

Torque and Rotational Dynamics

AP Physics C: Mechanics

Rotational Kinematics

A fan can keep turning while it slows down. Its velocity and acceleration then have opposite signs. Choose one fixed axis and a positive sense, such as counterclockwise. A **rigid body** holds its shape; its points share the same angular motion about that axis.

Angular displacement is $\Delta\theta = \theta - \theta_0$, measured in **radians**. One revolution is 2π radians. **Angular velocity** and **angular acceleration** are rates:

$$\omega = \frac{d\theta}{dt}, \quad \alpha = \frac{d\omega}{dt}.$$

The AP rotational-kinematics descriptions use clockwise/counterclockwise signs; three-dimensional directions of these kinematic vectors are outside this unit's assessment boundary. Torque directions are considered separately below.

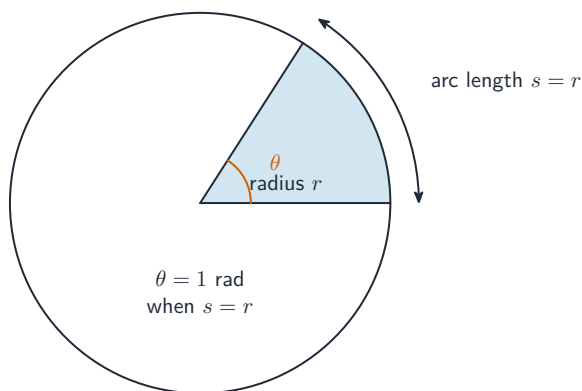
For constant α only,

$$\omega = \omega_0 + \alpha t, \quad \Delta\theta = \omega_0 t + \frac{1}{2}\alpha t^2, \quad \omega^2 = \omega_0^2 + 2\alpha\Delta\theta.$$

For varying acceleration, retain the initial conditions:

$$\omega(t) = \omega_0 + \int_0^t \alpha(t') dt', \quad \theta(t) = \theta_0 + \int_0^t \omega(t') dt'.$$

Slope of a θ - t graph gives ω ; slope of an ω - t graph gives α . Signed area under $\omega(t)$ gives angular displacement; it need not equal total angle travelled after reversal.



Use radians in the arc-length and linear-angular relations

Original constant-acceleration task. A fan slows from 30 rad/s to rest at constant $\alpha = -6.0 \text{ rad/s}^2$. From $\omega = \omega_0 + \alpha t$, $t = (0 - 30)/(-6.0) = 5.0 \text{ s}$. Then $\Delta\theta = \omega_0 t + \frac{1}{2}\alpha t^2 = 75 \text{ rad}$, or $75/(2\pi) \approx 12$ revolutions. Stop the model at rest unless the driving torque is specified to continue.

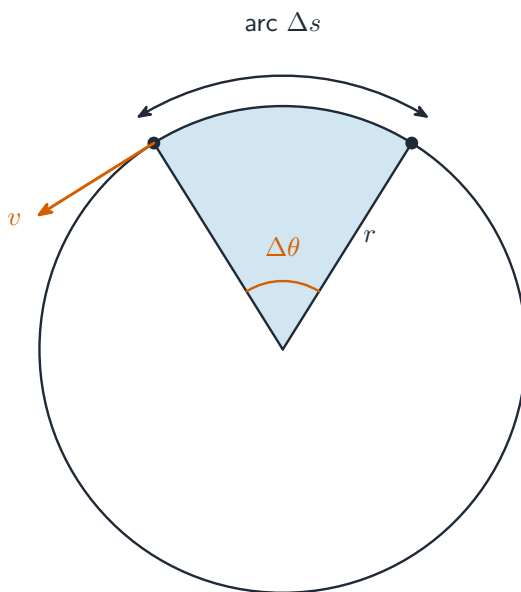
Original variable-acceleration task. Let $\alpha = ct$, with $c = 4.0 \text{ rad/s}^3$, $\omega_0 = 3.0 \text{ rad/s}$ and $\theta_0 = 1.0 \text{ rad}$. Integration gives $\omega = \omega_0 + \frac{1}{2}ct^2$ and $\theta = \theta_0 + \omega_0 t + ct^3/6$. At $t = 2.0 \text{ s}$, these are 11 rad/s and 12.3 rad. Omitting the integration constants changes the physical initial state.

Connecting Linear and Rotational Motion

For a point at perpendicular distance r from a fixed axis,

$$\Delta s = r\Delta\theta, \quad v = r|\omega|, \quad a_t = r\alpha, \quad a_c = r\omega^2.$$

Here v is speed; a_t is signed relative to the chosen tangential direction. The **tangential acceleration** changes speed and may oppose motion. The **centripetal acceleration** points toward the axis and changes direction. Their perpendicular magnitudes give $a = \sqrt{a_t^2 + a_c^2}$. Constant nonzero angular speed still gives centripetal acceleration.



Points at different radii share angular velocity but have different linear speeds

Original wheel task. At one instant $\omega = 4.0 \text{ rad/s}$ and $\alpha = -2.0 \text{ rad/s}^2$. At $r = 0.50 \text{ m}$, $v = r|\omega| = 2.0 \text{ m/s}$, $a_t = r\alpha = -1.0 \text{ m/s}^2$ and $a_c = r\omega^2 = 8.0 \text{ m/s}^2$. Thus $a = \sqrt{(-1.0)^2 + 8.0^2} = 8.1 \text{ m/s}^2$. At twice the radius all three component magnitudes double, although ω and α are unchanged.

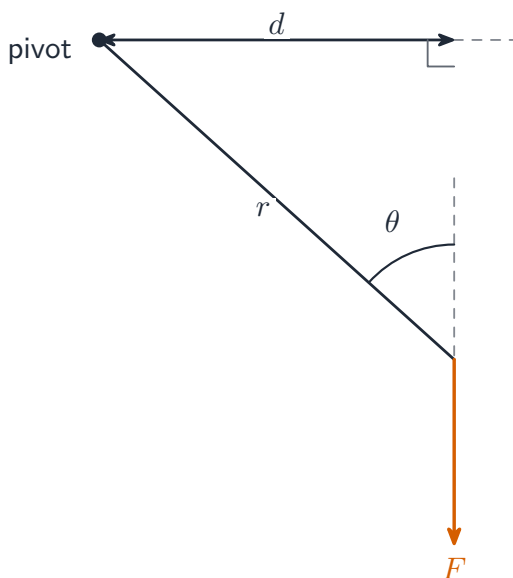
Torque

Torque product describes a force's turning effect about a chosen axis. It is a **vector** :

$$\vec{\tau} = \vec{r} \times \vec{F}, \quad |\tau| = rF \sin \phi = Fr_{\perp}.$$

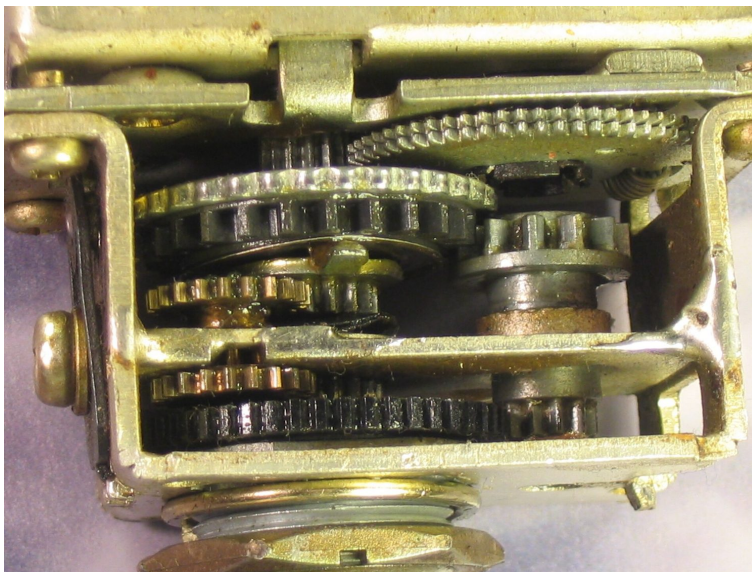
The **moment arm** $r_{\perp}$ is the perpendicular distance from the axis to the force's line of action, not necessarily the distance to its application point. A force whose line of action passes through the axis has zero torque about that axis; it can still exert a force and cause translation.

The **right-hand rule** gives the torque direction perpendicular to the plane of $\vec{r}$ and $\vec{F}$: curl from $\vec{r}$ toward $\vec{F}$. Counterclockwise torque is out of the page; clockwise is into it. For planar calculations choose signs consistently. Torque has unit N m, but it is not the scalar work measured in joules.



Measure the perpendicular distance to the force's line of action

Original lever task. A 20 N force acts at $r = 0.30$ m and angle 30° to the radius, tending counterclockwise. Its torque is $\tau = rF \sin \phi = (0.30)(20) \sin 30^\circ = +3.0$ N m. Equivalently, $r_{\perp} = r \sin \phi = 0.15$ m. A perpendicular 10 N force at the same radius gives the same torque. More force alone does not determine more torque.



Visible gear sizes suggest different angular speeds; the photo does not supply forces, inertias or losses

Rotational Inertia

Rotational inertia depends on the mass distribution relative to a specified axis:

$$I = \sum_i m_i r_i^2, \quad I = \int r^2 dm.$$

Use perpendicular distance from each mass element to that axis. The unit is kg m^2 . For a point-mass model, moving the same mass from r to $2r$ multiplies its contribution by four. Inertias add only when defined about the same axis.

Uniform rod about its centre. For mass M , length L and **linear density** $\lambda = M/L$, take $x = 0$ at the centre. Then $dm = (M/L)dx$ and

$$I_{\text{cm}} = \int_{-L/2}^{L/2} x^2 \frac{M}{L} dx = \frac{ML^2}{12}.$$

Nonuniform rod about its light end. If $\lambda(x) = cx$ on $0 \leq x \leq L$, c has unit kg/m^2 . First normalise the density:

$$M = \int_0^L cx dx = \frac{1}{2}cL^2, \quad c = \frac{2M}{L^2}.$$

Then $I = \int_0^L x^2(cx)dx = cL^4/4 = ML^2/2$. This is not the uniform-rod result; the mass is concentrated farther from the axis.

Disk from coaxial rings. For a uniform thin disk, surface density is $\sigma = M/(\pi R^2)$. A thin ring has $dm = \sigma 2\pi r dr$, so

$$I = \int_0^R r^2 \sigma 2\pi r dr = \frac{1}{2}MR^2.$$

A thin hoop of the same mass and radius has $I = MR^2$. For a uniform annular disk with inner radius a and outer radius b , use $\sigma = M/[\pi(b^2 - a^2)]$ and limits a to b : $I = M(a^2 + b^2)/2$. AP's calculus scope includes rods and bodies built from coaxial rings/shells; arbitrary three-dimensional shapes are not required here.

The **parallel-axis theorem** relates an axis through the centre of mass to a parallel axis a distance d away:

$$I' = I_{\text{cm}} + Md^2.$$

For the uniform rod about one end, $I = ML^2/12 + M(L/2)^2 = ML^2/3$. The starting axis must pass through the centre of mass. For $M = 3.0$ kg and $L = 2.0$ m, $I_{\text{cm}} = 1.0$ kg m² and $I_{\text{end}} = 4.0$ kg m².

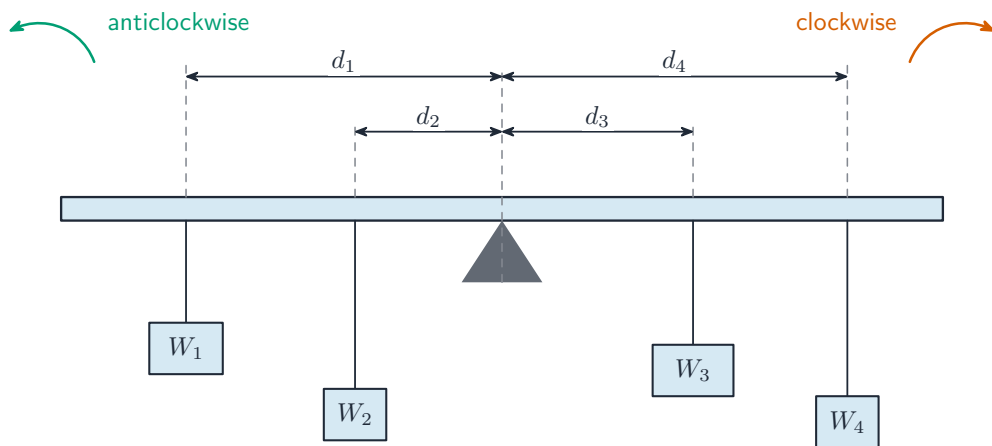
Verified exam connection – 2023 Set 1 Q2(a). The question gives $T = 2\pi\sqrt{I/\kappa}$ for a torsional pendulum and $I_{\text{disk}} = MR^2/2$. For N identical coaxial disks moving together, $I_{\text{total}} = NMR^2/2$, so $T = 2\pi\sqrt{NMR^2/(2\kappa)}$. Four times as many disks doubles the period, not quadruples it. The official scheme checks the summed inertia and consistent substitution. The period relation is supplied here; its oscillation derivation belongs to the oscillations unit.

Rotational Equilibrium

For a rigid system with constant inertia about a fixed axis, zero **net torque** gives $\alpha = 0$, so angular velocity is constant. This is **rotational equilibrium**, which can include steady rotation. It does not by itself establish zero net force or rest. Full **static equilibrium** also requires rest and

$$\sum \vec{F} = 0, \quad \sum \tau = 0.$$

Draw forces at their actual application points. In a static beam problem, choosing the axis through an unknown support removes that force's torque, but its force remains in the force-balance equation. Equilibrium torque balance can be taken about any axis.

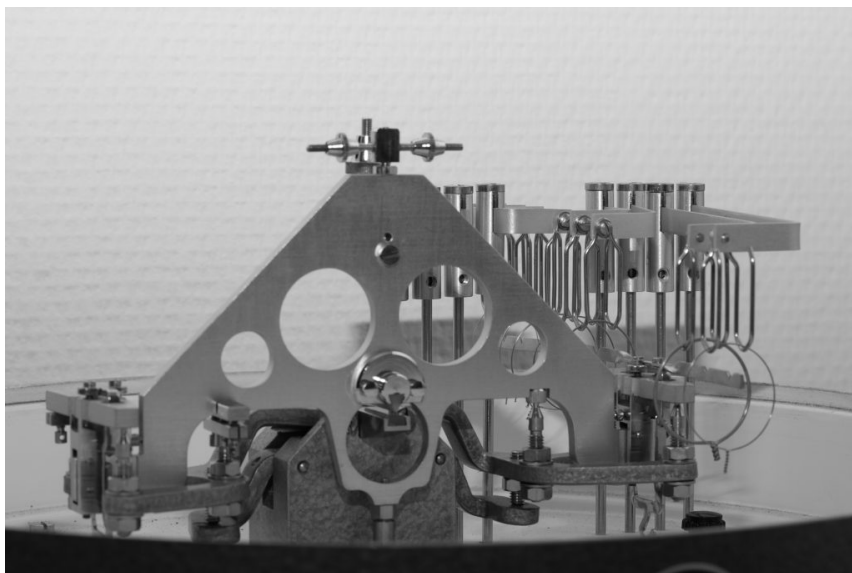


Torque balance is one condition; force balance is also needed for static equilibrium

Original beam task. A horizontal uniform 20 kg beam of length 4.0 m is pivoted at its left end and held by a vertical rope at its right end. A 40 kg child sits 1.0 m from the pivot. With $g = 9.8 \text{ m/s}^2$, counterclockwise positive, torque balance gives

$$0 = TL - Mg(L/2) - mgx, \quad T = \frac{Mg(L/2) + mgx}{L}.$$

Thus $T = [(20)(9.8)(2.0) + (40)(9.8)(1.0)]/4.0 = 196 \text{ N}$. Vertical force balance gives $F_p + T - (M + m)g = 0$, so $F_p = (M + m)g - T = 392 \text{ N}$ upward. Without the child, the tension would be 98 N; these are different physical problems.



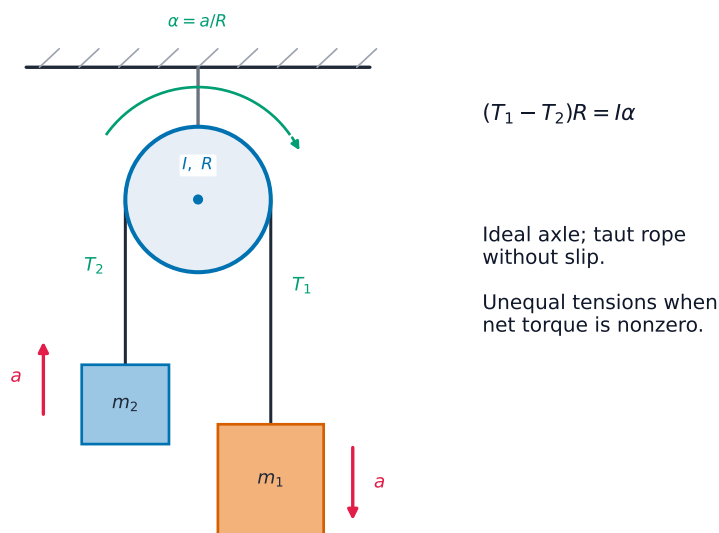
A mechanical balance uses torque balance; the photograph alone gives no numerical load

Newton's Second Law in Rotational Form

For a rigid body with constant I about a fixed axis, the component along that axis obeys

$$\sum \tau = I\alpha.$$

The torque changes angular velocity, not necessarily in the direction of existing rotation. At fixed net torque, doubling I halves α . For $\sum \tau = 6.0 \text{ N m}$ and $I = 2.0 \text{ kg m}^2$, $\alpha = 3.0 \text{ rad/s}^2$. If I changes, the simple constant- I model is insufficient; angular-momentum accounting is developed in Unit 6.



For the accelerating ideal-axle pulley shown, the tension difference provides its net torque

Original massive-pulley task. Masses $m_1 = 4.0 \text{ kg}$ and $m_2 = 2.0 \text{ kg}$ hang from a light inextensible rope over a **pulley** with $I = 0.50 \text{ kg m}^2$ and $R = 0.20 \text{ m}$. The axle has negligible friction, the rope remains taut and does not slip, and m_1 accelerates downward. Use down positive for m_1 , up positive for m_2 and clockwise positive for the pulley. Draw each body's forces separately. The pulley has downward rope forces at opposite rims, its weight at the centre and an axle reaction; the last two give zero torque about its centre.

$$m_1g - T_1 = m_1a, \quad T_2 - m_2g = m_2a.$$

$$\sum \tau = (T_1 - T_2)R = I\alpha, \quad a = R\alpha.$$

$$a = \frac{(m_1 - m_2)g}{m_1 + m_2 + I/R^2}.$$

Here $I/R^2 = 12.5 \text{ kg}$, so $a = 19.6/18.5 = 1.059 \dots \text{ m/s}^2$, $T_1 = m_1(g - a) = 34.96 \dots \text{ N}$ and $T_2 = m_2(g + a) = 21.72 \dots \text{ N}$. Keep unrounded a until the tensions are found. The acceleration is less than the ideal massless-pulley value 3.27 m/s^2 . For this model, $T_1 - T_2 = Ia/R^2 > 0$. A massive pulley at zero angular acceleration can have equal tensions; finite inertia alone does not prove unequal tension. Axle friction would add a torque term.

Exam tips

Choose the axis, object/system and signs before writing equations. Use initial conditions with integrals, and constant-acceleration formulas only when justified. For inertia, state dm , the perpendicular distance and integration limits; use the centre-of-mass axis correctly in the parallel-axis theorem. For combined translation and rotation, write separate laws and the no-slip constraint before eliminating unknowns. Read the actual exam demand; a universal free-body-diagram or full-mark checklist is not a substitute for its prompt.