

Part A (BC): Graphing calculator required**Question 2****9 points****General Scoring Notes**

The model solution is presented using standard mathematical notation.

Answers (numeric or algebraic) need not be simplified. Answers given as a decimal approximation should be correct to three places after the decimal point. Within each individual free-response question, at most one point is not earned for inappropriate rounding.

A particle moving along a curve in the xy -plane has position $(x(t), y(t))$ at time t seconds, where $x(t)$ and $y(t)$ are measured in centimeters. It is known that $x'(t) = 8t - t^2$ and $y'(t) = -t + \sqrt{t^{1.2} + 20}$. At time $t = 2$ seconds, the particle is at the point $(3, 6)$.

Model Solution	Scoring
(a) Find the speed of the particle at time $t = 2$ seconds. Show the setup for your calculations.	
$\sqrt{(x'(2))^2 + (y'(2))^2}$	Setup for speed 1 point
$= 12.3048506$	Answer 1 point
The speed of the particle at time $t = 2$ seconds is 12.305 (or 12.304) centimeters per second.	

Scoring notes:

- The first point is earned for the expression $\sqrt{(x'(2))^2 + (y'(2))^2}$, $\sqrt{(x'(t))^2 + (y'(t))^2}$, or equivalent.
- A response that presents just the exact answer, $\sqrt{144 + (-2 + \sqrt{2^{1.2} + 20})^2}$, earns both points.
- The second point is earned only for the answer 12.305 (or 12.304) regardless of whether the first point is earned.
- A response that includes a linkage error, such as $\sqrt{(x'(t))^2 + (y'(t))^2} = \sqrt{144 + (-2 + \sqrt{2^{1.2} + 20})^2}$ or $\sqrt{(x'(t))^2 + (y'(t))^2} = 12.305$ (or 12.304), earns at most 1 of the 2 points.
- Missing or incorrect units will not affect scoring in this part.

Total for part (a) 2 points

- (b) Find the total distance traveled by the particle over the time interval $0 \leq t \leq 2$. Show the setup for your calculations.

$\int_0^2 \sqrt{(x'(t))^2 + (y'(t))^2} dt$	Integral 1 point
$= 15.901715$	Answer 1 point
The total distance traveled by the particle over the time interval $0 \leq t \leq 2$ is 15.902 (or 15.901) centimeters.	

Scoring notes:

- The first point is earned only for an integral of $\int_0^2 \sqrt{(x'(t))^2 + (y'(t))^2} dt$ (or the mathematical equivalent), with or without the differential.
 - Note: $\int_0^2 \sqrt{x'(t)^2 + y'(t)^2} dt$ is not read as a parenthesis error.
- The second point is earned only for an answer of 15.902 (or 15.901), regardless of whether the first point is earned.
- Missing or incorrect units will not affect scoring in this part.

Total for part (b) 2 points

- (c) Find the y -coordinate of the position of the particle at the time $t = 0$. Show the setup for your calculations.

$y(0) = 6 + \int_2^0 y'(t) dt = 6 - 7.173613 = -1.173613$	Definite integral 1 point
	Uses initial condition 1 point
The y -coordinate of the position of the particle at time $t = 0$ is -1.174 (or -1.173).	Answer 1 point

Scoring notes:

- An answer of -1.174 (or -1.173) with no supporting work does not earn any points.
- The first point is earned for either of the definite integrals $\int_2^0 y'(t) dt$ or $\int_0^2 y'(t) dt$.
- The second point is earned for any of:
 - $y(2) \pm \int_2^0 y'(t) dt$, $6 \pm \int_2^0 y'(t) dt$,
 - $y(2) \pm \int_0^2 y'(t) dt$, $6 \pm \int_0^2 y'(t) dt$,
 - $y(2) \pm 7.173613$, or 6 ± 7.173613 .
- A response that attempts to evaluate $\int y'(t) dt$ does not earn the first or the third point.
 - Such a response can earn the second point by attempting to solve for the constant of integration by presenting an expression as an antiderivative for $y'(t)$, evaluating this expression at $t = 2$, and setting this expression equal to 6.

- A response that reverses the limits of integration, e.g., $y(0) = 6 + \int_0^2 y'(t) dt$ or $6 + 7.173613$, earns the second point but does not earn the third point.
- In order to earn the third point, a response must have earned at least 1 of the first 2 points.
- A response containing any linkage error can earn at most 2 of the 3 points. For example:
 - Equating two unequal quantities: $\int_2^0 y'(t) dt = -1.174$, $\int_2^0 y'(t) dt = 6 - 7.173613$,
 $6 + \int_0^2 y'(t) dt = 6 - 7.173613$, or $6 + \int_0^2 y'(t) dt = 6 + 7.173613 = -1.173613$
 - Equating an expression to a numerical value: $y(t) = 6 + \int_2^0 y'(t) dt = -1.174$
- Missing differentials (dt):
 - Unambiguous responses of $y(2) + \int_2^0 y'(t)$, $y(2) - \int_0^2 y'(t)$, $6 + \int_2^0 y'(t)$, or $6 - \int_0^2 y'(t)$ earn the first 2 points and would earn the third point for the correct numerical answer.
 - Unambiguous responses of $y(2) + \int_0^2 y'(t)$ or $6 + \int_0^2 y'(t)$ with reversed limits of integration and missing differential earn the first 2 points but cannot earn the third point.
 - Ambiguous responses of $\int_2^0 y'(t) + y(2)$, $-\int_0^2 y'(t) + y(2)$, $\int_2^0 y'(t) + 6$, or $-\int_0^2 y'(t) + 6$ earn the first point, do not earn the second point, but do earn the third point if a correct numeric answer is provided. If no numeric answer is given, none of these responses earn the third point.
 - Ambiguous responses of $\int_0^2 y'(t) + y(2)$ or $\int_0^2 y'(t) + 6$ with reversed limits of integration and no differential earn 1 out of 3 points.
- If a response provides work for both the x - and y -coordinates, the work for the x -coordinate will not affect scoring.
- However, a response that reports only a completely correct x -coordinate of the particle's position at time $t = 0$ with all supporting work, e.g., $x(0) = 3 + \int_2^0 x'(t) dt = -\frac{31}{3} = -10.333$, earns 2 out of 3 points.

Total for part (c) 3 points

- (d) For $2 \leq t \leq 8$, the particle remains in the first quadrant. Find all times t in the interval $2 \leq t \leq 8$ when the particle is moving toward the x -axis. Give a reason for your answer.

Because $y(t) > 0$ when $2 \leq t \leq 8$, the particle will be moving toward the x -axis when $y'(t) < 0$. This occurs when 5.222 (or 5.221) $< t < 8$.	Considers sign of $y'(t)$	1 point
	Answer with reason	1 point

Scoring notes:

- The first point can be earned by stating $y'(t) = 0$, $y'(t) < 0$, $y'(t) > 0$, or $t = 5.222$ (or 5.221).
 Note: $y'(t)$ may be written as $\frac{dy}{dt}$.
- The second point cannot be earned without the first.
- To earn the second point, a response must identify the correct interval (and no additional intervals in $[2, 8]$) and explicitly state the need for $y'(t) < 0$. The interval can be open, closed, or half-open.

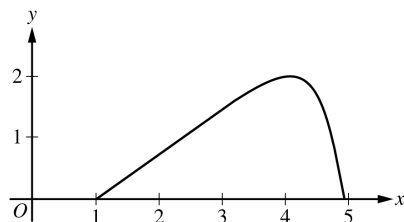
Total for part (d) 2 points

Total for question 2 9 points

Part A (BC): Graphing calculator required**Question 2****9 points****General Scoring Notes**

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For $0 \leq t \leq \pi$, a particle is moving along the curve shown so that its position at time t is $(x(t), y(t))$, where $x(t)$ is not explicitly given and $y(t) = 2\sin t$. It is known that $\frac{dx}{dt} = e^{\cos t}$. At time $t = 0$, the particle is at position $(1, 0)$.

Model Solution**Scoring**

- (a) Find the acceleration vector of the particle at time $t = 1$. Show the setup for your calculations.

$$x''(1) = \left. \frac{d}{dt}(e^{\cos t}) \right|_{t=1} = -1.444407$$

$x''(1)$ with setup **1 point**

$$y(t) = 2\sin t \Rightarrow y'(t) = 2\cos t$$

$$y''(1) = \left. \frac{d}{dt}(2\cos t) \right|_{t=1} = -1.682942$$

$y''(1)$ with setup **1 point**

The acceleration vector at time $t = 1$ is

$$a(1) = \langle -1.444, -1.683 \text{ (or } -1.682) \rangle.$$

Scoring notes:

- The exact answer is $\langle x''(1), y''(1) \rangle = \langle -e^{\cos 1} \sin 1, -2\sin 1 \rangle$.
- $\langle -e^{\cos t} \sin t, -2\sin t \rangle$ together with an incorrect or missing evaluation at $t = 1$ earns 1 of the 2 points.

- A response of $\langle -e^{\cos t} \sin t, -2\sin t \rangle = \langle -e^{\cos 1} \sin 1, -2\sin 1 \rangle$ or equivalent earns only 1 of the 2 points because it equates an expression to a numerical value.
- An unsupported correct acceleration vector earns 1 of the 2 points.
- The acceleration vector may be presented with other symbols, for example $(\ , \)$ or $[\ , \]$.
- The components may be listed separately, as long as they are labeled.
- Degree mode: A response that presents answers obtained by using a calculator in degree mode does not earn the first point it would have otherwise earned. The response is generally eligible for all subsequent points (unless no answer is possible in degree mode or the question is made simpler by using degree mode). In degree mode, $x''(1) = -0.000828$ or -0.047433 and $y''(1) = -0.000609$ or -0.034905 . A response that presents one of these values with correct setups earns 1 of the 2 points.

Total for part (a) 2 points

- (b) For $0 \leq t \leq \pi$, find the first time t at which the speed of the particle is 1.5. Show the work that leads to your answer.

$$\text{Speed} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{(e^{\cos t})^2 + (2\cos t)^2}$$

$$\sqrt{(e^{\cos t})^2 + (2\cos t)^2} = 1.5 \quad \text{1 point}$$

$$0 \leq t \leq \pi \text{ and } \sqrt{(e^{\cos t})^2 + (2\cos t)^2} = 1.5$$

$$\Rightarrow t = 1.254472, t = 2.358077$$

Answer **1 point**

The first time at which the speed of the particle is 1.5 is $t = 1.254$.

Scoring notes:

- A response with an implied equation is eligible for both points. For example, a response of “Speed = $\sqrt{(e^{\cos t})^2 + (2\cos t)^2}$ and is first equal to 1.5 at $t = 1.254$ ” earns both points.
- $\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = 1.5$ earns the first point. Speed = 1.5 by itself does not earn the first point. Both of these responses are eligible to earn the second point.
- A response need not consider the value $t = 2.358077$.
- A response of $t = 1.254$ alone does not earn either point.
- A response with a parenthesis error(s) in either $(e^{\cos t})^2$ or $(2\cos t)^2$ does not earn the first point but does earn the second point for the correct answer. Note: $\sqrt{\frac{dx^2}{dt} + \frac{dy^2}{dt}}$ is not considered a parenthesis error.

- Degree mode: In degree mode, $\sqrt{(e^{\cos t})^2 + (2 \cos t)^2} = 1.5$ has no solution for $0 \leq t \leq \pi$.

A response that finds no time t at which the speed of the particle is 1.5 cannot be assumed to be working in degree mode.

Total for part (b) 2 points

- (c) Find the slope of the line tangent to the path of the particle at time $t = 1$. Find the x -coordinate of the position of the particle at time $t = 1$. Show the work that leads to your answers.

$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2 \cos t}{e^{\cos t}}$ $\left. \frac{dy}{dx} \right _{t=1} = \frac{2 \cos 1}{e^{\cos 1}} = 0.629530$	Slope with supporting work	1 point
The slope of the line tangent to the curve at $t = 1$ is 0.630 (or 0.629).		
$x(1) = x(0) + \int_0^1 \frac{dx}{dt} dt = 1 + \int_0^1 e^{\cos t} dt = 3.341575$	$\int_0^1 e^{\cos t} dt$	1 point
The x -coordinate of the position at $t = 1$ is 3.342 (or 3.341).	$x(1)$	1 point

Scoring notes:

- To earn the first point, the response must communicate $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$; for example:
 - $\frac{dy}{dx} = \frac{2 \cos 1}{e^{\cos 1}}$
 - $\frac{dy/dt}{dx/dt} = 0.63$
 - $x'(1) = 1.716526$, $y'(1) = 1.080605$, slope = 0.63
 - $\frac{dy}{dt} = 2 \cos t$, slope = 0.63
- A response may import an incorrect expression for $y'(t)$ or value of $y'(1)$ from part (a), provided it was declared in part (a).
- The second point is earned for a response that presents the definite integral $\int_0^1 e^{\cos t} dt$ or $\int_0^1 \frac{dx}{dt} dt$ with or without the initial condition.

- For the second point, if the differential is missing:

- $\int_0^1 e^{\cos t} dt$ earns the second point and is eligible for the third point.
- $x(1) = \int_0^1 e^{\cos t} dt$ earns the second point but is not eligible for the third point.
- $x(1) = 1 + \int_0^1 e^{\cos t} dt$ earns the second point and is eligible for the third point.
- $x(1) = \int_0^1 e^{\cos t} + 1$ does not earn the second point but earns the third point for the correct answer.
- The third point is not earned for a response that presents an incorrect statement, such as $x(1) = \int_0^1 e^{\cos t} dt = 1 + 2.342$.

- Degree mode: In degree mode, $\frac{dy}{dx} = 0.735759$ or 0.012841 and $1 + \int_0^1 e^{\cos t} dt = 3.718144$.

Total for part (c) 3 points

- (d) Find the total distance traveled by the particle over the time interval $0 \leq t \leq \pi$. Show the setup for your calculations.

$\int_0^\pi \sqrt{(e^{\cos t})^2 + (2 \cos t)^2} dt$	Integral	1 point
$= 6.034611$	Answer	1 point
The total distance traveled by the particle over $0 \leq t \leq \pi$ is 6.035 (or 6.034).		

Scoring notes:

- The first point is earned for presenting the correct integrand in a definite integral.
- Parentheses errors were assessed in part (b) and, therefore, will not affect the scoring in part (d).
- If the integrand is an incorrect speed function imported from part (b), the response earns the first point and does not earn the second point.
- An unsupported answer of 6.035 (or 6.034) does not earn either point.
- Degree mode: In degree mode, the total distance is 10.596835 or 8.536161.

Total for part (d) 2 points

Total for question 2 9 points

AP[®] Calculus BC

2015 Scoring Guidelines

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2015 SCORING GUIDELINES

Question 1

The rate at which rainwater flows into a drainpipe is modeled by the function R , where $R(t) = 20\sin\left(\frac{t^2}{35}\right)$ cubic feet per hour, t is measured in hours, and $0 \leq t \leq 8$. The pipe is partially blocked, allowing water to drain out the other end of the pipe at a rate modeled by $D(t) = -0.04t^3 + 0.4t^2 + 0.96t$ cubic feet per hour, for $0 \leq t \leq 8$. There are 30 cubic feet of water in the pipe at time $t = 0$.

- How many cubic feet of rainwater flow into the pipe during the 8-hour time interval $0 \leq t \leq 8$?
- Is the amount of water in the pipe increasing or decreasing at time $t = 3$ hours? Give a reason for your answer.
- At what time t , $0 \leq t \leq 8$, is the amount of water in the pipe at a minimum? Justify your answer.
- The pipe can hold 50 cubic feet of water before overflowing. For $t > 8$, water continues to flow into and out of the pipe at the given rates until the pipe begins to overflow. Write, but do not solve, an equation involving one or more integrals that gives the time w when the pipe will begin to overflow.

(a) $\int_0^8 R(t) dt = 76.570$

2 : $\begin{cases} 1 : \text{integrand} \\ 1 : \text{answer} \end{cases}$

(b) $R(3) - D(3) = -0.313632 < 0$

Since $R(3) < D(3)$, the amount of water in the pipe is decreasing at time $t = 3$ hours.

2 : $\begin{cases} 1 : \text{considers } R(3) \text{ and } D(3) \\ 1 : \text{answer and reason} \end{cases}$

(c) The amount of water in the pipe at time t , $0 \leq t \leq 8$, is $30 + \int_0^t [R(x) - D(x)] dx$.

$$R(t) - D(t) = 0 \Rightarrow t = 0, 3.271658$$

t	Amount of water in the pipe
0	30
3.271658	27.964561
8	48.543686

The amount of water in the pipe is a minimum at time $t = 3.272$ (or 3.271) hours.

3 : $\begin{cases} 1 : \text{considers } R(t) - D(t) = 0 \\ 1 : \text{answer} \\ 1 : \text{justification} \end{cases}$

(d) $30 + \int_0^w [R(t) - D(t)] dt = 50$

2 : $\begin{cases} 1 : \text{integral} \\ 1 : \text{equation} \end{cases}$

2015 SCORING GUIDELINES

Question 2

At time $t \geq 0$, a particle moving along a curve in the xy -plane has position $(x(t), y(t))$ with velocity vector $v(t) = (\cos(t^2), e^{0.5t})$. At $t = 1$, the particle is at the point $(3, 5)$.

- (a) Find the x -coordinate of the position of the particle at time $t = 2$.
- (b) For $0 < t < 1$, there is a point on the curve at which the line tangent to the curve has a slope of 2. At what time is the object at that point?
- (c) Find the time at which the speed of the particle is 3.
- (d) Find the total distance traveled by the particle from time $t = 0$ to time $t = 1$.

(a) $x(2) = 3 + \int_1^2 \cos(t^2) dt = 2.557$ (or 2.556)

3 : $\begin{cases} 1 : \text{integral} \\ 1 : \text{uses initial condition} \\ 1 : \text{answer} \end{cases}$

(b) $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{e^{0.5t}}{\cos(t^2)}$

$$\frac{e^{0.5t}}{\cos(t^2)} = 2$$

$$t = 0.840$$

2 : $\begin{cases} 1 : \text{slope in terms of } t \\ 1 : \text{answer} \end{cases}$

(c) Speed = $\sqrt{\cos^2(t^2) + e^t}$

$$\sqrt{\cos^2(t^2) + e^t} = 3$$

$$t = 2.196$$
 (or 2.195)

2 : $\begin{cases} 1 : \text{speed in terms of } t \\ 1 : \text{answer} \end{cases}$

(d) Distance = $\int_0^1 \sqrt{\cos^2(t^2) + e^t} dt = 1.595$ (or 1.594)

2 : $\begin{cases} 1 : \text{integral} \\ 1 : \text{answer} \end{cases}$

2015 SCORING GUIDELINES

Question 3

t (minutes)	0	12	20	24	40
$v(t)$ (meters per minute)	0	200	240	-220	150

Johanna jogs along a straight path. For $0 \leq t \leq 40$, Johanna's velocity is given by a differentiable function v . Selected values of $v(t)$, where t is measured in minutes and $v(t)$ is measured in meters per minute, are given in the table above.

- (a) Use the data in the table to estimate the value of $v'(16)$.

- (b) Using correct units, explain the meaning of the definite integral $\int_0^{40} |v(t)| dt$ in the context of the problem.

Approximate the value of $\int_0^{40} |v(t)| dt$ using a right Riemann sum with the four subintervals indicated in the table.

- (c) Bob is riding his bicycle along the same path. For $0 \leq t \leq 10$, Bob's velocity is modeled by $B(t) = t^3 - 6t^2 + 300$, where t is measured in minutes and $B(t)$ is measured in meters per minute.

Find Bob's acceleration at time $t = 5$.

- (d) Based on the model B from part (c), find Bob's average velocity during the interval $0 \leq t \leq 10$.

(a) $v'(16) \approx \frac{240 - 200}{20 - 12} = 5$ meters/min²

1 : approximation

- (b) $\int_0^{40} |v(t)| dt$ is the total distance Johanna jogs, in meters, over the time interval $0 \leq t \leq 40$ minutes.

3 : $\begin{cases} 1 : \text{explanation} \\ 1 : \text{right Riemann sum} \\ 1 : \text{approximation} \end{cases}$

$$\begin{aligned} \int_0^{40} |v(t)| dt &\approx 12 \cdot |v(12)| + 8 \cdot |v(20)| + 4 \cdot |v(24)| + 16 \cdot |v(40)| \\ &= 12 \cdot 200 + 8 \cdot 240 + 4 \cdot 220 + 16 \cdot 150 \\ &= 2400 + 1920 + 880 + 2400 \\ &= 7600 \text{ meters} \end{aligned}$$

- (c) Bob's acceleration is $B'(t) = 3t^2 - 12t$.

$$B'(5) = 3(25) - 12(5) = 15 \text{ meters/min}^2$$

2 : $\begin{cases} 1 : \text{uses } B'(t) \\ 1 : \text{answer} \end{cases}$

(d) Avg vel = $\frac{1}{10} \int_0^{10} (t^3 - 6t^2 + 300) dt$

$$= \frac{1}{10} \left[\frac{t^4}{4} - 2t^3 + 300t \right]_0^{10}$$

$$= \frac{1}{10} \left[\frac{10000}{4} - 2000 + 3000 \right] = 350 \text{ meters/min}$$

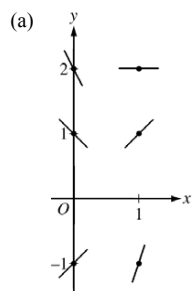
3 : $\begin{cases} 1 : \text{integral} \\ 1 : \text{antiderivative} \\ 1 : \text{answer} \end{cases}$

2015 SCORING GUIDELINES

Question 4

Consider the differential equation $\frac{dy}{dx} = 2x - y$.

- (a) On the axes provided, sketch a slope field for the given differential equation at the six points indicated.
- (b) Find $\frac{d^2y}{dx^2}$ in terms of x and y . Determine the concavity of all solution curves for the given differential equation in Quadrant II. Give a reason for your answer.
- (c) Let $y = f(x)$ be the particular solution to the differential equation with the initial condition $f(2) = 3$. Does f have a relative minimum, a relative maximum, or neither at $x = 2$? Justify your answer.
- (d) Find the values of the constants m and b for which $y = mx + b$ is a solution to the differential equation.



- 2 : $\begin{cases} 1 : \text{slopes where } x = 0 \\ 1 : \text{slopes where } x = 1 \end{cases}$

(b) $\frac{d^2y}{dx^2} = 2 - \frac{dy}{dx} = 2 - (2x - y) = 2 - 2x + y$

In Quadrant II, $x < 0$ and $y > 0$, so $2 - 2x + y > 0$.
Therefore, all solution curves are concave up in Quadrant II.

- 2 : $\begin{cases} 1 : \frac{d^2y}{dx^2} \\ 1 : \text{concave up with reason} \end{cases}$

(c) $\left. \frac{dy}{dx} \right|_{(x,y)=(2,3)} = 2(2) - 3 = 1 \neq 0$

Therefore, f has neither a relative minimum nor a relative maximum at $x = 2$.

- 2 : $\begin{cases} 1 : \text{considers } \left. \frac{dy}{dx} \right|_{(x,y)=(2,3)} \\ 1 : \text{conclusion with justification} \end{cases}$

(d) $y = mx + b \Rightarrow \frac{dy}{dx} = \frac{d}{dx}(mx + b) = m$

$$2x - y = m$$

$$2x - (mx + b) = m$$

$$(2 - m)x - (m + b) = 0$$

$$2 - m = 0 \Rightarrow m = 2$$

$$b = -m \Rightarrow b = -2$$

- 3 : $\begin{cases} 1 : \frac{d}{dx}(mx + b) = m \\ 1 : 2x - y = m \\ 1 : \text{answer} \end{cases}$

2015 SCORING GUIDELINES

Question 5

Consider the function $f(x) = \frac{1}{x^2 - kx}$, where k is a nonzero constant. The derivative of f is given by

$$f'(x) = \frac{k - 2x}{(x^2 - kx)^2}.$$

- (a) Let $k = 3$, so that $f(x) = \frac{1}{x^2 - 3x}$. Write an equation for the line tangent to the graph of f at the point whose x -coordinate is 4.
- (b) Let $k = 4$, so that $f(x) = \frac{1}{x^2 - 4x}$. Determine whether f has a relative minimum, a relative maximum, or neither at $x = 2$. Justify your answer.
- (c) Find the value of k for which f has a critical point at $x = -5$.
- (d) Let $k = 6$, so that $f(x) = \frac{1}{x^2 - 6x}$. Find the partial fraction decomposition for the function f .

Find $\int f(x) dx$.

(a) $f(4) = \frac{1}{4^2 - 3 \cdot 4} = \frac{1}{4}$ $f'(4) = \frac{3 - 2 \cdot 4}{(4^2 - 3 \cdot 4)^2} = -\frac{5}{16}$

An equation for the line tangent to the graph of f at the point whose x -coordinate is 4 is $y = -\frac{5}{16}(x - 4) + \frac{1}{4}$.

- 2 : $\begin{cases} 1 : \text{slope} \\ 1 : \text{tangent line equation} \end{cases}$

(b) $f'(x) = \frac{4 - 2x}{(x^2 - 4x)^2}$ $f'(2) = \frac{4 - 2 \cdot 2}{(2^2 - 4 \cdot 2)^2} = 0$

$f'(x)$ changes sign from positive to negative at $x = 2$.
Therefore, f has a relative maximum at $x = 2$.

- 2 : $\begin{cases} 1 : \text{considers } f'(2) \\ 1 : \text{answer with justification} \end{cases}$

(c) $f'(-5) = \frac{k - 2 \cdot (-5)}{((-5)^2 - k \cdot (-5))^2} = 0 \Rightarrow k = -10$

- 1 : answer

(d) $\frac{1}{x^2 - 6x} = \frac{1}{x(x - 6)} = \frac{A}{x} + \frac{B}{x - 6} \Rightarrow 1 = A(x - 6) + Bx$

$$x = 0 \Rightarrow 1 = A \cdot (-6) \Rightarrow A = -\frac{1}{6}$$

$$x = 6 \Rightarrow 1 = B \cdot (6) \Rightarrow B = \frac{1}{6}$$

$$\frac{1}{x(x - 6)} = \frac{-1/6}{x} + \frac{1/6}{x - 6}$$

$$\int f(x) dx = \int \left(\frac{-1/6}{x} + \frac{1/6}{x - 6} \right) dx$$

$$= -\frac{1}{6} \ln|x| + \frac{1}{6} \ln|x - 6| + C = \frac{1}{6} \ln \left| \frac{x - 6}{x} \right| + C$$

- 4 : $\begin{cases} 2 : \text{partial fraction decomposition} \\ 2 : \text{general antiderivative} \end{cases}$

2015 SCORING GUIDELINES

Question 6

The Maclaurin series for a function f is given by $\sum_{n=1}^{\infty} \frac{(-3)^{n-1}}{n} x^n = x - \frac{3}{2}x^2 + 3x^3 - \dots + \frac{(-3)^{n-1}}{n} x^n + \dots$ and converges to $f(x)$ for $|x| < R$, where R is the radius of convergence of the Maclaurin series.

- (a) Use the ratio test to find R .
- (b) Write the first four nonzero terms of the Maclaurin series for f' , the derivative of f . Express f' as a rational function for $|x| < R$.
- (c) Write the first four nonzero terms of the Maclaurin series for e^x . Use the Maclaurin series for e^x to write the third-degree Taylor polynomial for $g(x) = e^x f(x)$ about $x = 0$.

- (a) Let a_n be the n th term of the Maclaurin series.

$$\frac{a_{n+1}}{a_n} = \frac{(-3)^n x^{n+1}}{n+1} \cdot \frac{n}{(-3)^{n-1} x^n} = \frac{-3n}{n+1} \cdot x$$

$$\lim_{n \rightarrow \infty} \left| \frac{-3n}{n+1} \cdot x \right| = 3|x|$$

$$3|x| < 1 \Rightarrow |x| < \frac{1}{3}$$

The radius of convergence is $R = \frac{1}{3}$.

- (b) The first four nonzero terms of the Maclaurin series for f' are

$$1 - 3x + 9x^2 - 27x^3.$$

$$f'(x) = \frac{1}{1 - (-3x)} = \frac{1}{1 + 3x}$$

- (c) The first four nonzero terms of the Maclaurin series for e^x are

$$1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}.$$

The product of the Maclaurin series for e^x and the Maclaurin series for f is

$$\begin{aligned} \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots\right) \left(x - \frac{3}{2}x^2 + 3x^3 - \dots\right) \\ = x - \frac{1}{2}x^2 + 2x^3 + \dots \end{aligned}$$

The third-degree Taylor polynomial for $g(x) = e^x f(x)$

about $x = 0$ is $T_3(x) = x - \frac{1}{2}x^2 + 2x^3$.

- 3 : $\begin{cases} 1 : \text{sets up ratio} \\ 1 : \text{computes limit of ratio} \\ 1 : \text{determines radius of convergence} \end{cases}$

- 3 : $\begin{cases} 2 : \text{first four nonzero terms} \\ 1 : \text{rational function} \end{cases}$

- 3 : $\begin{cases} 1 : \text{first four nonzero terms} \\ \text{of the Maclaurin series for } e^x \\ 2 : \text{Taylor polynomial} \end{cases}$