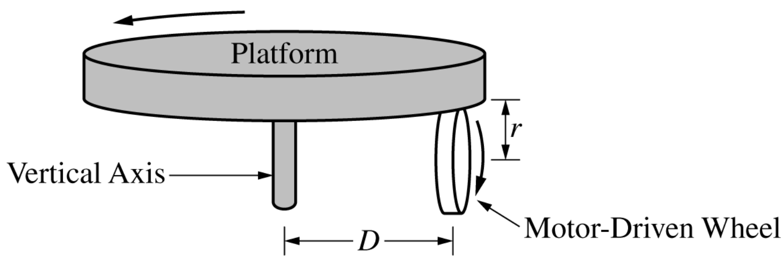
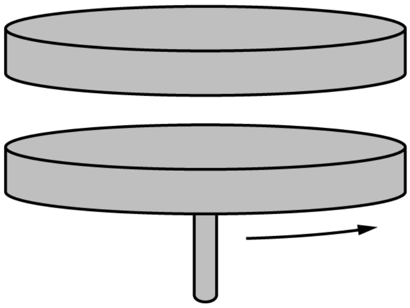


3. A horizontal circular platform with rotational inertia I_P rotates freely without friction on a vertical axis. A small motor-driven wheel that is used to rotate the platform is mounted under the platform and touches it. The wheel has radius r and touches the platform a distance D from the vertical axis of the platform, as shown above. The platform starts at rest, and the wheel exerts a constant horizontal force of magnitude F tangent to the wheel until the platform reaches an angular speed ω_P after time Δt. During time Δt, the wheel stays in contact with the platform without slipping.

   (a) Derive an expression for the angular speed ω_P of the platform. Express your answer in terms of I_P, r, D, F, Δt, and physical constants, as appropriate.

   (b) Determine an expression for the kinetic energy of the platform at the moment it reaches angular speed ω_P. Express your answer in terms of I_P, r, D, F, Δt, and physical constants, as appropriate.

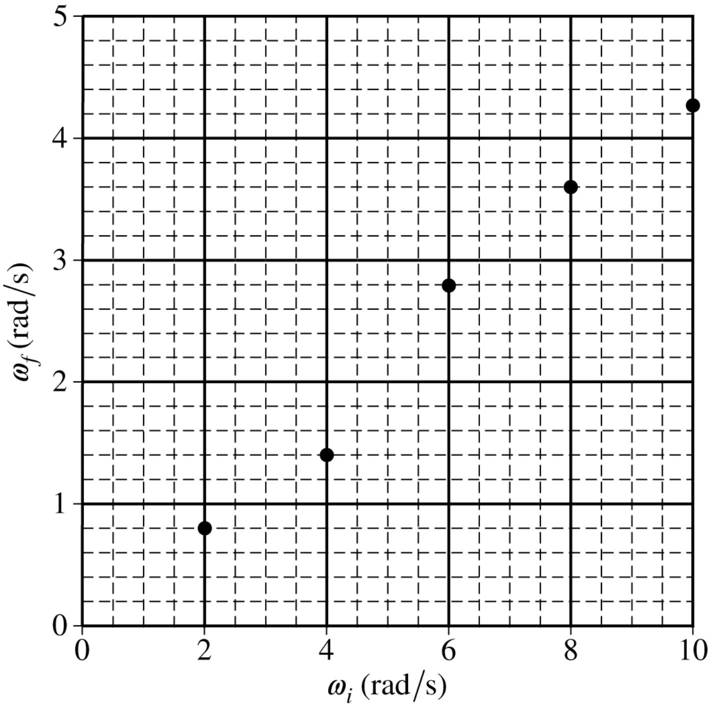
   (c) Derive an expression for the angular speed of the wheel ω_W when the platform has reached angular speed ω_P. Express your answer in terms of D, r, ω_P, and physical constants, as appropriate.



When the platform is spinning at angular speed ω_P, the motor-driven wheel is removed. A student holds a disk directly above and concentric with the platform, as shown above. The disk has the same rotational inertia I_P as the platform. The student releases the disk from rest, and the disk falls onto the platform. After a short time, the disk and platform are observed to be rotating together at angular speed ω_f.

   (d) Derive an expression for ω_f. Express your answer in terms of ω_P, I_P, and physical constants, as appropriate.

A student now uses the rotating platform $\left(I_P = 3.1\ \text{kg}\bullet\text{m}^2\right)$ to determine the rotational inertia I_U of an unknown object about a vertical axis that passes through the object's center of mass. The platform is rotating at an initial angular speed ω_i when the unknown object is dropped with its center of mass directly above the center of the platform. The platform and object are observed to be rotating together at angular speed ω_f. Trials are repeated for different values of ω_i. A graph of ω_f as a function of ω_i is shown on the axes below.



(e)

i. On the graph on the previous page, draw a best-fit line for the data.

ii. Using the straight line, calculate the rotational inertia of the unknown object I_U about a vertical axis passing through its center of mass.

(f) The kinetic energy of the spinning platform before the object is dropped on it is K_i. The total kinetic energy of the platform-object system when it reaches angular speed ω_f is K_f. Which of the following expressions is true?

____ $K_f < K_i$ ____ $K_f = K_i$ ____ $K_f > K_i$

Justify your answer.

(g) One of the students observes that the center of mass of the object is not actually aligned with the axis of the platform. Is the experimental value of I_U obtained in part (e) greater than, less than, or equal to the actual value of the rotational inertia of the unknown object about a vertical axis that passes through its center of mass?

____ Greater than ____ Less than ____ Equal to

Justify your answer.

STOP

END OF EXAM

5.4 Rotational Inertia

Name: ____________ Class: ________ Date: __________ **Total: 14 marks**

KEY WORDS rotational inertia 转动惯量

Objective

Build the skills to answer exam questions on **rotational inertia**.

You must be able to:

- describe **rotational inertia 转动惯量** I as resistance to a change in rotation
- use $I = \sum m_i r_i^2$ for point masses
- explain that mass farther from the axis contributes more
- recall that I depends on the mass distribution and the axis chosen

1 Worked examples

Study these first. Each one shows the method for a question type used later — follow the steps and you can do the Practice and Exam-style questions yourself.

■ Rotational inertia of point masses

$$I = \sum m_i r_i^2.$$

A 2 kg mass at $r = 3$ m from the axis has $I = 2 \times 3^2 = 18$ kg m^2.

■ Distance matters most

Because r is squared, mass far from the axis dominates. Moving a mass from 1 m to 2 m quadruples its contribution.

■ It depends on the axis

The same object has different I about different axes. A rod is easier to spin about its centre than about one end.

2 Practice

Now apply the methods above.

2.1 A 3 kg mass sits 2 m from the axis. Find its rotational inertia. [2]

2.2 State what rotational inertia measures. [1]

2.3 Two 1 kg masses sit at $r = 1$ m and $r = 2$ m. Find the total I. [2]

2.4 If a mass is moved twice as far from the axis, by what factor does its contribution to I change? [1]

3 Exam-style questions

3.1 Rotational inertia depends on [1]

- **A** only the total mass
- **B** the mass and how it is distributed about the axis
- **C** only the speed
- **D** the colour of the object

3.2 A point mass's contribution to rotational inertia is [1]

- **A** mr
- **B** mr^2
- **C** $\frac{1}{2}mr^2$
- **D** m/r

3.3 A 4 kg mass sits at $r = 1.5$ m and a 2 kg mass at $r = 3$ m on a light rod.

(a) Find the total rotational inertia. [2]

(b) State which mass contributes more, and why. [1]

3.4 A solid disc and a ring have the same mass and radius.

(a) State which has the larger rotational inertia. [1]

(b) Explain why, in terms of mass distribution. [2]

Solutions

2.1 $I = mr^2 = 3 \times 2^2 = 12$ kg m^2.

2.2 An object's resistance to a change in its rotation.

2.3 $I = 1(1)^2 + 1(2)^2 = 1 + 4 = 5$ kg m^2.

2.4 Four times (r is squared).

3.1 B — mass and its distribution about the axis.

3.2 B — mr^2.

3.3 (a) $I = 4(1.5)^2 + 2(3)^2 = 9 + 18 = 27$ kg m^2. (b) The 2 kg mass, because it is farther out (r^2 dominates).

3.4 (a) The ring. (b) All the ring's mass is at the rim (maximum r), while the disc's mass is spread inward, lowering its I.

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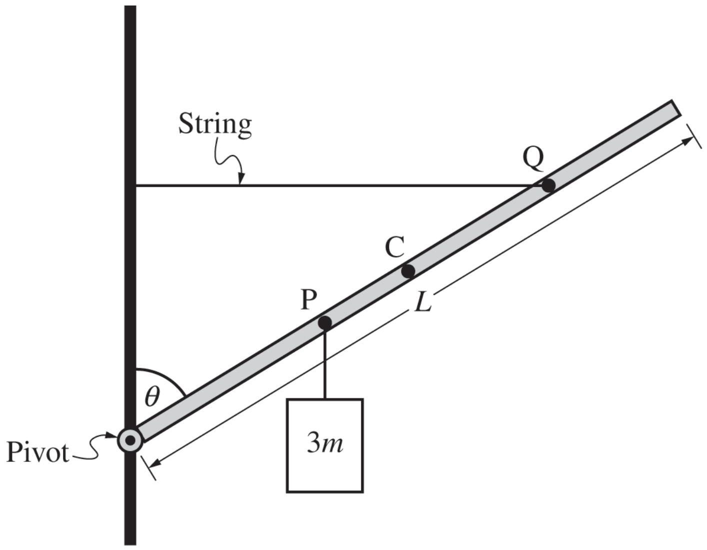
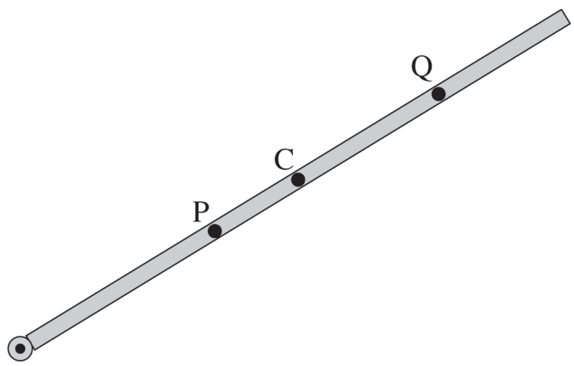


Figure 1

Note: Figure not drawn to scale.

3. A uniform rod of length L and mass m is attached to a pivot on a vertical pole, as shown in Figure 1. There is negligible friction between the rod and the pivot. A horizontal string connects Point Q on the rod to the pole. The rod makes an angle θ with the pole. A block of mass $3m$ hangs from the rod at Point P. The center of mass of the rod is located at Point C.

(a) On the following representation of the rod, **draw** and **label** the forces (not components) that are exerted on the rod. Each force must be represented by a distinct arrow that starts on and points away from the point at which the force is exerted on the rod.



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(b) In Figure 1, Point P is located $\frac{3}{8}L$ from the pivot and Point Q is located $\frac{6}{8}L$ from the pivot. **Derive** an equation for the tension F_T in the horizontal string in terms of L, m, θ, and physical constants, as appropriate.

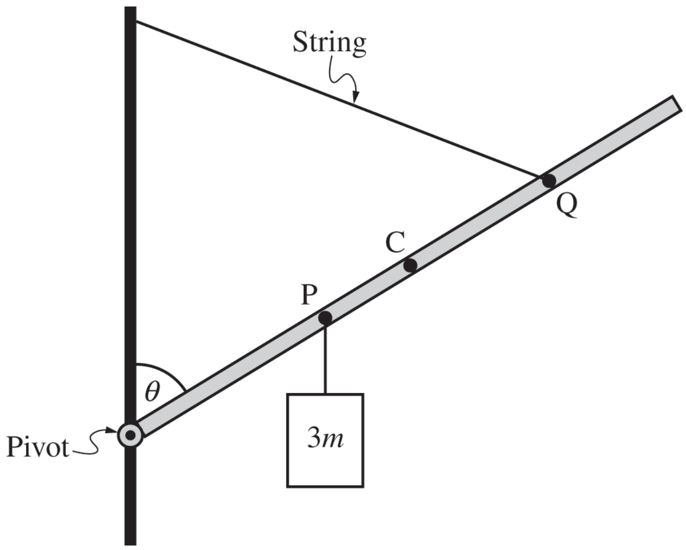

Figure 2

Note: Figure not drawn to scale.

(c) The original string is replaced with a longer string that connects Point Q to a higher location on the vertical pole, as shown in Figure 2. The angle θ remains the same. How does the new tension $F_{T,\text{ new}}$ compare with the original tension F_T from part (b) ? **Justify** your reasoning.

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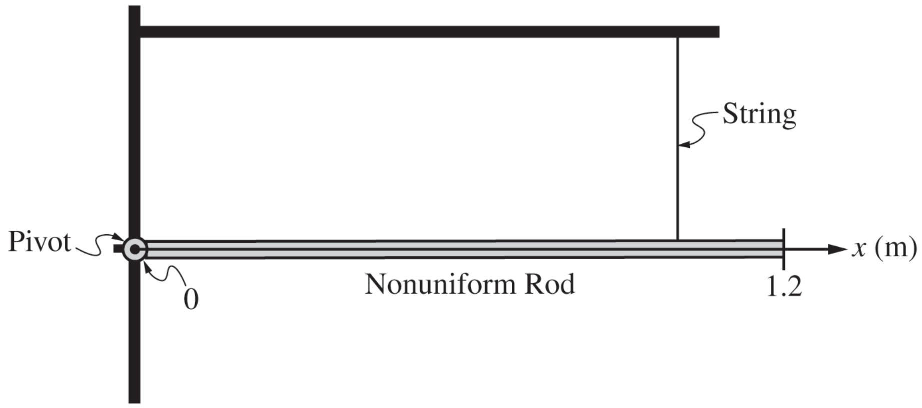

Figure 3

Note: Figure not drawn to scale.

(d) A nonuniform rod is now attached to the pivot, as shown in Figure 3. There is negligible friction between the nonuniform rod and the pivot. The rod has a length of 1.2 m and a linear mass density $\lambda(x) = A + Bx$, where x is the distance from the pivot, $A = 6.0$ kg/m, and $B = 10.0$ kg/m^2.

i. **Calculate** the mass of the rod.

ii. **Calculate** the rotational inertia of the rod about the pivot.

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END OF EXAM

Begin your response to **QUESTION 3** on this page.

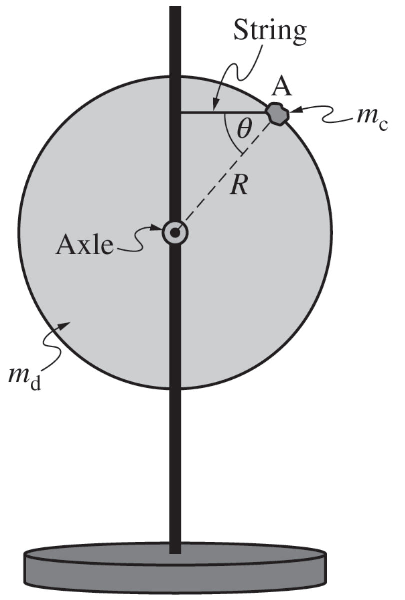
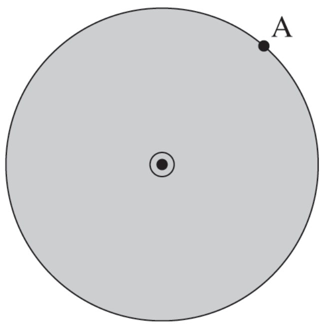


Figure 1

Note: Figure not drawn to scale.

3. A uniform disk of radius R and mass m_d is attached to a vertical pole by a horizontal axle that passes through the center of the disk. Friction between the axle and the disk is negligible. A lump of clay of mass m_c is attached to the edge of the disk at Point A. The size of the lump of clay is small compared with the radius of the disk. A horizontal string is connected from the pole to the edge of the disk at Point A. The string makes an angle θ with the line between Point A and the axle, as shown in Figure 1.

(a) On the following representation of the clay-disk system, **draw** and **label** the external forces (not components) exerted on the system. Each force must be represented by a distinct arrow that starts on, and points away from, the point at which the force is exerted on the system.



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Continue your response to **QUESTION 3** on this page.

(b) **Derive** an expression for the tension F_T in the string when the clay is at Point A, as shown in Figure 1, in terms of R, m_d, m_c, θ, and physical constants, as appropriate.

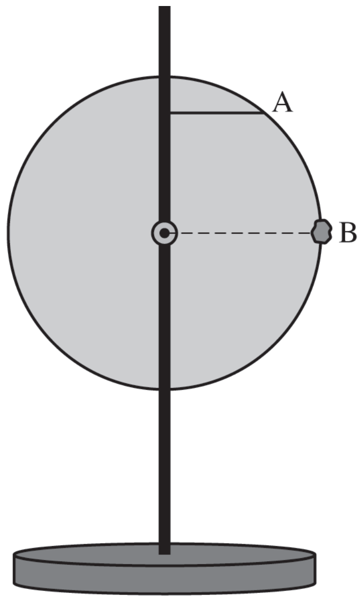

Figure 2

Note: Figure not drawn to scale.

(c) The string remains connected to the edge of the disk at Point A. The clay is moved to Point B, which is horizontally in line with the axle, as shown in Figure 2. How does the new tension $F_{T,\text{ new}}$ compare with tension F_T from part (b)? **Justify** your reasoning.

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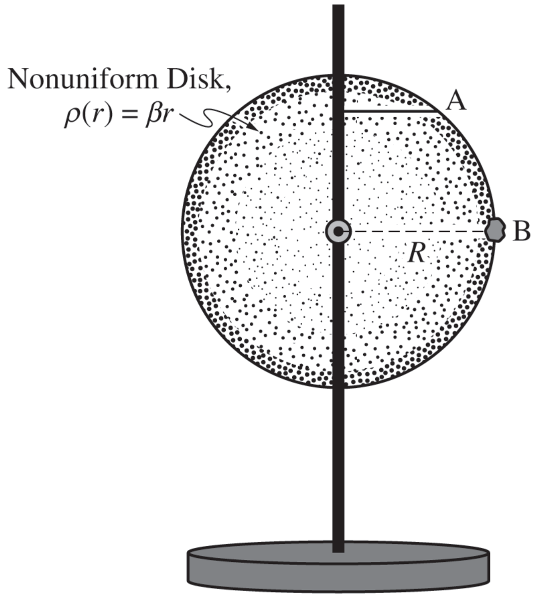

Figure 3

<u>Note:</u> Figure not drawn to scale.

(d) A nonuniform disk is now attached to the axle. The lump of clay is attached to the disk at Point B, as shown in Figure 3. The clay has mass $m_c = 0.60\ \text{kg}$ and the disk has a radius $R = 0.30\ \text{m}$. The mass density of the disk varies radially and can be modeled by $\rho(r) = \beta r$, where r is the radial distance from the axle and $\beta = 4.0\ \text{kg/m}^3$.

i. **Calculate** the rotational inertia of the disk about the axle.

ii. The string connecting the disk to the pole is cut. **Calculate** the magnitude of the initial angular acceleration of the clay-disk system.

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END OF EXAM

Begin your response to **QUESTION 2** on this page.

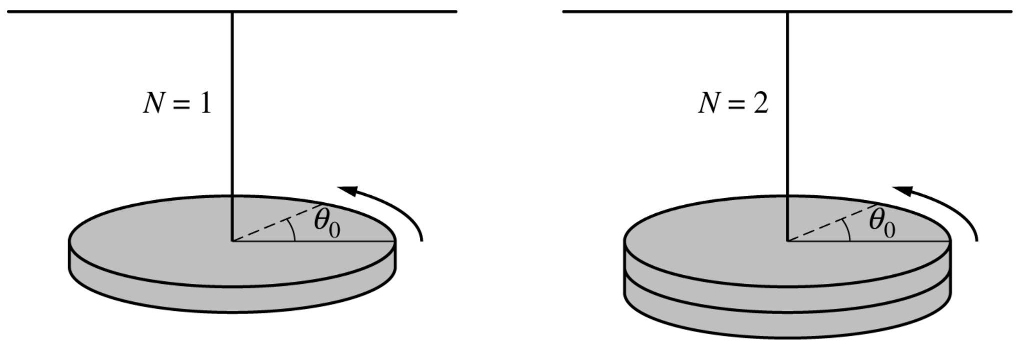

Figure 1 Figure 2

2. A student makes a torsional pendulum by suspending a uniform disk of mass M and radius R from a light wire with torsion constant κ that is attached to the center of the disk as shown in Figure 1. The rotational inertia of the disk is given by $I = \frac{1}{2}MR^2$. The student conducts an investigation to determine the relationship between the period of oscillation T of the torsional pendulum and the number N of identical disks that are suspended from the wire.

The student starts with a single disk. Holding the disk at a small initial angular displacement θ_0 from the untwisted position, the student releases the disk from rest and the pendulum oscillates. The student records the period of oscillation for a single disk. An additional identical disk is attached, as shown in Figure 2, and the procedure is repeated for $N = 2$ disks. This procedure is repeated through $N = 10$ identical disks. Assume the disks move together as one system.

(a) Using $T = 2\pi\sqrt{\frac{I}{\kappa}}$, derive an expression for T as a function of N. Express your answer in terms of M, R, κ, N, and physical constants, as appropriate.

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Continue your response to **QUESTION 2** on this page.

(b) The potential energy stored in the torsional pendulum when the disks are displaced is $U = \frac{1}{2}\kappa(\Delta\theta)^2$. On the following axes, sketch a graph of the maximum kinetic energy K_{max} of the torsional pendulum as a function of N for $N \geq 1$.

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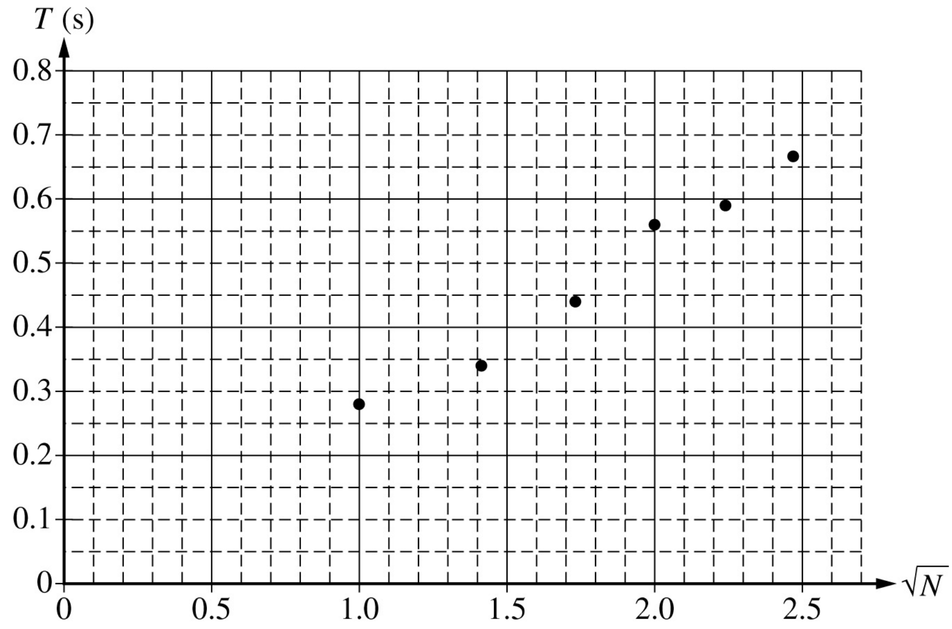
(c) The student plots the data for T as a function of $\sqrt{N}$, as shown.



i. Draw the best-fit line for the data.

ii. The student previously determined that the radius of a disk is $R = 0.2$ m and found that $\kappa = 1.6$ N·m. Using the graph, calculate the mass M of a single disk.

iii. The student finds that the value given by the manufacturer for the mass of the disk is less than the value determined experimentally in part (c)(ii). Determine a single source of experimental error that could result in the observed difference in the value of M. Justify your answer.

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Continue your response to **QUESTION 2** on this page.

(d) The student repeats the experiment, but now the disks have a density that varies as a function of the radius of the disk according to $\rho = 0.3r$.

i. Would the slope of the best-fit line for this new data be greater than, less than, or the same as the slope of the best-fit line in part (c)(i) ?

___ greater than   ___less than   ___the same as

Justify your answer.

ii. When $N = 1$, the maximum angular speed of the torsional pendulum with a uniform disk is found to be ω_U. When $N = 1$, the maximum angular speed of the torsional pendulum with a nonuniform disk is $\omega_{\text{non-U}}$. Which of the following correctly compares ω_U and $\omega_{\text{non-U}}$?

____$\omega_U > \omega_{\text{non-U}}$ ____$\omega_U < \omega_{\text{non-U}}$ ____ $\omega_U = \omega_{\text{non-U}}$

Briefly justify your answer.

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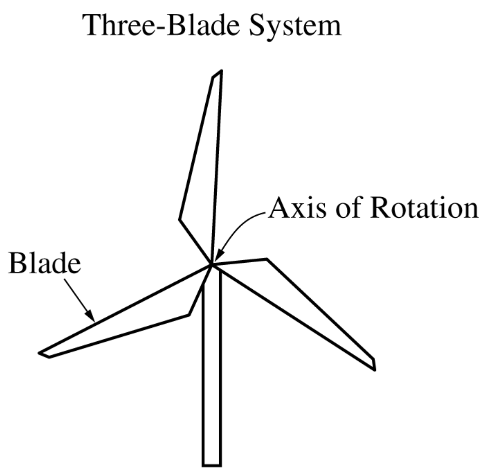

Figure 1

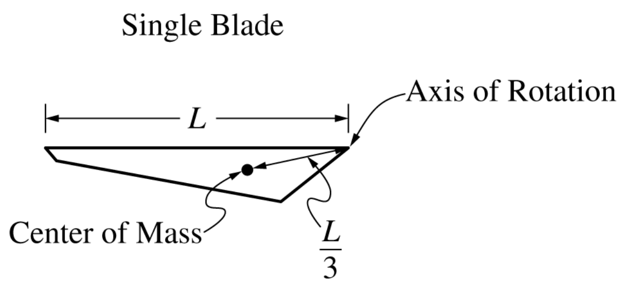

Figure 2

. A wind turbine includes a three-blade system that rotates about an axis through the end of each blade, as shown in Figure 1. Each blade has a length L and mass M, with a center of mass located at a distance $\frac{L}{3}$ from the axis of rotation, as shown in Figure 2.

(a) Derive an expression for the rotational inertia of the three-blade system. Express your answer in terms of M, L, and physical constants, as appropriate. The rotational inertia of each blade about an axis through its center of mass is given by the equation $I_{cm} = \frac{1}{18}ML^2$.

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Continue your response to **QUESTION 3** on this page.

(b) While the wind blows, the three-blade system operates at a constant angular speed $\omega_0 = 2.6$ rad/s. The length of one blade is $L = 36$ m. The numerical value of the rotational inertia of the system is $I_{sys} = 6.7 \times 10^6$ kg·m^2. Calculate the time T it takes the outer edge of a single blade to complete one revolution.

(c) When the wind stops blowing, the angular speed of the system decreases. The angular speed ω of the system while slowing down is given as a function of time t by the equation $\omega = \omega_0 e^{-\beta_0 t}$, where β_0 is a constant with appropriate units, as shown on the graph in Figure 3.

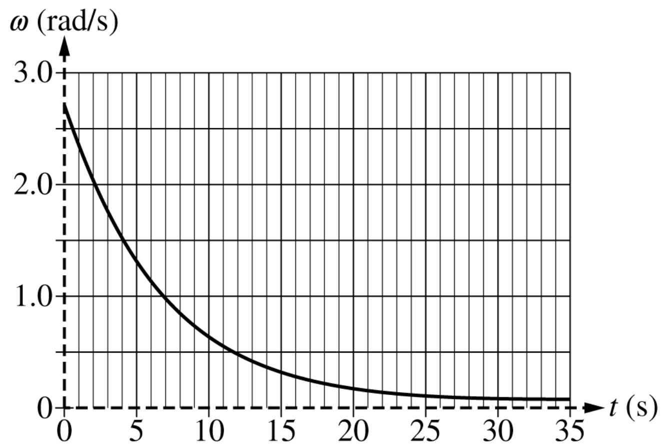

Figure 3

i. Calculate the amount of energy dissipated from $t = 0$, when the wind stops blowing, until the system comes to rest.

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Continue your response to **QUESTION 3** on this page.

ii. Derive an expression for the net torque exerted on the system as a function of t as the system slows down. Express your answers in terms of β_0, ω_0, M, L, I_{sys}, and physical constants, as appropriate.

iii. Derive an expression for the angular displacement of the system $\Delta\theta$ as a function of t. Express your answer in terms of β_0, ω_0, M, L, I_{sys}, and physical constants, as appropriate.

The three-blade system is now replaced with a second three-blade system identical to the first, except that the second three-blade system slows down according to the equation $\omega = \omega_0 e^{-\beta t}$, where $\omega_0 = 2.6$ rad/s and $\beta > \beta_0$. The original angular speed function is shown as a dashed line in Figure 4.

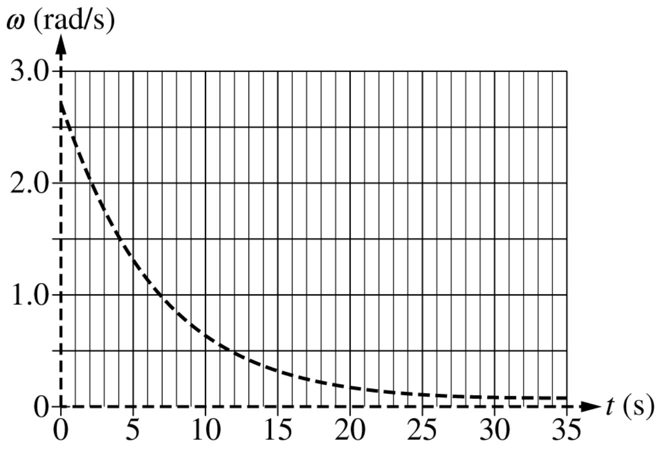

Figure 4

(d) On the graph in Figure 4, sketch the angular speed of the second three-blade system as a function of time t.

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END OF EXAM

Begin your response to **QUESTION 3** on this page.

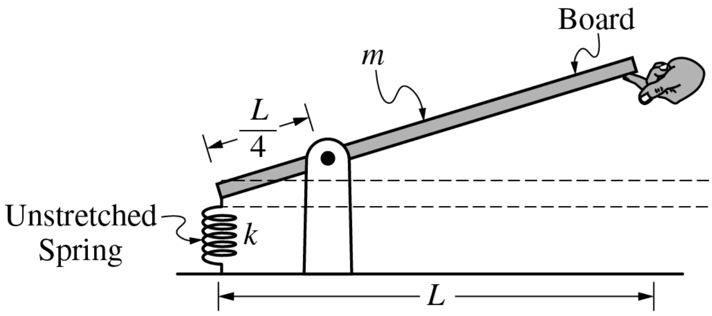


Figure 1

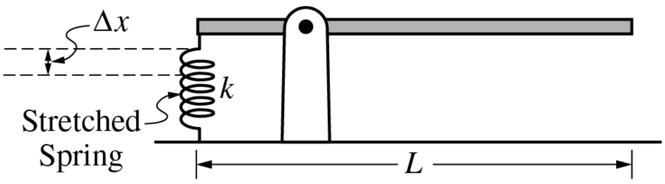
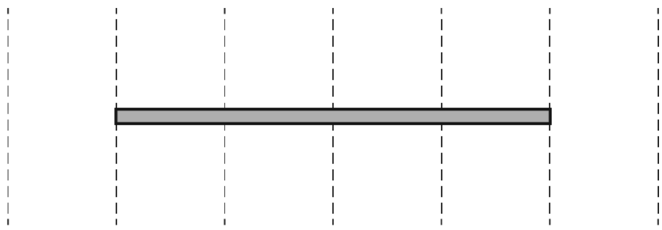


Figure 2

<u>Note</u>: Figures not drawn to scale.

. A uniform board of length L and mass m is attached to a pivot $\frac{L}{4}$ from the left end of the board. The left end of the board is attached to an ideal spring of spring constant k that is attached to the ground. The right end of the board is initially held by a student so that the spring is unstretched, as shown in Figure 1. The student slowly lowers and then releases the board. The board remains at rest in the horizontal position, with the spring stretched, as shown in Figure 2. The rotational inertia of the board about the pivot is I.

(a) On the rectangle below, which represents the board, draw and label the forces (not components) that act on the board while the board-spring system is in equilibrium. Each force should be represented by an arrow that starts on, and points away from, the board, and should represent the location at which that force acts.



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Continue your response to **QUESTION 3** on this page.

(b) Derive an expression for the distance the spring stretches, Δx, when the board is in equilibrium. Express your answer in terms of k, L, m, and physical constants, as appropriate.

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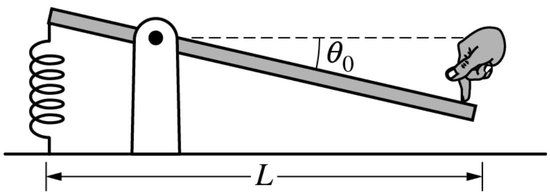
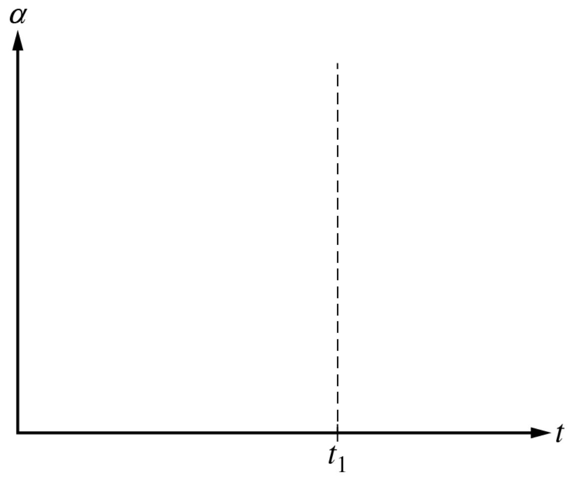


Note: Figure not drawn to scale.

(c) A student pushes the board down on the right side, stretching the spring a new distance Δx_2 from the unstretched position. The board is held at a small angle θ_0 with the horizontal, as shown. The student then releases the board from rest.

i. At time $t = 0$, the board is released. At $t = t_1$, the board first crosses the horizontal. Sketch a graph of the magnitude of the angular acceleration α of the board as a function of time t from $t = 0$ to $t = t_1$.



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ii. Derive an expression for the angular acceleration α_0 of the board immediately after the board is released. Express your answer in terms of k, L, m, I, Δx_2, θ_0, and physical constants, as appropriate.

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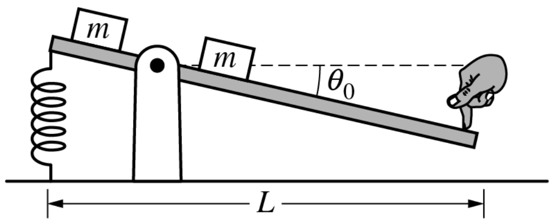
Continue your response to **QUESTION 3** on this page.



Note: Figure not drawn to scale.

(d) Two blocks of equal mass m are attached to the board equal distances from the pivot point, as shown. The board is again pushed down on the right side so that the spring stretches the same distance Δx_2 as in part (c). The board is then released. How does the new angular acceleration α' when the blocks are attached compare to the angular acceleration α_0 from part (c) ?

___ $\alpha' > \alpha_0$ ___ $\alpha' < \alpha_0$ ___ $\alpha' = \alpha_0$

Justify your answer.

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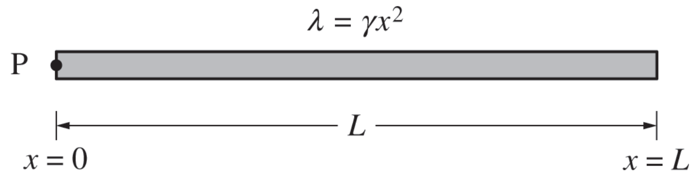
END OF EXAM



3. A triangular rod of length L and mass M has a nonuniform linear mass density given by the equation $\lambda = \gamma x^2$, where $\gamma = \dfrac{3M}{L^3}$ and x is the distance from point P at the left end of the rod.

(a) Using integral calculus, show that the rotational inertia I of the rod about an axis perpendicular to the page and through point P is $\frac{3}{5}ML^2$.

(b) Determine the horizontal location of the center of mass of the rod relative to point P. Express your answer in terms of L.

(c) For an axis perpendicular to the page, is the value of the rotational inertia of the rod around point P greater than, less than, or equal to the value of the rotational inertia of the rod around the rod's center of mass?

_____ Greater than   _____ Less than   _____ Equal to

Justify your answer.

GO ON TO THE NEXT PAGE.

The rod is released from rest in the position shown, and the rod begins to rotate about a horizontal axis perpendicular to the page and through point P.

(d) On the axes below, sketch graphs of the magnitude of the net torque τ on the rod and the angular speed ω of the rod as functions of time t from the time the rod is released until the time its center of mass reaches its lowest point.

(e) As the rod rotates from the horizontal position down through vertical, is the magnitude of the angular acceleration on the rod increasing, decreasing, or not changing?

_____ Increasing _____ Decreasing _____ Not changing

Justify your answer.

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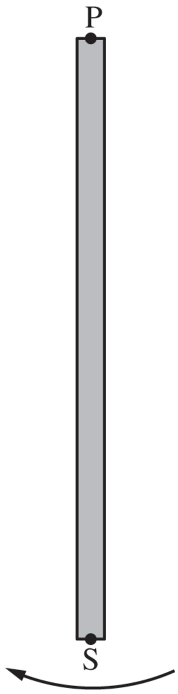

(f) The mass of the rod is $3.0\,\text{kg}$, and the length of the rod is $1.0\,\text{m}$. Calculate the linear speed v of point S as the rod swings through the vertical position shown.

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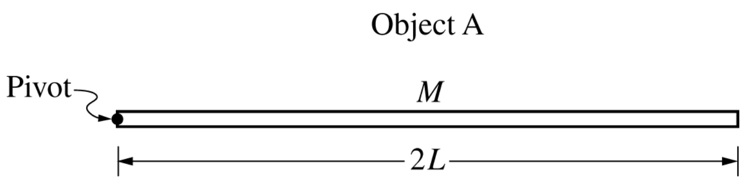
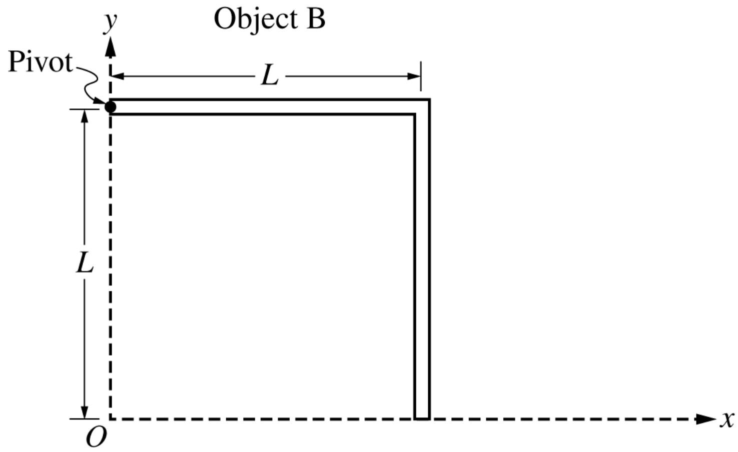


2. Object A is a long, thin, uniform rod of mass M and length $2L$ that is free to rotate about a pivot of negligible friction at its left end, as shown above.

(a) Using integral calculus, derive an expression to show that the rotational inertia I_A of object A about the pivot is given by $\frac{4}{3}ML^2$.



Object B of total mass M is formed by attaching two thin, uniform, identical rods of length L at a right angle to each other. Object B is held in place, as shown above. Express your answers in part (b) in terms of L .

(b) Determine the following for the given coordinate system shown in the figure.

i. The x-coordinate of the center of mass of object B

ii. The y-coordinate of the center of mass of object B

GO ON TO THE NEXT PAGE.

Object B has a rotational inertia of I_B about its pivot.

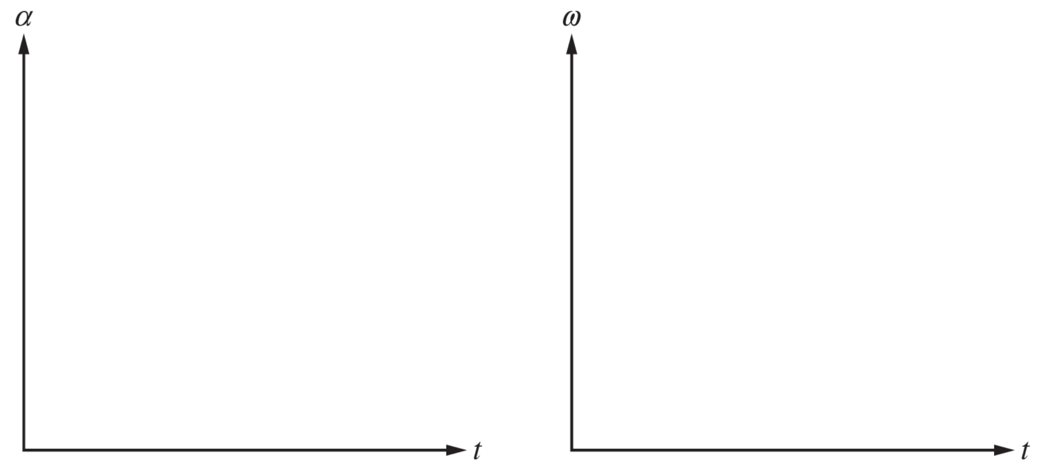
(c) Is the value of I_B greater than, less than, or equal to I_A ?

_____ Greater than   _____ Less than   _____ Equal to

Justify your answer.

Object B is released from rest and begins to rotate about its pivot.

(d) On the axes below, sketch graphs of the magnitude of the angular acceleration α and the angular speed ω of object B as functions of time t from the time it is released to the time its center of mass reaches its lowest point.



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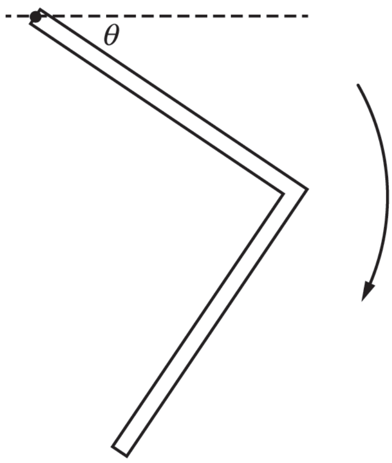
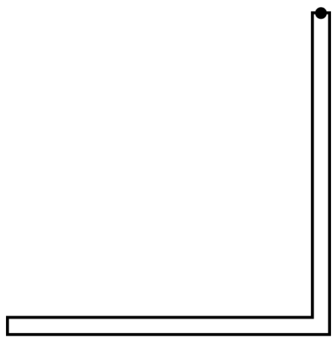


(e) While object B rotates from the horizontal position down through the angle θ shown above, is the magnitude of its angular acceleration increasing, decreasing, or not changing?

_____ Increasing _____ Decreasing _____ Not changing

Justify your answer.



Object B rotates through the position shown above.

(f) Derive an expression for the angular speed of object B when it is in the position shown above. Express your answer in terms of M, L, I_B, and physical constants, as appropriate.

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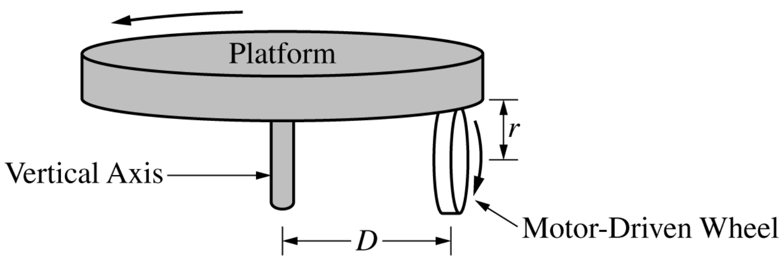
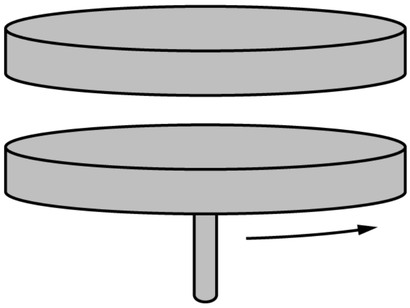


3. A horizontal circular platform with rotational inertia I_P rotates freely without friction on a vertical axis. A small motor-driven wheel that is used to rotate the platform is mounted under the platform and touches it. The wheel has radius r and touches the platform a distance D from the vertical axis of the platform, as shown above. The platform starts at rest, and the wheel exerts a constant horizontal force of magnitude F tangent to the wheel until the platform reaches an angular speed ω_P after time Δt. During time Δt, the wheel stays in contact with the platform without slipping.

   (a) Derive an expression for the angular speed ω_P of the platform. Express your answer in terms of I_P, r, D, F, Δt, and physical constants, as appropriate.

   (b) Determine an expression for the kinetic energy of the platform at the moment it reaches angular speed ω_P. Express your answer in terms of I_P, r, D, F, Δt, and physical constants, as appropriate.

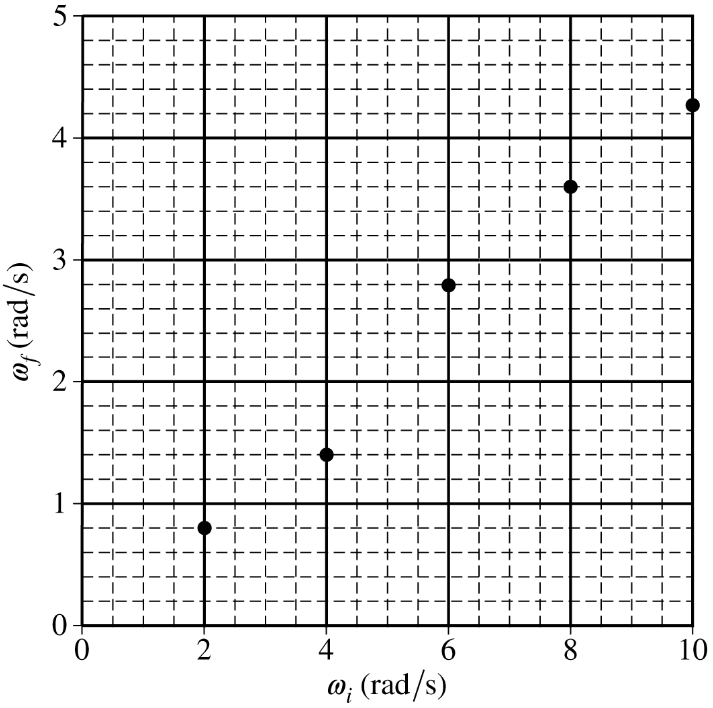
   (c) Derive an expression for the angular speed of the wheel ω_W when the platform has reached angular speed ω_P. Express your answer in terms of D, r, ω_P, and physical constants, as appropriate.



When the platform is spinning at angular speed ω_P, the motor-driven wheel is removed. A student holds a disk directly above and concentric with the platform, as shown above. The disk has the same rotational inertia I_P as the platform. The student releases the disk from rest, and the disk falls onto the platform. After a short time, the disk and platform are observed to be rotating together at angular speed ω_f.

   (d) Derive an expression for ω_f. Express your answer in terms of ω_P, I_P, and physical constants, as appropriate.

A student now uses the rotating platform $\left(I_P = 3.1 \text{ kg}\bullet\text{m}^2\right)$ to determine the rotational inertia I_U of an unknown object about a vertical axis that passes through the object's center of mass. The platform is rotating at an initial angular speed ω_i when the unknown object is dropped with its center of mass directly above the center of the platform. The platform and object are observed to be rotating together at angular speed ω_f. Trials are repeated for different values of ω_i. A graph of ω_f as a function of ω_i is shown on the axes below.



(e)

i. On the graph on the previous page, draw a best-fit line for the data.

ii. Using the straight line, calculate the rotational inertia of the unknown object I_U about a vertical axis passing through its center of mass.

(f) The kinetic energy of the spinning platform before the object is dropped on it is K_i. The total kinetic energy of the platform-object system when it reaches angular speed ω_f is K_f. Which of the following expressions is true?

____ $K_f < K_i$ ____ $K_f = K_i$ ____ $K_f > K_i$

Justify your answer.

(g) One of the students observes that the center of mass of the object is not actually aligned with the axis of the platform. Is the experimental value of I_U obtained in part (e) greater than, less than, or equal to the actual value of the rotational inertia of the unknown object about a vertical axis that passes through its center of mass?

____ Greater than ____ Less than ____ Equal to

Justify your answer.

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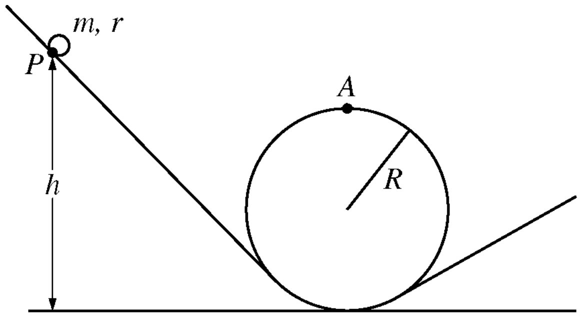
END OF EXAM



Note: Figure not drawn to scale.

3. The rotational inertia of a rolling object may be written in terms of its mass m and radius r as $I = bmr^2$, where b is a numerical value based on the distribution of mass within the rolling object. Students wish to conduct an experiment to determine the value of b for a partially hollowed sphere. The students use a looped track of radius $R >> r$, as shown in the figure above. The sphere is released from rest a height h above the floor and rolls around the loop.

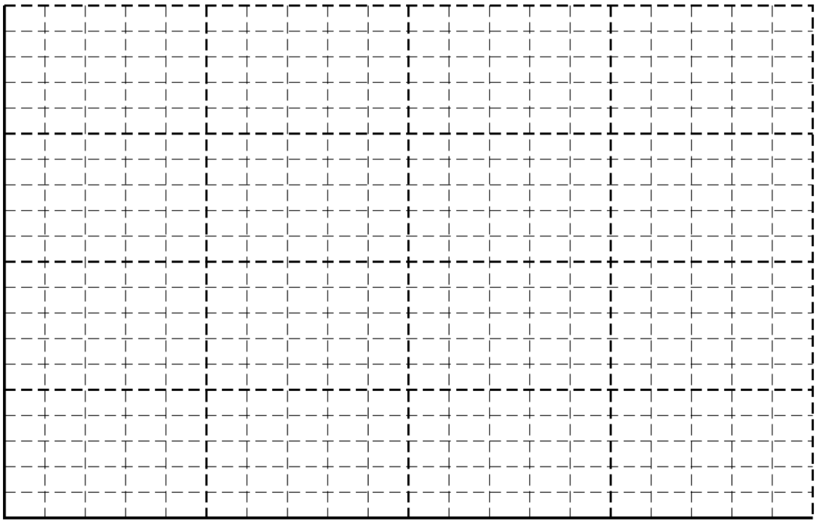
   (a) Derive an expression for the minimum speed of the sphere's center of mass that will allow the sphere to just pass point A without losing contact with the track. Express your answer in terms of b, m, R, and fundamental constants, as appropriate.

   (b) Suppose the sphere is released from rest at some point P and rolls without slipping. Derive an equation for the minimum release height h that will allow the sphere to pass point A without losing contact with the track. Express your answer in terms of b, m, R, and fundamental constants, as appropriate.

   The students perform an experiment by determining the minimum release height h for various other objects of radius r and known values of b. They collect the following data.

Object	b	h (m)
Solid sphere	0.40	1.08
Hollow sphere	0.67	1.13
Solid cylinder	0.50	1.10
Hollow cylinder	1.0	1.20

(c) On the grid below, plot the release height h as a function of b. Clearly scale and label all axes, including units, if appropriate. Draw a straight line that best represents the data.



(d) The students repeat the experiment with the partially hollowed sphere and determine the minimum release height to be 1.16 m. Using the straight line from part (c), determine the value of b for the partially hollowed sphere.

(e) Calculate R, the radius of the loop.

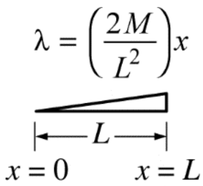
(f) In part (b), the radius r of the rolling sphere was assumed to be much smaller than the radius R of the loop. If the radius r of the rolling sphere was not negligible, would the value of the minimum release height h be greater, less, or the same?

____ Greater ____ Less ____ The same

Justify your answer.

STOP

END OF EXAM



3. A triangular rod, shown above, has length L, mass M, and a nonuniform linear mass density given by the equation $\lambda = \frac{2M}{L^2}x$, where x is the distance from one end of the rod.

(a) Using integral calculus, show that the rotational inertia of the rod about its left end is $ML^2/2$.

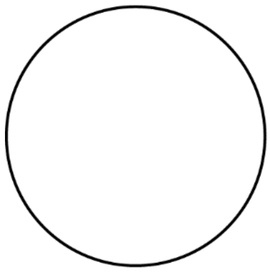


Figure 1

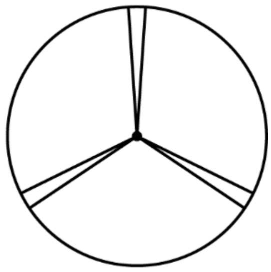


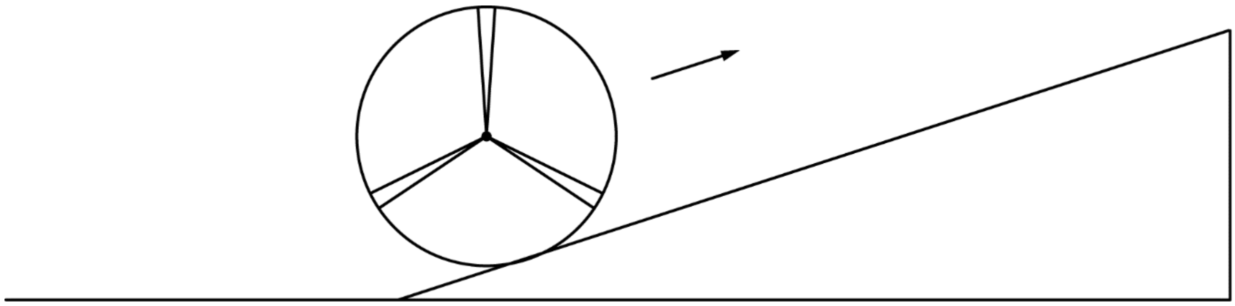
Figure 2

The thin hoop shown above in Figure 1 has a mass M, radius L, and a rotational inertia around its center of ML^2. Three rods identical to the rod from part (a) are now fastened to the thin hoop, as shown in Figure 2 above.

(b) Derive an expression for the rotational inertia I_{tot} of the hoop-rods system about the center of the hoop. Express your answer in terms of M, L, and physical constants, as appropriate.

The hoop-rods system is initially at rest and held in place but is free to rotate around its center. A constant force F is exerted tangent to the hoop for a time Δt.

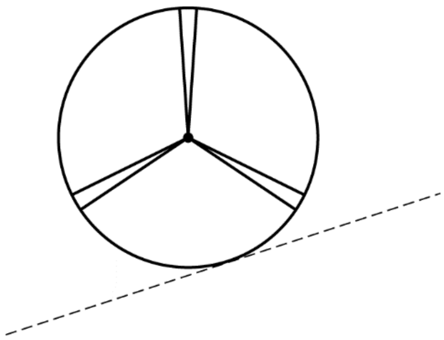
(c) Derive an expression for the final angular speed ω of the hoop-rods system. Express your answer in terms of M, L, F, Δt, and physical constants, as appropriate.



The hoop-rods system is rolling without slipping along a level horizontal surface with the angular speed ω found in part (c). At time $t = 0$, the system begins rolling without slipping up a ramp, as shown in the figure above.

(d)

i. On the figure of the hoop-rods system below, draw and label the forces (not components) that act on the system. Each force must be represented by a distinct arrow starting at, and pointing away from, the point at which the force is exerted on the system.



ii. Justify your choice for the direction of each of the forces drawn in part (d)i.

(e) Derive an expression for the change in height of the center of the hoop from the moment it reaches the bottom of the ramp until the moment it reaches its maximum height. Express your answer in terms of M, L, I_{tot}, ω, and physical constants, as appropriate.

STOP

END OF EXAM

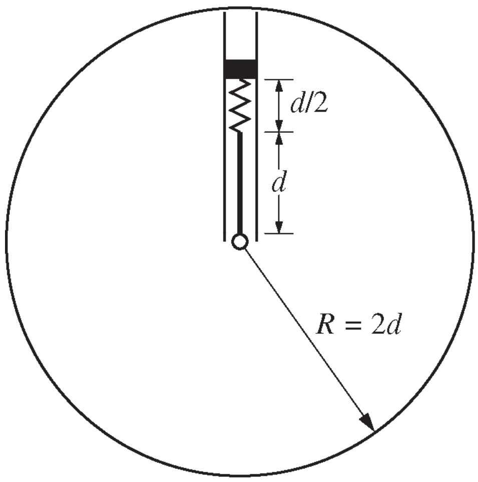


Figure 1

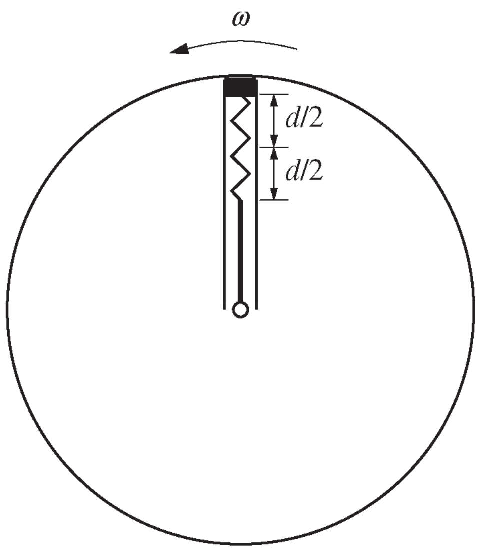


Figure 2

Mech.3.

A uniform rod of length d has one end fixed to the central axis of a horizontal, frictionless circular platform of radius $R = 2d$. Fixed at the other end of the rod is an ideal spring of negligible mass to which a block is attached. The block is set in frictionless grooves so that it can only move along a radius of the platform, as shown in Figure 1 above. The equilibrium length of the spring is $d/2$. Below is a table showing the mass of the block and the masses and rotational inertias of the rod and platform.

	Mass	Rotational Inertia
Block	m	
Rod	$m_R = 3m$	$\frac{m_R d^2}{3}$ (about the end of the rod)
Platform	$m_P = 5m$	$\frac{m_P R^2}{2}$ (about the central axis)

A motor begins to slowly rotate the platform counterclockwise as viewed from above until the platform reaches a constant angular speed ω. Under these conditions, the spring has stretched by an additional length $d/2$, as shown in Figure 2.

Answer the following questions for the platform rotating at constant angular speed ω. Express all algebraic answers in terms of m, d, ω, and physical constants, as appropriate.

(a) Derive an expression for the spring constant of the spring.

(b)

i. Determine an expression for the rotational inertia of the block around the axis of the platform.

ii. Derive an expression for the rotational inertia of the entire system about the axis of the platform.

(c) Determine an expression for the angular momentum of the entire system about the axis of the platform.

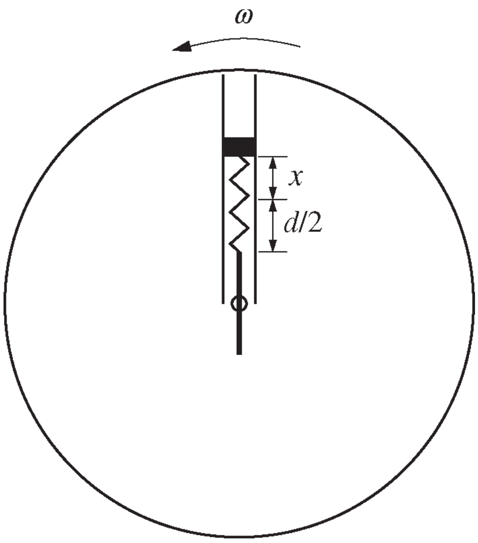


Figure 3

While the system continues to rotate, a small mechanism in the pivot moves the rod slowly until the center of the rod is positioned on the axis, as shown in Figure 3 above. The same constant angular speed ω is maintained by the motor driving the platform.

(d) Derive an expression for the distance x that the spring is stretched when the rod reaches the position shown in Figure 3 above.

For parts (e), (f), and (g), assume the center of the rod is still moving toward the axis of the platform.

(e) Is the angular momentum of the entire system increasing, decreasing, or staying the same?

_____ Increasing   _____ Decreasing   _____ Staying the same

Justify your answer.

(f) In order to keep the system rotating with constant angular speed ω, is the motor doing positive work, negative work, or no work on the rotating system?

_____ Positive   _____ Negative   _____ No work

Justify your answer.

(g) On the block in Figure 4 below, draw a single vector representing the direction of the acceleration of the block. Draw the vector so that it is starting on, and pointing away from, the block.

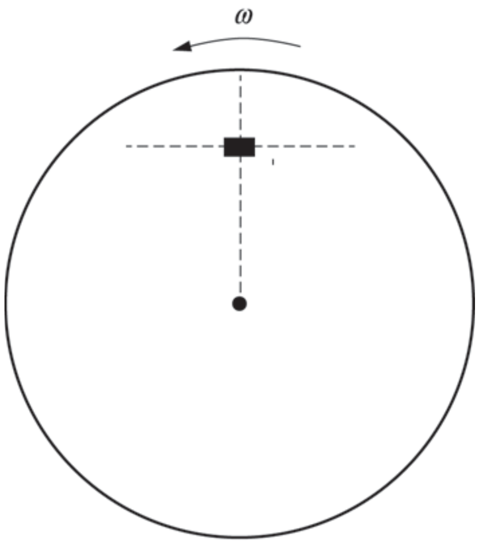


Figure 4

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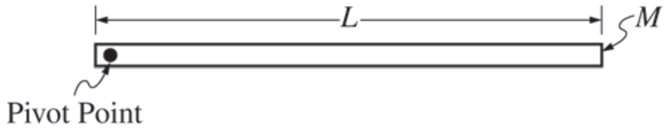
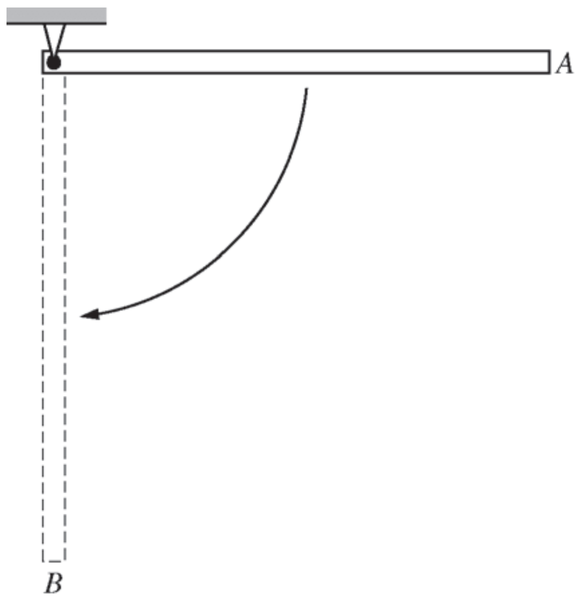
END OF EXAM



Mech.3.

A uniform, thin rod of length L and mass M is allowed to pivot about its end, as shown in the figure above.

(a) Using integral calculus, derive the rotational inertia for the rod around its end to show that it is $ML^2/3$.



The rod is fixed at one end and allowed to fall from the horizontal position A through the vertical position B.

(b) Derive an expression for the velocity of the free end of the rod at position B. Express your answer in terms of M, L, and physical constants, as appropriate.

An experiment is designed to test the validity of the expression found in part (b). A student uses rods of various lengths that all have a uniform mass distribution. The student releases each of the rods from the horizontal position A and uses photogates to measure the velocity of the free end at position B. The data are recorded below.

Length (m)	0.25	0.50	0.75	1.00	1.25	1.50
Velocity (m/s)	2.7	3.8	4.6	5.2	5.8	6.3

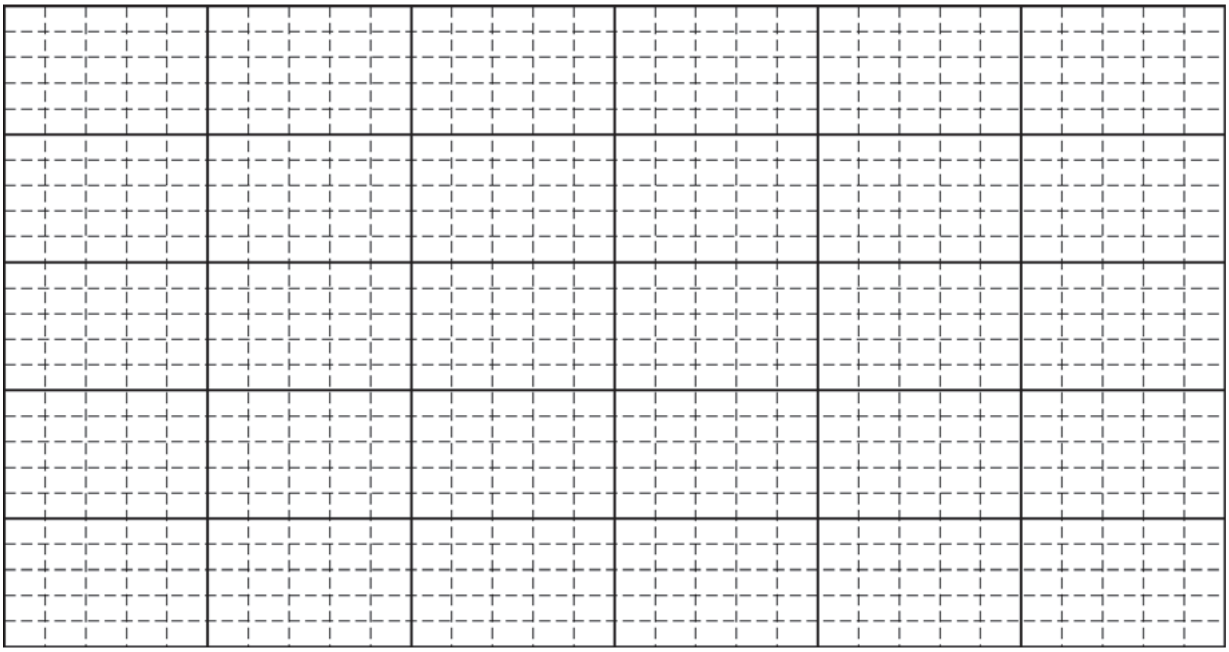
(c) Indicate below which quantities should be graphed to yield a straight line whose slope could be used to calculate a numerical value for the acceleration due to gravity g.

Horizontal axis: ____________________

Vertical axis: ____________________

Use the remaining rows in the table above, as needed, to record any quantities that you indicated that are not given. Label each row you use and include units.

(d) Plot the straight line data points on the grid below. Clearly scale and label all axes, including units as appropriate. Draw a straight line that best represents the data.



(e)

i. Using your straight line, determine an experimental value for g.

ii. Describe two ways in which the effects of air resistance could be reduced.

STOP

END OF EXAM

5.5 Rotational Equilibrium and Newton's First Law in Rotational Form

Name: ______________ Class: __________ Date: ____________ **Total: 15 marks**

KEY WORDS rotational equilibrium 转动平衡 • torque 力矩 • angular acceleration 角加速度

Objective

Build the skills to answer exam questions on **rotational equilibrium**.

You must be able to:

- state the condition for **rotational equilibrium 转动平衡**: the net torque is zero
- combine it with translational equilibrium (net force zero)
- balance clockwise and anticlockwise torques about a pivot
- solve for an unknown force or distance in a balanced system

1 Worked examples

Study these first. Each one shows the method for a question type used later — follow the steps and you can do the Practice and Exam-style questions yourself.

■ The condition

An object is in rotational equilibrium when

$$\sum \tau = 0,$$

i.e. the clockwise torques equal the anticlockwise torques about any axis.

■ A balanced beam

A pivot supports a beam. A 20 N weight sits 0.5 m left of the pivot; an unknown weight W sits 1 m right. Balance:

$$20 \times 0.5 = W \times 1 \Rightarrow W = 10 \text{ N}.$$

■ Full equilibrium

Complete equilibrium needs **both** $\sum F = 0$ and $\sum \tau = 0$: no linear acceleration and no angular acceleration.

2 Practice

Now apply the methods above.

2.1 State the condition for rotational equilibrium. [1]

2.2 A 30 N weight sits 0.4 m from a pivot. Find its torque. [2]

2.3 For full equilibrium, name the two conditions that must both hold. [2]

2.4 A beam balances with 40 N m clockwise. State the anticlockwise torque. [1]

3 Exam-style questions

3.1 An object is in rotational equilibrium when [1]

- **A** the net force is zero
- **B** the net torque is zero
- **C** it is at rest
- **D** it has no mass

3.2 A seesaw balances. The clockwise and anticlockwise torques are [1]

- **A** both zero
- **B** equal in size
- **C** unequal
- **D** in the same direction

3.3 A uniform beam pivots at its centre. A 50 N weight hangs 0.6 m to the left.

(a) Find the torque it produces. [2]

(b) A 30 N weight is placed on the right to balance it. Find its distance from the pivot. [2]

3.4 A 2 m uniform plank of weight 60 N rests on a pivot 0.5 m from its left end.

(a) State where the plank's weight effectively acts. [1]

(b) Explain, using torque, why the plank would tip without extra support. [2]

Solutions

2.1 The net torque about any axis is zero.

2.2 $\tau = 30 \times 0.4 = 12$ N m.

2.3 Net force zero ($\sum F = 0$) and net torque zero ($\sum \tau = 0$).

2.4 40 N m.

3.1 **B** — the net torque is zero.

3.2 **B** — the two torques are equal in size (and opposite in sense).

3.3 (a) $\tau = 50 \times 0.6 = 30$ N m. (b) $30 = 30 \times d \Rightarrow d = 1.0$ m.

3.4 (a) At its centre, 1 m from each end. (b) The weight acts 0.5 m right of the pivot, giving an unbalanced clockwise torque, so it rotates unless supported.

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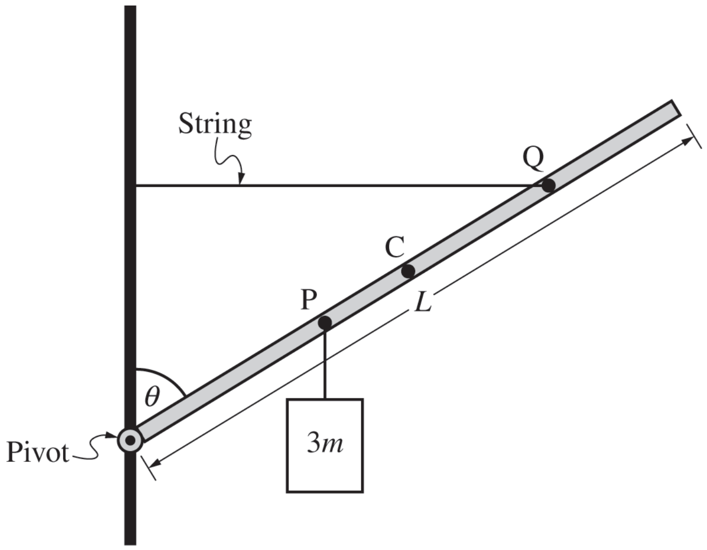
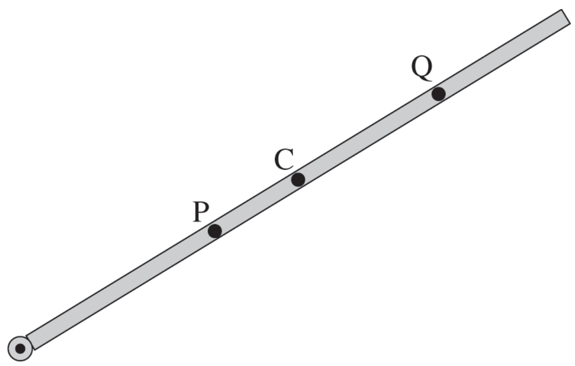


Figure 1

<u>Note</u>: Figure not drawn to scale.

3. A uniform rod of length L and mass m is attached to a pivot on a vertical pole, as shown in Figure 1. There is negligible friction between the rod and the pivot. A horizontal string connects Point Q on the rod to the pole. The rod makes an angle θ with the pole. A block of mass $3m$ hangs from the rod at Point P. The center of mass of the rod is located at Point C.

   (a) On the following representation of the rod, **draw** and **label** the forces (not components) that are exerted on the rod. Each force must be represented by a distinct arrow that starts on and points away from the point at which the force is exerted on the rod.



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(b) In Figure 1, Point P is located $\frac{3}{8}L$ from the pivot and Point Q is located $\frac{6}{8}L$ from the pivot. **Derive** an equation for the tension F_T in the horizontal string in terms of L, m, θ, and physical constants, as appropriate.

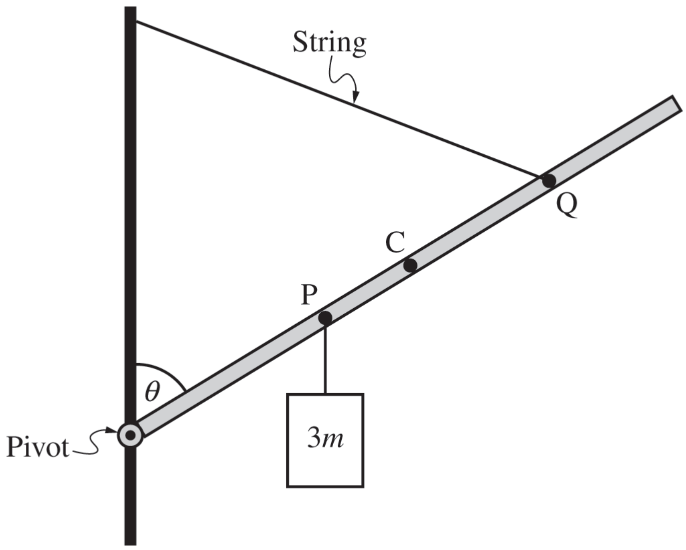

Figure 2

<u>Note:</u> Figure not drawn to scale.

(c) The original string is replaced with a longer string that connects Point Q to a higher location on the vertical pole, as shown in Figure 2. The angle θ remains the same. How does the new tension $F_{T,\,new}$ compare with the original tension F_T from part (b) ? **Justify** your reasoning.

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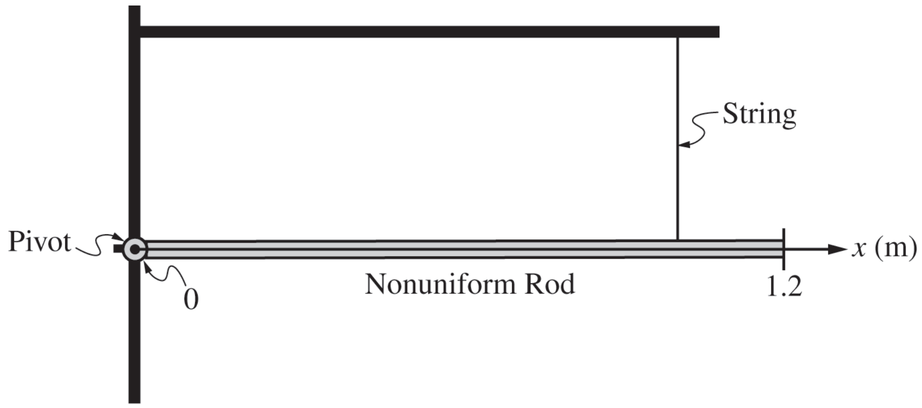

Figure 3

Note: Figure not drawn to scale.

(d) A nonuniform rod is now attached to the pivot, as shown in Figure 3. There is negligible friction between the nonuniform rod and the pivot. The rod has a length of 1.2 m and a linear mass density $\lambda(x) = A + Bx$, where x is the distance from the pivot, $A = 6.0$ kg/m, and $B = 10.0$ kg/m^2.

i. **Calculate** the mass of the rod.

ii. **Calculate** the rotational inertia of the rod about the pivot.

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END OF EXAM

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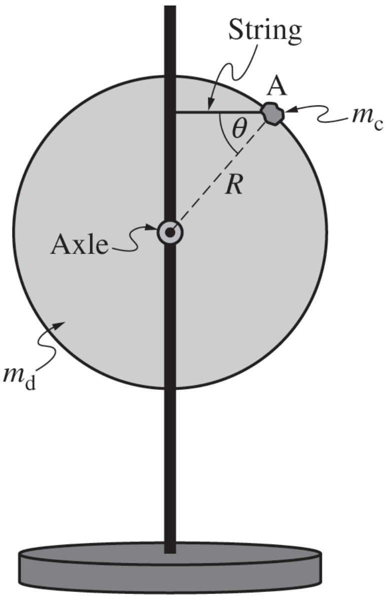
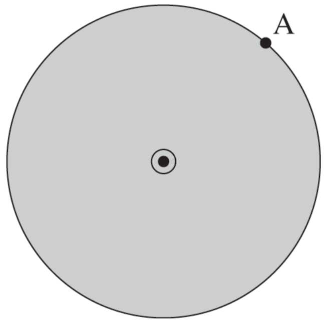


Figure 1

Note: Figure not drawn to scale.

3. A uniform disk of radius R and mass m_d is attached to a vertical pole by a horizontal axle that passes through the center of the disk. Friction between the axle and the disk is negligible. A lump of clay of mass m_c is attached to the edge of the disk at Point A. The size of the lump of clay is small compared with the radius of the disk. A horizontal string is connected from the pole to the edge of the disk at Point A. The string makes an angle θ with the line between Point A and the axle, as shown in Figure 1.

(a) On the following representation of the clay-disk system, **draw** and **label** the external forces (not components) exerted on the system. Each force must be represented by a distinct arrow that starts on, and points away from, the point at which the force is exerted on the system.



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Continue your response to **QUESTION 3** on this page.

(b) **Derive** an expression for the tension F_T in the string when the clay is at Point A, as shown in Figure 1, in terms of R, m_d, m_c, θ, and physical constants, as appropriate.

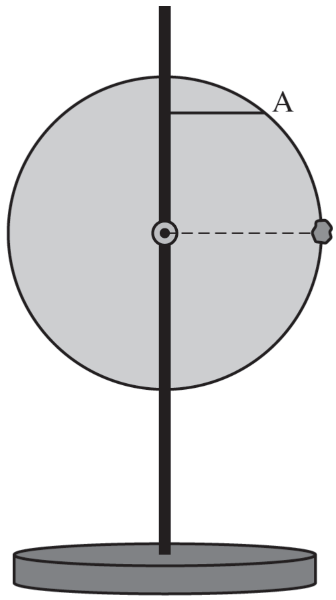

Figure 2

<u>Note:</u> Figure not drawn to scale.

(c) The string remains connected to the edge of the disk at Point A. The clay is moved to Point B, which is horizontally in line with the axle, as shown in Figure 2. How does the new tension $F_{T,\text{ new}}$ compare with tension F_T from part (b)? **Justify** your reasoning.

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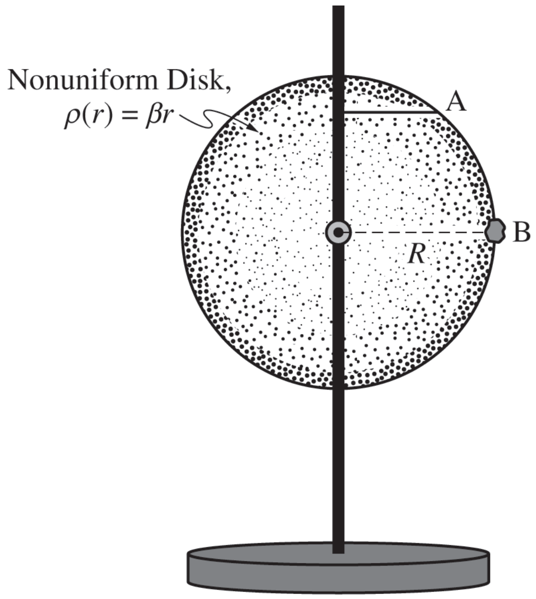

Figure 3

Note: Figure not drawn to scale.

(d) A nonuniform disk is now attached to the axle. The lump of clay is attached to the disk at Point B, as shown in Figure 3. The clay has mass $m_c = 0.60\ \text{kg}$ and the disk has a radius $R = 0.30\ \text{m}$. The mass density of the disk varies radially and can be modeled by $\rho(r) = \beta r$, where r is the radial distance from the axle and $\beta = 4.0\ \text{kg/m}^3$.

i. **Calculate** the rotational inertia of the disk about the axle.

ii. The string connecting the disk to the pole is cut. **Calculate** the magnitude of the initial angular acceleration of the clay-disk system.

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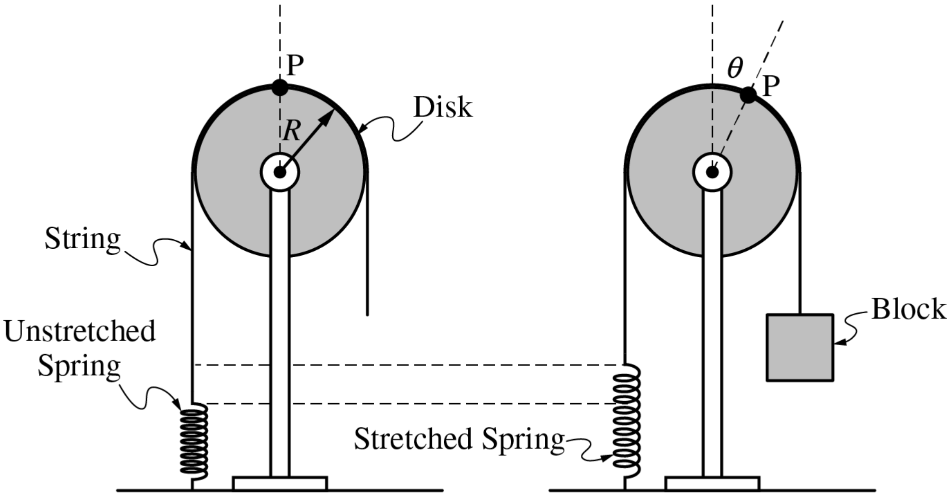

Figure 1 Figure 2

Note: Figures not drawn to scale.

. A solid uniform disk is supported by a vertical stand. The disk is able to rotate with negligible friction about an axle that passes through the center of the disk. The mass and radius of the disk are given by M_d and R, respectively. The rotational inertia of the disk is $I_d = \frac{1}{2} M_d R^2$. A string of negligible mass is draped over the disk and attached to the top of the disk at point P. One end of the string is connected to an unstretched ideal spring of spring constant k, which is fixed to the ground as shown in Figure 1.

A block of mass m_B is then attached to the string on the right side of the disk. The block is slowly lowered until the spring-disk-block system reaches equilibrium, as shown in Figure 2. In this equilibrium position, the disk has rotated clockwise through a small angle θ.

Give all algebraic answers in terms of M_d, R, k, θ, and physical constants, as appropriate.

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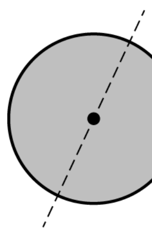
Continue your response to **QUESTION 3** on this page.

(a) Derive an expression for the mass m_B of the block.

(b) At time $t = 0$, the string on the right side of the disk is cut and the block falls to the ground. On the circle below, which represents the disk, draw and label the forces (not components) that act on the disk immediately after the string is cut and the block is falling to the ground. Each force should be represented by an arrow that starts on and is directed away from the point of application.



(c) Derive an expression for the angular acceleration α of the disk immediately after the string is cut.

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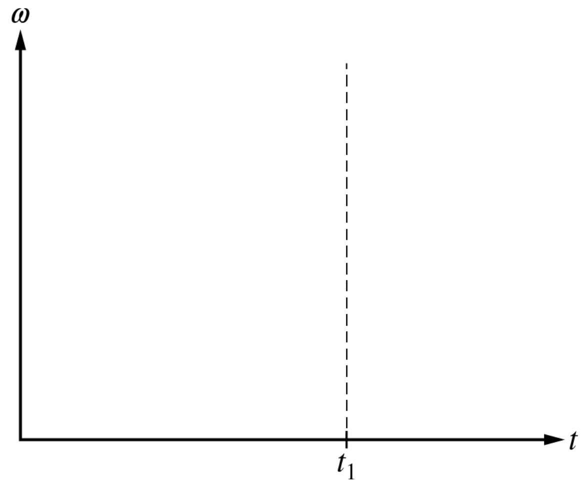
(d) At $t = t_1$, the disk has rotated and point P is again directly above the axle. Sketch a graph of the magnitude of the angular velocity ω of the disk as a function of time t from $t = 0$ to $t = t_1$.



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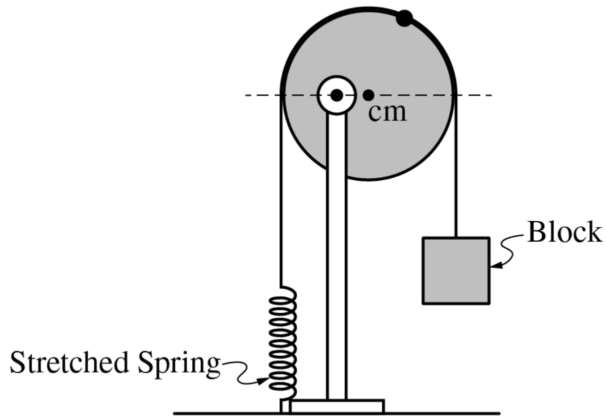

Figure 3

<u>Note</u>: Figure not drawn to scale.

(e) The disk is adjusted on the support so that the axle does <u>not</u> pass through the center of mass of the disk. The block is again hung on the right side of the disk and the spring-disk-block system comes to equilibrium, as shown in Figure 3. The axle does not exert a torque on the disk. For each force on the disk, indicate whether the magnitude of the torque about the axle caused by that force increases, decreases, or stays the same relative to part (b).

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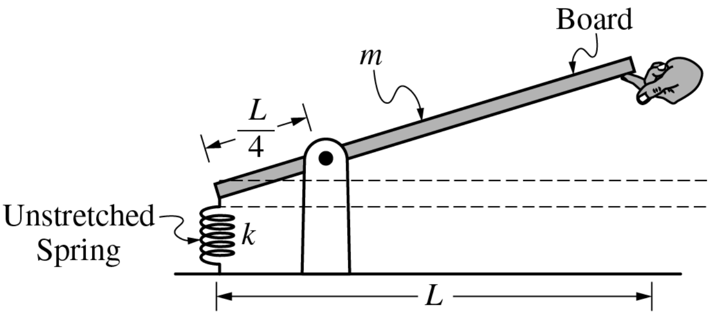


Figure 1

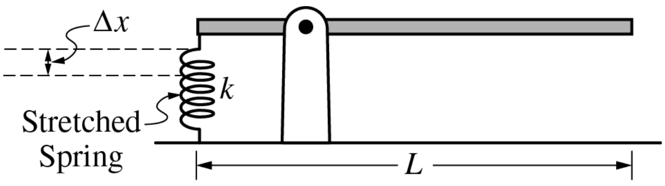
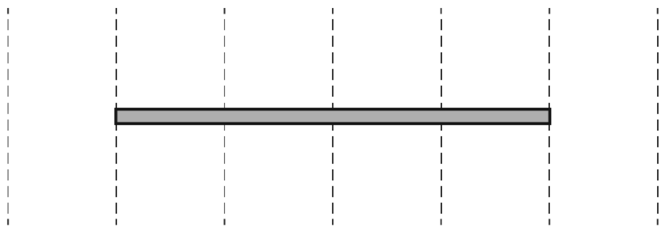


Figure 2

Note: Figures not drawn to scale.

. A uniform board of length L and mass m is attached to a pivot $\frac{L}{4}$ from the left end of the board. The left end of the board is attached to an ideal spring of spring constant k that is attached to the ground. The right end of the board is initially held by a student so that the spring is unstretched, as shown in Figure 1. The student slowly lowers and then releases the board. The board remains at rest in the horizontal position, with the spring stretched, as shown in Figure 2. The rotational inertia of the board about the pivot is I.

(a) On the rectangle below, which represents the board, draw and label the forces (not components) that act on the board while the board-spring system is in equilibrium. Each force should be represented by an arrow that starts on, and points away from, the board, and should represent the location at which that force acts.



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(b) Derive an expression for the distance the spring stretches, Δx, when the board is in equilibrium. Express your answer in terms of k, L, m, and physical constants, as appropriate.

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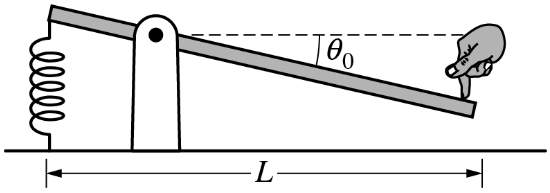
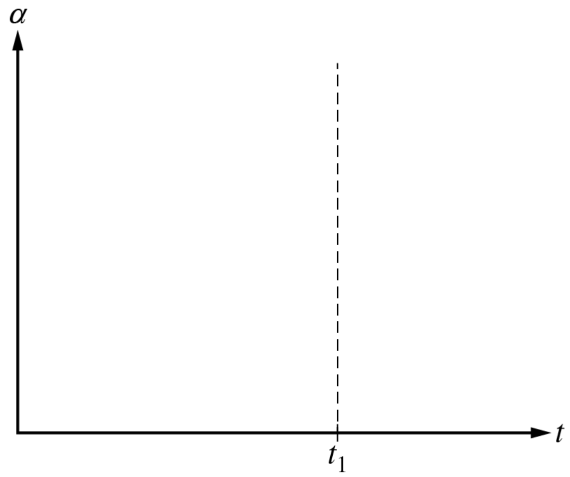


Note: Figure not drawn to scale.

(c) A student pushes the board down on the right side, stretching the spring a new distance Δx_2 from the unstretched position. The board is held at a small angle θ_0 with the horizontal, as shown. The student then releases the board from rest.

i. At time $t = 0$, the board is released. At $t = t_1$, the board first crosses the horizontal. Sketch a graph of the magnitude of the angular acceleration α of the board as a function of time t from $t = 0$ to $t = t_1$.



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ii. Derive an expression for the angular acceleration α_0 of the board immediately after the board is released. Express your answer in terms of k, L, m, I, Δx_2, θ_0, and physical constants, as appropriate.

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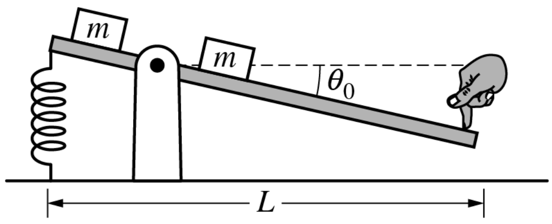

Note: Figure not drawn to scale.

(d) Two blocks of equal mass m are attached to the board equal distances from the pivot point, as shown. The board is again pushed down on the right side so that the spring stretches the same distance Δx_2 as in part (c). The board is then released. How does the new angular acceleration α' when the blocks are attached compare to the angular acceleration α_0 from part (c) ?

___ $\alpha' > \alpha_0$ ___ $\alpha' < \alpha_0$ ___ $\alpha' = \alpha_0$

Justify your answer.

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END OF EXAM

AP PHYSICS C: MECHANICS • EXERCISE SHEET

5.6 Newton's Second Law in Rotational Form

Name: ______________ Class: __________ Date: ____________ **Total: 15 marks**

KEY WORDS angular acceleration 角加速度 • rotational inertia 转动惯量 • torque 力矩

Objective

Build the skills to answer exam questions on **Newton's second law in rotational form**.

You must be able to:

- apply the rotational second law: $\tau_{net} = I\alpha$
- recognise it as the analogue of $F = ma$ (torque ↔ force, I ↔ mass, α ↔ acceleration)
- find any one of τ, I, α given the other two
- explain why a larger rotational inertia gives a smaller angular acceleration

1 Worked examples

Study these first. Each one shows the method for a question type used later — follow the steps and you can do the Practice and Exam-style questions yourself.

■ The rotational second law

$$\tau_{net} = I\alpha.$$

A net torque of 8 N m on a wheel with $I = 2$ kg m^2 gives

$$\alpha = \frac{\tau}{I} = \frac{8}{2} = 4 \text{ rad s}^{-2}.$$

■ The analogy with F = ma

Torque plays the role of force, rotational inertia the role of mass, and angular acceleration the role of acceleration.

■ Bigger inertia, gentler response

For a fixed torque, a larger I gives a smaller α – a heavy flywheel is hard to spin up.

2 Practice

Now apply the methods above.

2.1 A torque of 12 N m acts on a body with $I = 3$ kg m^2. Find the angular acceleration. [2]

2.2 State the rotational analogue of mass. [1]

2.3 What net torque gives a 5 kg m^2 wheel an angular acceleration of 2 rad s^{-2}? [2]

2.4 For a fixed torque, how does a larger rotational inertia affect α? [1]

3 Exam-style questions

3.1 The rotational form of Newton's second law is [1]

- **A** $F = ma$
- **B** $\tau = I\alpha$
- **C** $p = mv$
- **D** $W = \tau\theta$

3.2 A 6 N m torque acts on a wheel with $I = 2$ kg m^2. Its angular acceleration is [1]

- **A** 3 rad s^{-2}
- **B** 12 rad s^{-2}
- **C** 0.33 rad s^{-2}
- **D** 8 rad s^{-2}

3.3 A grindstone with $I = 0.5$ kg m^2 is driven by a net torque of 4 N m.

(a) Find its angular acceleration. [2]

(b) State what happens to α if the torque is doubled. [1]

3.4 A wheel with $I = 4$ kg m^2 speeds up from rest to 6 rad s^{-1} in 3 s.

(a) Find the angular acceleration. [2]

(b) Find the net torque needed. [2]

Solutions

2.1 $\alpha = \tau/I = 12/3 = 4$ rad s^{-2}.

2.2 Rotational inertia I.

2.3 $\tau = I\alpha = 5 \times 2 = 10$ N m.

2.4 It makes α smaller.

3.1 **B** — $\tau = I\alpha$.

3.2 **A** — $\alpha = 6/2 = 3$ rad s^{-2}.

3.3 (a) $\alpha = 4/0.5 = 8$ rad s^{-2}. (b) It doubles.

3.4 (a) $\alpha = 6/3 = 2$ rad s^{-2}. (b) $\tau = I\alpha = 4 \times 2 = 8$ N m.

Question 4

4. A uniform disk and ring, each of mass M and radius R, roll without slipping along a horizontal surface, as shown in Figure 1. The outer edges of the disk and ring are made of the same material. The center of mass of the disk and the center of mass of the ring each initially move with the same constant speed v.

The disk and the ring then smoothly transition to a ramp that is inclined at an angle θ above the horizontal. Both the disk and the ring continue to roll without slipping as they move up the ramp, as shown in Figure 2.

The ring travels a greater distance along the ramp than the disk travels before each momentarily comes to rest.

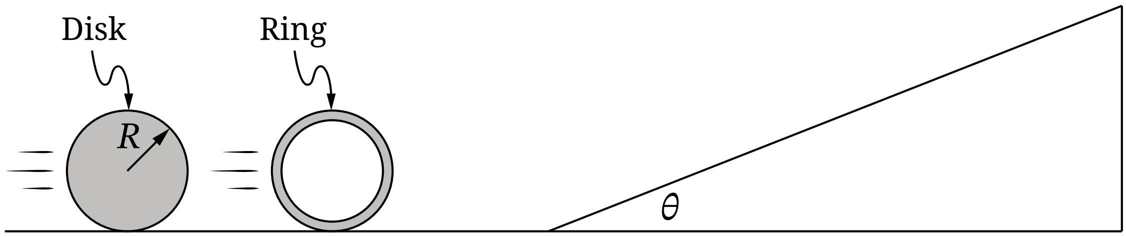

Figure 1

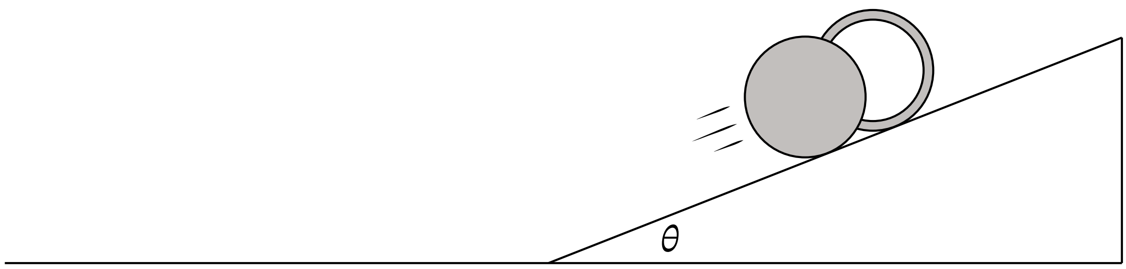

Figure 2

A. While the disk and the ring are rolling on the ramp without slipping, the magnitudes of the static frictional force exerted on the disk and on the ring by the ramp are f_D and f_R, respectively.

Indicate whether f_D is greater than, less than, or equal to f_R by writing one of the following.

- $f_D > f_R$
- $f_D < f_R$
- $f_D = f_R$

Justify your answer using qualitative reasoning beyond referencing equations.

B. A cylinder has mass M, radius R, and rotational inertia I about its central axis. The cylinder rolls without slipping up a ramp that is inclined at an angle θ above the horizontal.

Derive an expression for the magnitude of the static frictional force f exerted on the cylinder by the ramp. Express your answer in terms of M, R, I, θ, and physical constants, as appropriate. Begin your derivation by writing a fundamental physics principle or an equation from the reference information.

C. In a different scenario, the centers of mass of the original disk and ring each have the same initial speed v as they did in the original scenario. The ramp is replaced by a new ramp on which the disk and the ring initially slip as they roll up the new ramp.

Indicate whether the magnitude of the kinetic frictional force exerted on the disk by the new ramp is greater than, less than, or equal to the magnitude of the kinetic frictional force exerted on the ring by the new ramp while both are slipping.

Briefly **justify** your answer.

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END OF EXAM

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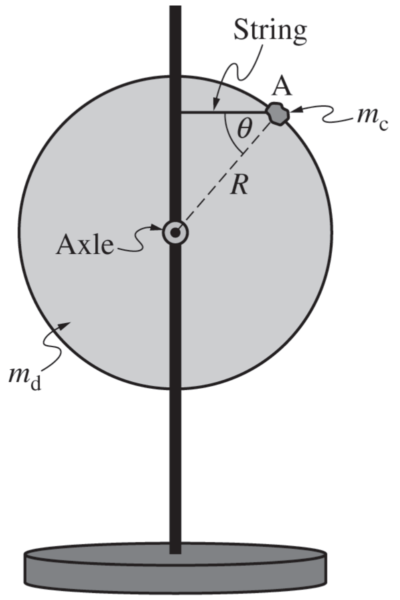
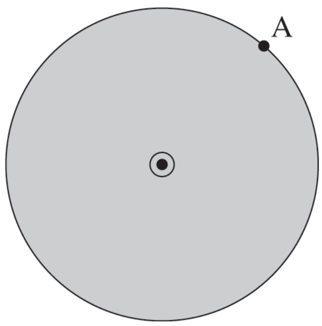


Figure 1

<u>Note:</u> Figure not drawn to scale.

3. A uniform disk of radius R and mass m_d is attached to a vertical pole by a horizontal axle that passes through the center of the disk. Friction between the axle and the disk is negligible. A lump of clay of mass m_c is attached to the edge of the disk at Point A. The size of the lump of clay is small compared with the radius of the disk. A horizontal string is connected from the pole to the edge of the disk at Point A. The string makes an angle θ with the line between Point A and the axle, as shown in Figure 1.

(a) On the following representation of the clay-disk system, **draw** and **label** the external forces (not components) exerted on the system. Each force must be represented by a distinct arrow that starts on, and points away from, the point at which the force is exerted on the system.



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(b) **Derive** an expression for the tension F_T in the string when the clay is at Point A, as shown in Figure 1, in terms of R, m_d, m_c, θ, and physical constants, as appropriate.

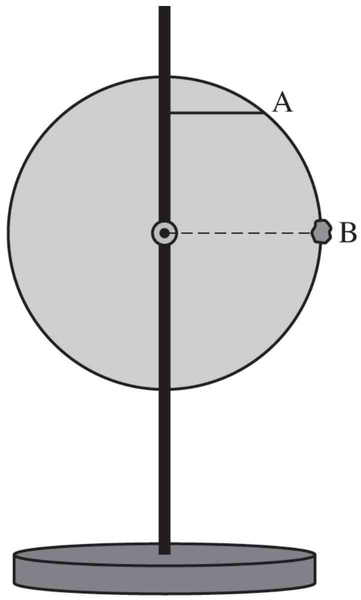

Figure 2

<u>Note:</u> Figure not drawn to scale.

(c) The string remains connected to the edge of the disk at Point A. The clay is moved to Point B, which is horizontally in line with the axle, as shown in Figure 2. How does the new tension $F_{T,\ new}$ compare with tension F_T from part (b)? **Justify** your reasoning.

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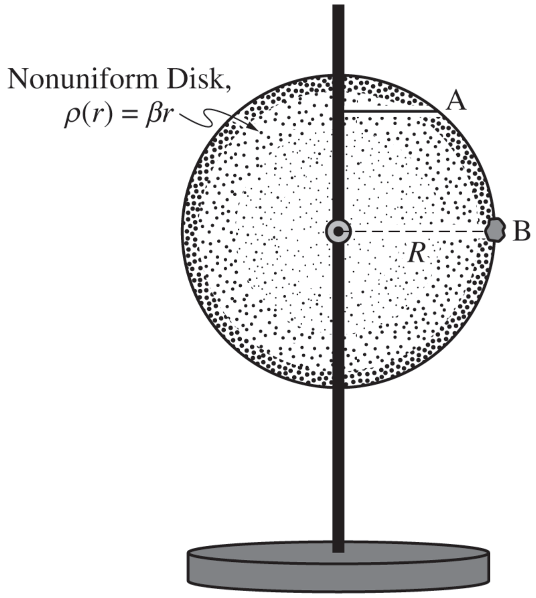

Figure 3

<u>Note:</u> Figure not drawn to scale.

(d) A nonuniform disk is now attached to the axle. The lump of clay is attached to the disk at Point B, as shown in Figure 3. The clay has mass $m_c = 0.60\text{ kg}$ and the disk has a radius $R = 0.30\text{ m}$. The mass density of the disk varies radially and can be modeled by $\rho(r) = \beta r$, where r is the radial distance from the axle and $\beta = 4.0\text{ kg/m}^3$.

i. **Calculate** the rotational inertia of the disk about the axle.

ii. The string connecting the disk to the pole is cut. **Calculate** the magnitude of the initial angular acceleration of the clay-disk system.

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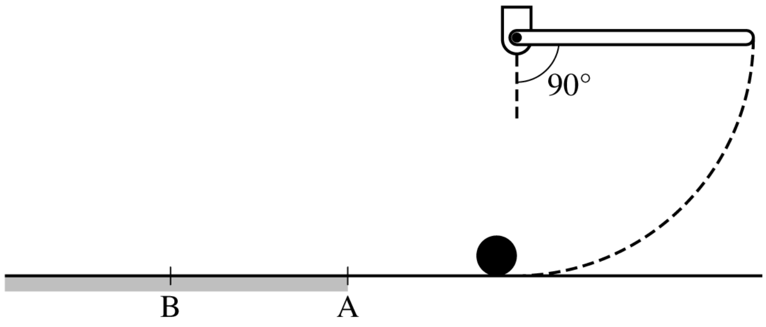
END OF EXAM



Note: Figure not drawn to scale.

Figure 1

3. A system consists of a small sphere of mass m and radius R at rest on a horizontal surface and a uniform rod of mass $M = 2m$ and length ℓ attached at one end to a pivot with negligible friction, where $R \ll \ell$. There is negligible friction between the surface and the sphere to the right of Point A and nonnegligible friction to the left of Point A. The rod is held horizontally as shown in Figure 1, then is released from rest. The total rotational inertia of the rod about the pivot is $\frac{1}{3}M\ell^2$ and the rotational inertia of the sphere about its center is $\frac{2}{5}mR^2$. After the rod is released, the rod swings down and strikes the sphere head-on. As a result of this collision, the rod is stopped, and the ball initially slides without rotating to the left across the horizontal surface.

(a) Derive an expression for the angular speed of the rod just before striking the sphere in terms of the length ℓ and physical constants as appropriate.

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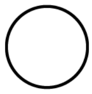
Continue your response to **QUESTION 3** on this page.

(b) Derive an expression for the linear speed v_0 of the sphere immediately after colliding with the rod in terms of the length ℓ and physical constants as appropriate.

After sliding a short distance, at time $t = 0$ the sphere encounters a region of the horizontal surface with a coefficient of kinetic friction μ, beginning at Point A as indicated in Figure 1. The sphere begins rotating while sliding and eventually begins rolling without sliding at Point B, also as indicated.

(c) In the following diagram, which represents the sphere while the sphere is traveling between Points A and B, draw and label the forces (not components) that act on the sphere. Each force must be represented by a distinct arrow starting on, and pointing away from, the point of application on the sphere.



(d) Derive an expression for each of the following as the sphere is rotating and sliding between points A and B in terms of v_0, μ, R, t, and physical constants as appropriate.

i. The linear velocity v of the center of mass of the sphere as a function of time t

ii. The angular velocity ω of the sphere as a function of time t

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Continue your response to **QUESTION 3** on this page.

(e)

i. Derive an expression for the time it takes the sphere to travel from Point A to Point B in terms of v_0, μ, and physical constants as appropriate.

ii. Derive an expression for the linear velocity of the sphere upon reaching Point B in terms of v_0.

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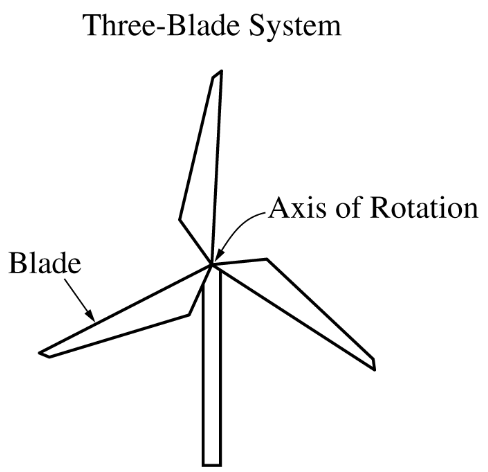

Figure 1

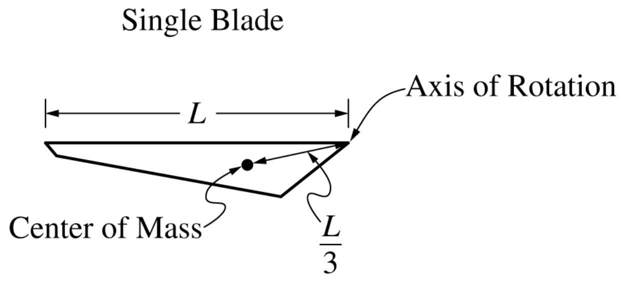

Figure 2

. A wind turbine includes a three-blade system that rotates about an axis through the end of each blade, as shown in Figure 1. Each blade has a length L and mass M, with a center of mass located at a distance $\frac{L}{3}$ from the axis of rotation, as shown in Figure 2.

(a) Derive an expression for the rotational inertia of the three-blade system. Express your answer in terms of M, L, and physical constants, as appropriate. The rotational inertia of each blade about an axis through its center of mass is given by the equation $I_{cm} = \frac{1}{18}ML^2$.

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(b) While the wind blows, the three-blade system operates at a constant angular speed $\omega_0 = 2.6$ rad/s. The length of one blade is $L = 36$ m. The numerical value of the rotational inertia of the system is $I_{sys} = 6.7 \times 10^6$ kg·m^2. Calculate the time T it takes the outer edge of a single blade to complete one revolution.

(c) When the wind stops blowing, the angular speed of the system decreases. The angular speed ω of the system while slowing down is given as a function of time t by the equation $\omega = \omega_0 e^{-\beta_0 t}$, where β_0 is a constant with appropriate units, as shown on the graph in Figure 3.

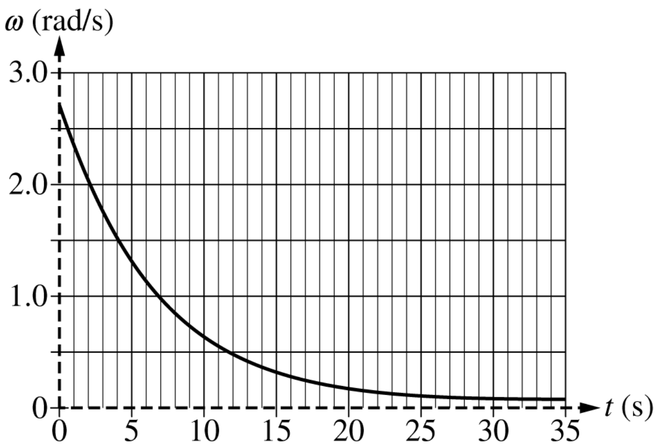

Figure 3

i. Calculate the amount of energy dissipated from $t = 0$, when the wind stops blowing, until the system comes to rest.

Continue your response to **QUESTION 3** on this page.

ii. Derive an expression for the net torque exerted on the system as a function of t as the system slows down. Express your answers in terms of β_0, ω_0, M, L, I_{sys}, and physical constants, as appropriate.

iii. Derive an expression for the angular displacement of the system $\Delta\theta$ as a function of t. Express your answer in terms of β_0, ω_0, M, L, I_{sys}, and physical constants, as appropriate.

The three-blade system is now replaced with a second three-blade system identical to the first, except that the second three-blade system slows down according to the equation $\omega = \omega_0 e^{-\beta t}$, where $\omega_0 = 2.6$ rad/s and $\beta > \beta_0$. The original angular speed function is shown as a dashed line in Figure 4.

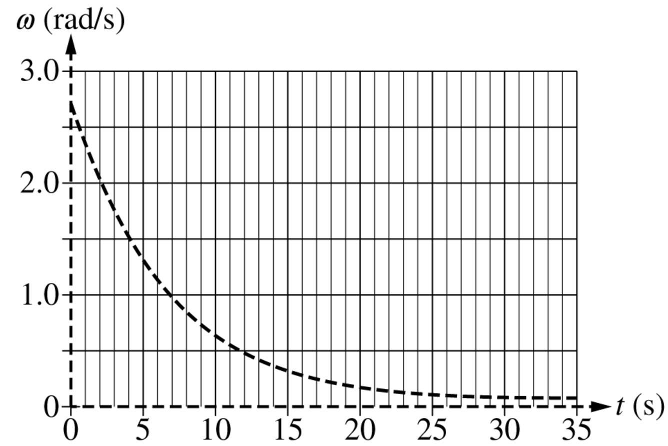

Figure 4

(d) On the graph in Figure 4, sketch the angular speed of the second three-blade system as a function of time t.

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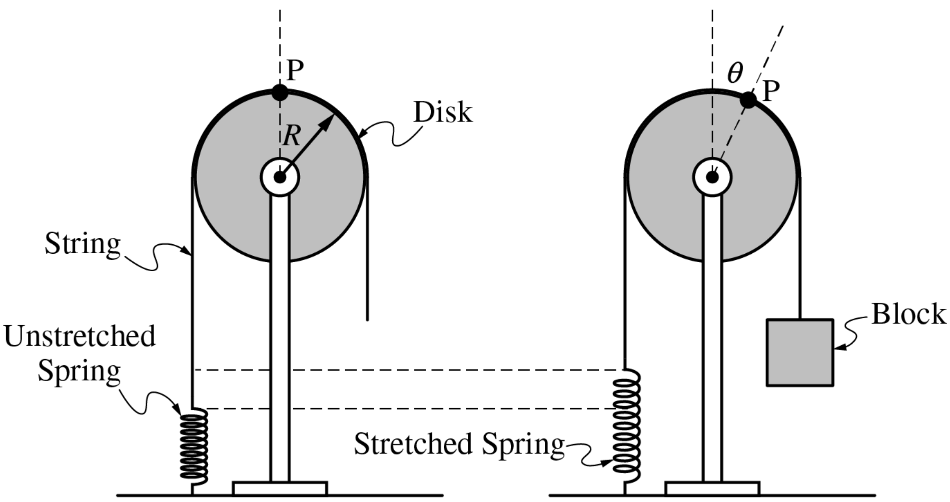

Figure 1 Figure 2

Note: Figures not drawn to scale.

. A solid uniform disk is supported by a vertical stand. The disk is able to rotate with negligible friction about an axle that passes through the center of the disk. The mass and radius of the disk are given by M_d and R, respectively. The rotational inertia of the disk is $I_d = \frac{1}{2}M_d R^2$. A string of negligible mass is draped over the disk and attached to the top of the disk at point P. One end of the string is connected to an unstretched ideal spring of spring constant k, which is fixed to the ground as shown in Figure 1.

A block of mass m_B is then attached to the string on the right side of the disk. The block is slowly lowered until the spring-disk-block system reaches equilibrium, as shown in Figure 2. In this equilibrium position, the disk has rotated clockwise through a small angle θ.

Give all algebraic answers in terms of M_d, R, k, θ, and physical constants, as appropriate.

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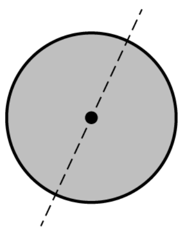
Continue your response to **QUESTION 3** on this page.

(a) Derive an expression for the mass m_B of the block.

(b) At time $t = 0$, the string on the right side of the disk is cut and the block falls to the ground. On the circle below, which represents the disk, draw and label the forces (not components) that act on the disk immediately after the string is cut and the block is falling to the ground. Each force should be represented by an arrow that starts on and is directed away from the point of application.



(c) Derive an expression for the angular acceleration α of the disk immediately after the string is cut.

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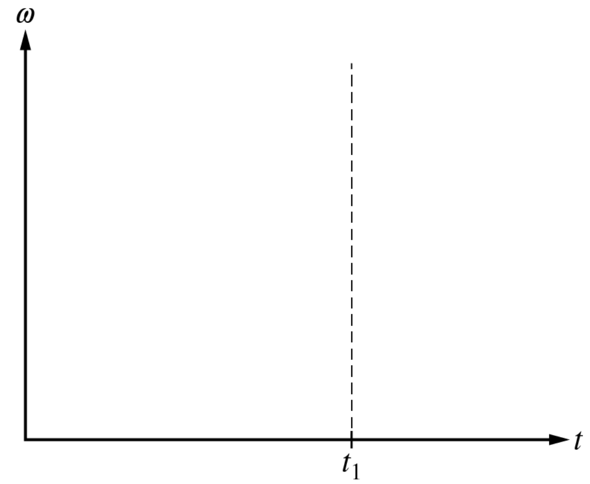
(d) At $t = t_1$, the disk has rotated and point P is again directly above the axle. Sketch a graph of the magnitude of the angular velocity ω of the disk as a function of time t from $t = 0$ to $t = t_1$.



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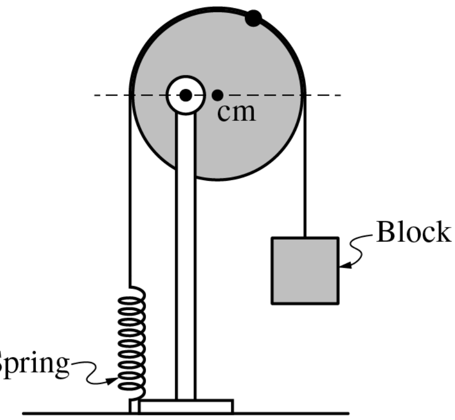

Figure 3

Note: Figure not drawn to scale.

(e) The disk is adjusted on the support so that the axle does not pass through the center of mass of the disk. The block is again hung on the right side of the disk and the spring-disk-block system comes to equilibrium, as shown in Figure 3. The axle does not exert a torque on the disk. For each force on the disk, indicate whether the magnitude of the torque about the axle caused by that force increases, decreases, or stays the same relative to part (b).

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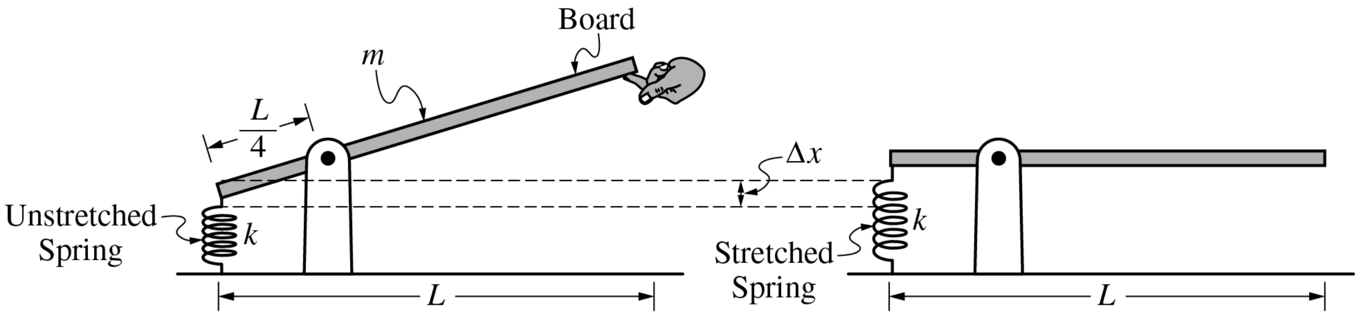


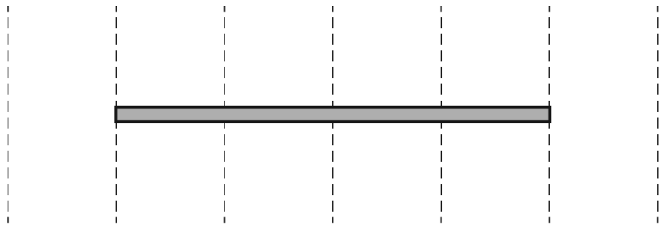
Figure 1

Figure 2

Note: Figures not drawn to scale.

. A uniform board of length L and mass m is attached to a pivot $\frac{L}{4}$ from the left end of the board. The left end of the board is attached to an ideal spring of spring constant k that is attached to the ground. The right end of the board is initially held by a student so that the spring is unstretched, as shown in Figure 1. The student slowly lowers and then releases the board. The board remains at rest in the horizontal position, with the spring stretched, as shown in Figure 2. The rotational inertia of the board about the pivot is I.

(a) On the rectangle below, which represents the board, draw and label the forces (not components) that act on the board while the board-spring system is in equilibrium. Each force should be represented by an arrow that starts on, and points away from, the board, and should represent the location at which that force acts.



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(b) Derive an expression for the distance the spring stretches, Δx, when the board is in equilibrium. Express your answer in terms of k, L, m, and physical constants, as appropriate.

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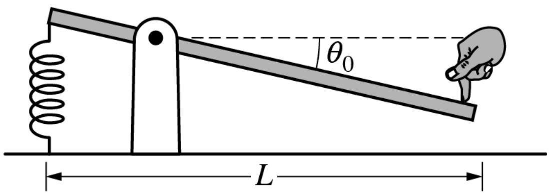
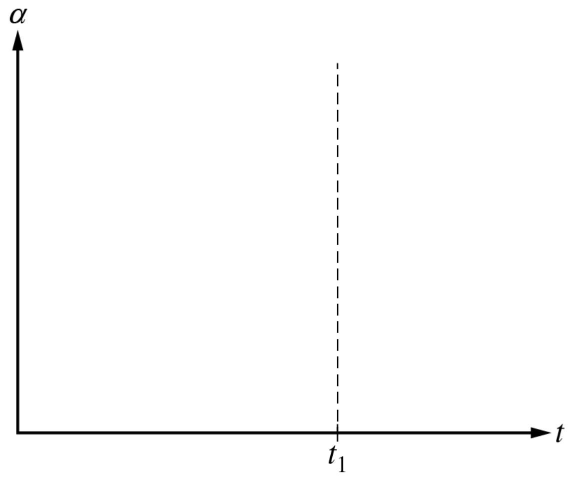


Note: Figure not drawn to scale.

(c) A student pushes the board down on the right side, stretching the spring a new distance Δx_2 from the unstretched position. The board is held at a small angle θ_0 with the horizontal, as shown. The student then releases the board from rest.

i. At time $t = 0$, the board is released. At $t = t_1$, the board first crosses the horizontal. Sketch a graph of the magnitude of the angular acceleration α of the board as a function of time t from $t = 0$ to $t = t_1$.



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ii. Derive an expression for the angular acceleration α_0 of the board immediately after the board is released. Express your answer in terms of k, L, m, I, Δx_2, θ_0, and physical constants, as appropriate.

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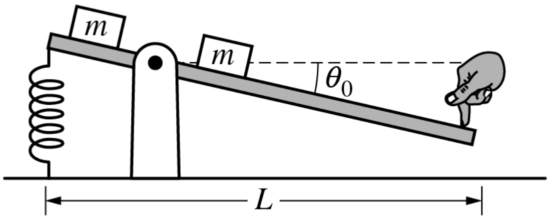

Note: Figure not drawn to scale.

(d) Two blocks of equal mass m are attached to the board equal distances from the pivot point, as shown. The board is again pushed down on the right side so that the spring stretches the same distance Δx_2 as in part (c). The board is then released. How does the new angular acceleration α' when the blocks are attached compare to the angular acceleration α_0 from part (c) ?

___ $\alpha' > \alpha_0$ ___ $\alpha' < \alpha_0$ ___ $\alpha' = \alpha_0$

Justify your answer.

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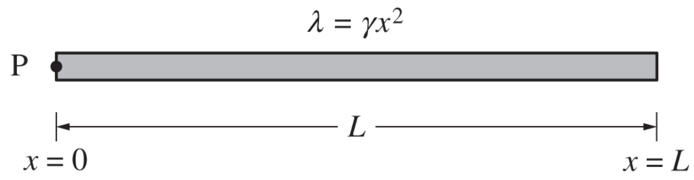

3. A triangular rod of length L and mass M has a nonuniform linear mass density given by the equation $\lambda = \gamma x^2$, where $\gamma = \frac{3M}{L^3}$ and x is the distance from point P at the left end of the rod.

(a) Using integral calculus, show that the rotational inertia I of the rod about an axis perpendicular to the page and through point P is $\frac{3}{5}ML^2$.

(b) Determine the horizontal location of the center of mass of the rod relative to point P. Express your answer in terms of L.

(c) For an axis perpendicular to the page, is the value of the rotational inertia of the rod around point P greater than, less than, or equal to the value of the rotational inertia of the rod around the rod's center of mass?

_____ Greater than _____ Less than _____ Equal to

Justify your answer.

The rod is released from rest in the position shown, and the rod begins to rotate about a horizontal axis perpendicular to the page and through point P.

(d) On the axes below, sketch graphs of the magnitude of the net torque τ on the rod and the angular speed ω of the rod as functions of time t from the time the rod is released until the time its center of mass reaches its lowest point.

(e) As the rod rotates from the horizontal position down through vertical, is the magnitude of the angular acceleration on the rod increasing, decreasing, or not changing?

_____ Increasing   _____ Decreasing   _____ Not changing

Justify your answer.

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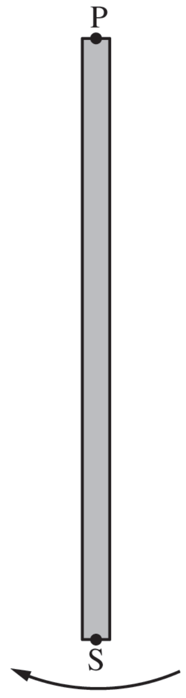

(f) The mass of the rod is 3.0 kg, and the length of the rod is 1.0 m. Calculate the linear speed v of point S as the rod swings through the vertical position shown.

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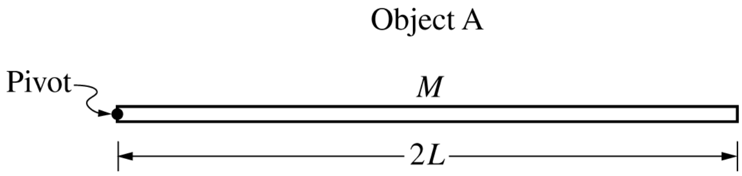
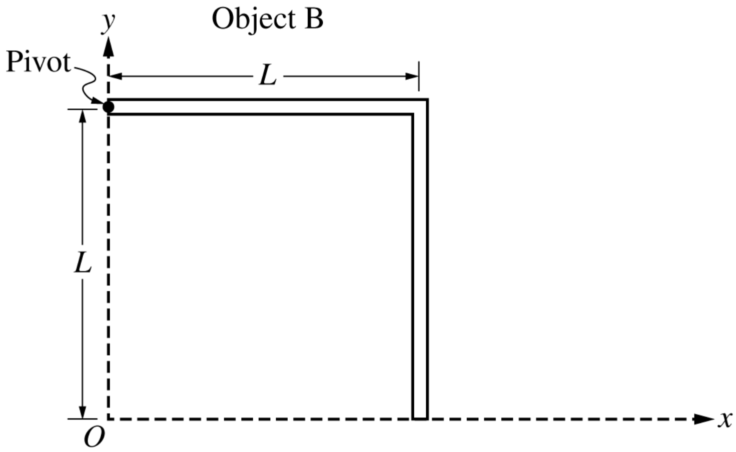


2. Object A is a long, thin, uniform rod of mass M and length $2L$ that is free to rotate about a pivot of negligible friction at its left end, as shown above.

(a) Using integral calculus, derive an expression to show that the rotational inertia I_A of object A about the pivot is given by $\frac{4}{3}ML^2$.



Object B of total mass M is formed by attaching two thin, uniform, identical rods of length L at a right angle to each other. Object B is held in place, as shown above. Express your answers in part (b) in terms of L .

(b) Determine the following for the given coordinate system shown in the figure.

i. The x-coordinate of the center of mass of object B

ii. The y-coordinate of the center of mass of object B

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Object B has a rotational inertia of I_B about its pivot.

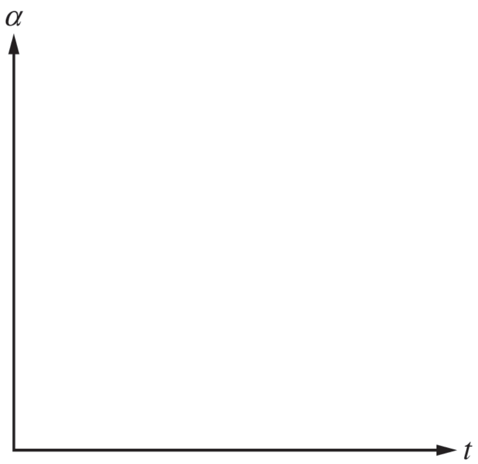
(c) Is the value of I_B greater than, less than, or equal to I_A ?

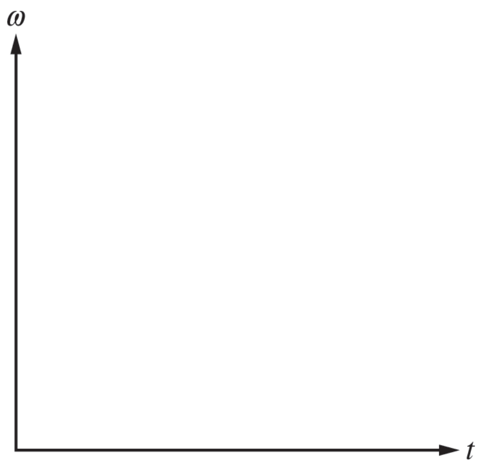
_____ Greater than _____ Less than _____ Equal to

Justify your answer.

Object B is released from rest and begins to rotate about its pivot.

(d) On the axes below, sketch graphs of the magnitude of the angular acceleration α and the angular speed ω of object B as functions of time t from the time it is released to the time its center of mass reaches its lowest point.





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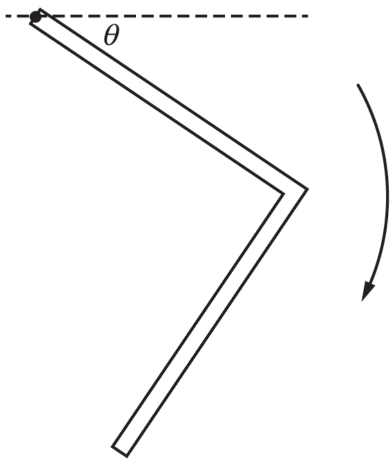
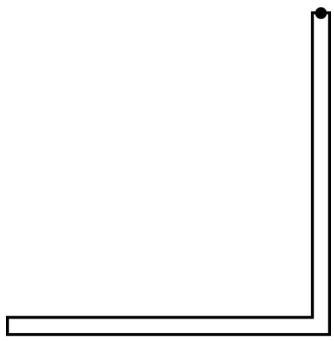


(e) While object B rotates from the horizontal position down through the angle θ shown above, is the magnitude of its angular acceleration increasing, decreasing, or not changing?

_____ Increasing _____ Decreasing _____ Not changing

Justify your answer.



Object B rotates through the position shown above.

(f) Derive an expression for the angular speed of object B when it is in the position shown above. Express your answer in terms of M, L, I_B, and physical constants, as appropriate.

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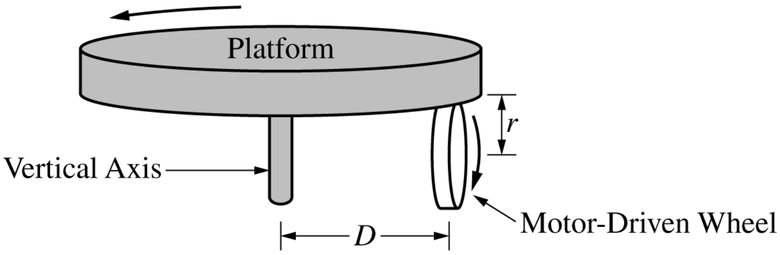
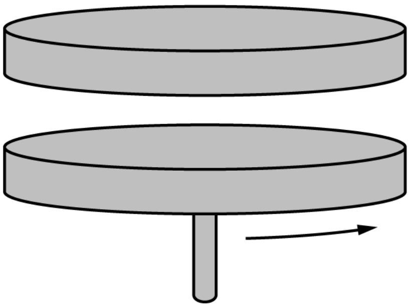


3. A horizontal circular platform with rotational inertia I_P rotates freely without friction on a vertical axis. A small motor-driven wheel that is used to rotate the platform is mounted under the platform and touches it. The wheel has radius r and touches the platform a distance D from the vertical axis of the platform, as shown above. The platform starts at rest, and the wheel exerts a constant horizontal force of magnitude F tangent to the wheel until the platform reaches an angular speed ω_P after time Δt. During time Δt, the wheel stays in contact with the platform without slipping.

   (a) Derive an expression for the angular speed ω_P of the platform. Express your answer in terms of I_P, r, D, F, Δt, and physical constants, as appropriate.

   (b) Determine an expression for the kinetic energy of the platform at the moment it reaches angular speed ω_P. Express your answer in terms of I_P, r, D, F, Δt, and physical constants, as appropriate.

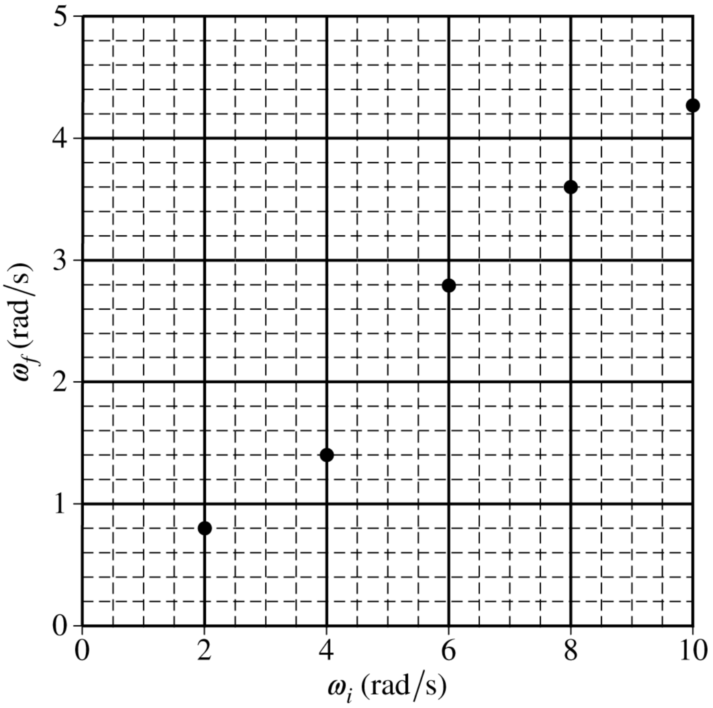
   (c) Derive an expression for the angular speed of the wheel ω_W when the platform has reached angular speed ω_P. Express your answer in terms of D, r, ω_P, and physical constants, as appropriate.



When the platform is spinning at angular speed ω_P, the motor-driven wheel is removed. A student holds a disk directly above and concentric with the platform, as shown above. The disk has the same rotational inertia I_P as the platform. The student releases the disk from rest, and the disk falls onto the platform. After a short time, the disk and platform are observed to be rotating together at angular speed ω_f.

   (d) Derive an expression for ω_f. Express your answer in terms of ω_P, I_P, and physical constants, as appropriate.

A student now uses the rotating platform $\left(I_P = 3.1\ \text{kg}\cdot\text{m}^2\right)$ to determine the rotational inertia I_U of an unknown object about a vertical axis that passes through the object's center of mass. The platform is rotating at an initial angular speed ω_i when the unknown object is dropped with its center of mass directly above the center of the platform. The platform and object are observed to be rotating together at angular speed ω_f. Trials are repeated for different values of ω_i. A graph of ω_f as a function of ω_i is shown on the axes below.



(e)

i. On the graph on the previous page, draw a best-fit line for the data.

ii. Using the straight line, calculate the rotational inertia of the unknown object I_U about a vertical axis passing through its center of mass.

(f) The kinetic energy of the spinning platform before the object is dropped on it is K_i. The total kinetic energy of the platform-object system when it reaches angular speed ω_f is K_f. Which of the following expressions is true?

____ $K_f < K_i$ ____ $K_f = K_i$ ____ $K_f > K_i$

Justify your answer.

(g) One of the students observes that the center of mass of the object is not actually aligned with the axis of the platform. Is the experimental value of I_U obtained in part (e) greater than, less than, or equal to the actual value of the rotational inertia of the unknown object about a vertical axis that passes through its center of mass?

____ Greater than ____ Less than ____ Equal to

Justify your answer.

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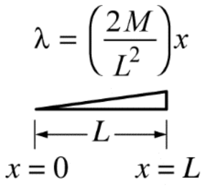


3. A triangular rod, shown above, has length L, mass M, and a nonuniform linear mass density given by the equation $\lambda = \dfrac{2M}{L^2}x$, where x is the distance from one end of the rod.

(a) Using integral calculus, show that the rotational inertia of the rod about its left end is $ML^2/2$.

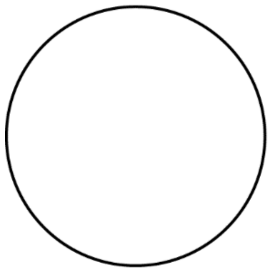


Figure 1

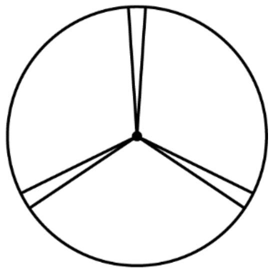


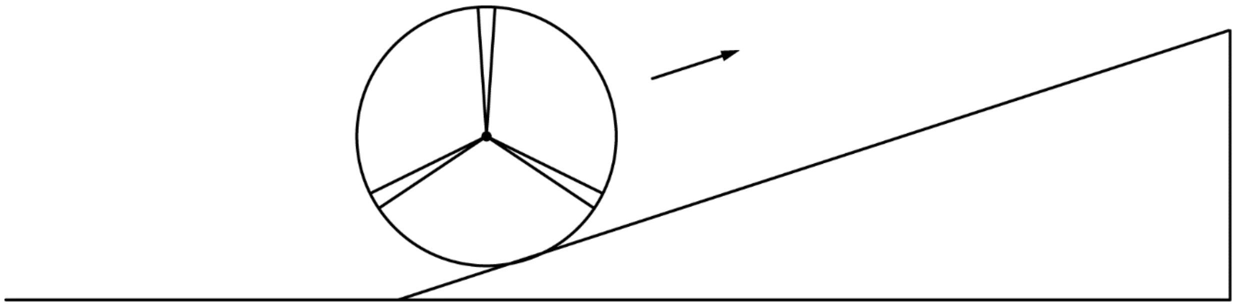
Figure 2

The thin hoop shown above in Figure 1 has a mass M, radius L, and a rotational inertia around its center of ML^2. Three rods identical to the rod from part (a) are now fastened to the thin hoop, as shown in Figure 2 above.

(b) Derive an expression for the rotational inertia I_{tot} of the hoop-rods system about the center of the hoop. Express your answer in terms of M, L, and physical constants, as appropriate.

The hoop-rods system is initially at rest and held in place but is free to rotate around its center. A constant force F is exerted tangent to the hoop for a time Δt.

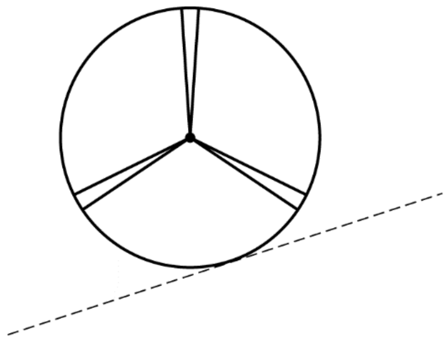
(c) Derive an expression for the final angular speed ω of the hoop-rods system. Express your answer in terms of M, L, F, Δt, and physical constants, as appropriate.



The hoop-rods system is rolling without slipping along a level horizontal surface with the angular speed ω found in part (c). At time $t = 0$, the system begins rolling without slipping up a ramp, as shown in the figure above.

(d)

i. On the figure of the hoop-rods system below, draw and label the forces (not components) that act on the system. Each force must be represented by a distinct arrow starting at, and pointing away from, the point at which the force is exerted on the system.



ii. Justify your choice for the direction of each of the forces drawn in part (d)i.

(e) Derive an expression for the change in height of the center of the hoop from the moment it reaches the bottom of the ramp until the moment it reaches its maximum height. Express your answer in terms of M, L, I_{tot} , ω, and physical constants, as appropriate.

STOP

END OF EXAM