

Part B (BC): Graphing calculator not allowed

Question 5

9 points

General Scoring Notes

- The model solution is presented using standard mathematical notation.
- Answers (numeric or algebraic) need not be simplified. Answers given as a decimal approximation should be accurate to three places after the decimal point. Within each individual free-response question, at most one point is not earned for inappropriate rounding.

Let $y = f(x)$ be the particular solution to the differential equation $\frac{dy}{dx} = (3 - x)y^2$ with initial condition $f(1) = -1$.

	Model Solution	Scoring
A	Find $f''(1)$, the value of $\frac{d^2y}{dx^2}$ at the point $(1, -1)$. Show the work that leads to your answer.	
	$\frac{d^2y}{dx^2} = -y^2 + (3 - x)2y \frac{dy}{dx}$	Product rule Point 1 (P1)
		Chain rule Point 2 (P2)
	$f'(1) = \frac{dy}{dx} \Big _{(x,y)=(1,-1)} = (3 - 1)(-1)^2 = 2$	$f''(1)$ Point 3 (P3)
	$f''(1) = \frac{d^2y}{dx^2} \Big _{(x,y)=(1,-1)} = -(-1)^2 + (3 - 1)(2)(-1)(2) = -9$	

Scoring Notes for Part A

- The expression $\frac{d^2y}{dx^2} = -y^2 + (3 - x)2y$ or $\frac{d^2y}{dx^2} = 6y - y^2 - 2xy$ earns **P1** but not **P2**. Such a response is not eligible for **P3**.
- The expression $\frac{d^2y}{dx^2} = -2y \frac{dy}{dx}$ or $\frac{d^2y}{dx^2} = 6y \left(\frac{dy}{dx} \right) - 2y \left(\frac{dy}{dx} \right)$ earns **P2** but not **P1**. Such a response is eligible for **P3** for a consistent answer of $f''(1) = 4$ or $f''(1) = -8$, respectively, which is found by correctly substituting correct values for x , y , and $\frac{dy}{dx}$.

- A response of $-(-1)^2 + (3 - 1)(2)(-1)(2)$ earns **P1**, **P2**, and **P3** regardless of any subsequent errors in simplification.

- Alternate approach (using separation of variables):

The particular solution for the differential equation that passes through the point $(1, -1)$ is

$$y = \frac{2}{x^2 - 6x + 3}. \text{ Therefore, } \frac{dy}{dx} = \frac{-2(2x - 6)}{(x^2 - 6x + 3)^2}.$$

This response has not yet earned **P1**, **P2**, or **P3**.

- A response that correctly applies the quotient rule (or product rule) and the chain rule to find that $\frac{d^2y}{dx^2} = \frac{-4(x^2 - 6x + 3)^2 + 4(2x - 6)(x^2 - 6x + 3)(2x - 6)}{(x^2 - 6x + 3)^4}$ earns **P1** and **P2** and is eligible to earn **P3** for the correct answer of $f''(1) = -9$.
- A response that correctly applies the quotient rule (or product rule) but does not correctly apply the chain rule (e.g., $\frac{d^2y}{dx^2} = \frac{-4(x^2 - 6x + 3)^2 + 4(2x - 6)(x^2 - 6x + 3)}{(x^2 - 6x + 3)^4}$) earns **P1**, does not earn **P2**, and is not eligible to earn **P3**.
- A response that does not correctly apply the quotient rule (or product rule) but does correctly apply the chain rule (e.g., $\frac{d^2y}{dx^2} = \frac{-4}{2(x^2 - 6x + 3)(2x - 6)}$) does not earn **P1**, earns **P2**, and is eligible to earn **P3** for a consistent answer.

B Write the second-degree Taylor polynomial for f about $x = 1$.

$$f'(1) = 2 \text{ and } f''(1) = -9$$

Two terms

Point 4 (P4)

$$P_2(x) = -1 + 2(x - 1) - \frac{9}{2}(x - 1)^2$$

Remaining term

Point 5 (P5)

Scoring Notes for Part B

- **P4** and **P5** can be earned with an answer consistent with incorrect values of $f'(1)$ and $f''(1)$ imported from part A.
- Any terms of degree greater than two or “+...” does not earn **P5**.
- A response of $-1 + 2(x - 1) - \frac{9}{2}(x - 1)^2$ earns **P4** and **P5**, regardless of any subsequent algebraic simplification.
- A response that does not present the polynomial as powers of $(x - 1)$ but instead presents a correct expanded/simplified form of the polynomial (e.g., $-\frac{9}{2}x^2 + 11x - \frac{15}{2}$) earns **P4** but not **P5**.

C The second-degree Taylor polynomial for f about $x = 1$ is used to approximate $f(1.1)$. Given that $|f'''(x)| \leq 60$ for all x in the interval $1 \leq x \leq 1.1$, use the Lagrange error bound to show that this approximation differs from $f(1.1)$ by at most 0.01.

$$|f(1.1) - P_2(1.1)| \leq \frac{\max_{1 \leq x \leq 1.1} |f'''(x)|}{3!} |1.1 - 1|^3 \leq \frac{60}{6} (0.1)^3 = 0.01$$

Form of error bound

Point 6 (P6)

Analysis

Point 7 (P7)

Scoring Notes for Part C

- **P6** is earned for presenting either $\frac{\max_{1 \leq x \leq 1.1} |f'''(x)|}{3!} |1.1 - 1|^3$ or $\frac{60}{6} (0.1)^3$. Subsequent errors in simplification will not earn **P7**.
- To earn **P7**, a response must have earned **P6** and must explicitly connect the error bound with 0.01; for example by communicating Error ≤ 0.01 , Error Bound = 0.01, or equivalent.
- A response that declares the error is equal to 0.01 (or any equivalent form of this value) does not earn **P7**.

D Use Euler's method, starting at $x = 1$ with two steps of equal size, to approximate $f(1.4)$. Show the work that leads to your answer.

$$f(1.2) \approx f(1) + (1.2 - 1) \cdot \frac{dy}{dx} \Big|_{(1, -1)}$$

$$= -1 + 0.2(2) = -0.6$$

First step of Euler's method

Point 8 (P8)

$$f(1.4) \approx f(1.2) + (1.4 - 1.2) \cdot \frac{dy}{dx} \Big|_{(1.2, -0.6)}$$

$$\approx -0.6 + 0.2(3 - 1.2)(-0.6)^2 = -0.4704$$

Answer with supporting work

Point 9 (P9)

An approximation of $f(1.4)$ is -0.47 .

Scoring Notes for Part D

- To earn **P8**, a response must demonstrate the first step of Euler's method, with the correct initial condition, correct step size, and correct (or imported) expression for the derivative.
Note: Any subsequent error in simplification or rounding will not affect the scoring for **P8**.
- The two steps of Euler's method may be explicit expressions or may be presented in a table. For example:

x	y	$\frac{dy}{dx} \cdot \Delta x$ (or $\frac{dy}{dx} \cdot 0.2$)
1	-1	0.4
1.2	-0.6	0.1296
1.4	-0.4704	

Note: In the presence of a correct answer, a table does not need to be labeled to earn both **P8** and **P9**. In the presence of no answer or an incorrect answer, such a table must be correctly labeled to earn **P8**.

- A response of $-0.6 + 0.2(3 - 1.2)(-0.6)^2$ earns **P9**, regardless of any subsequent errors in simplification or rounding.
- A response that imports an incorrect value for $f'(1)$ from part A or part B is eligible to earn **P9** with a consistent answer.
- A response may report the final answer as -0.47 , $(1.4, -0.47)$, $-\frac{294}{625}$, or equivalent.

Part A (AB or BC): Graphing calculator required**Question 1****9 points****General Scoring Notes**

The model solution is presented using standard mathematical notation.

Answers (numeric or algebraic) need not be simplified. Answers given as a decimal approximation should be correct to three places after the decimal point. Within each individual free-response question, at most one point is not earned for inappropriate rounding.

t (seconds)	0	60	90	120	135	150
$f(t)$ (gallons per second)	0	0.1	0.15	0.1	0.05	0

A customer at a gas station is pumping gasoline into a gas tank. The rate of flow of gasoline is modeled by a differentiable function f , where $f(t)$ is measured in gallons per second and t is measured in seconds since pumping began. Selected values of $f(t)$ are given in the table.

Model Solution**Scoring**

- (a) Using correct units, interpret the meaning of $\int_{60}^{135} f(t) dt$ in the context of the problem. Use a right Riemann sum with the three subintervals $[60, 90]$, $[90, 120]$, and $[120, 135]$ to approximate the value of $\int_{60}^{135} f(t) dt$.

$\int_{60}^{135} f(t) dt$ represents the total number of gallons of gasoline pumped into the gas tank from time $t = 60$ seconds to time $t = 135$ seconds.	Interpretation with units	1 point
$\int_{60}^{135} f(t) dt \approx f(90)(90 - 60) + f(120)(120 - 90) + f(135)(135 - 120) = (0.15)(30) + (0.1)(30) + (0.05)(15) = 8.25$	Form of Riemann sum	1 point
	Answer	1 point

Scoring notes:

- To earn the first point the response must reference gallons of gasoline added/pumped and the time interval $t = 60$ to $t = 135$.
- To earn the second point at least five of the six factors in the Riemann sum must be correct.
- If there is any error in the Riemann sum, the response does not earn the third point.
- A response of $(0.15)(30) + (0.1)(30) + (0.05)(15)$ earns both the second and third points, unless there is a subsequent error in simplification, in which case the response would earn only the second point.

- A response that presents a correct value with accompanying work that shows the three products in the Riemann sum but does not show all six of the factors and/or the sum process, does not earn the second point but does earn the third point. For example, responses of either $4.5 + 3.0 + 0.75$ or $(0.15)(30) + 0.1(30) + 0.05(15) \rightarrow 8.25$ earn the third point but not the second.
- A response of $f(90)(90 - 60) + f(120)(120 - 90) + f(135)(135 - 120) = 8.25$ earns both the second and the third points.
- A response that presents an answer of only 8.25 does not earn either the second or third point.
- A response that provides a completely correct left Riemann sum with accompanying work, $f(60)(30) + f(90)(30) + f(120)(15) = 9$, or $(0.1)(30) + (0.15)(30) + (0.1)(15)$ earns 1 of the last 2 points. A response with any errors or missing factors in a left Riemann sum earns neither of the last 2 points.

Total for part (a) 3 points

- (b) Must there exist a value of c , for $60 < c < 120$, such that $f'(c) = 0$? Justify your answer.

f is differentiable. $\Rightarrow f$ is continuous on $[60, 120]$.	$f(120) - f(60) = 0$	1 point
$\frac{f(120) - f(60)}{120 - 60} = \frac{0.1 - 0.1}{60} = 0$	Answer with justification	1 point
By the Mean Value Theorem, there must exist a c , for $60 < c < 120$, such that $f'(c) = 0$.		

Scoring notes:

- To earn the first point a response must present either $f(120) - f(60) = 0$, $0.1 - 0.1 = 0$ (perhaps as the numerator of a quotient), or $f(60) = f(120)$.
- To earn the second point a response must:
 - have earned the first point,
 - state that f is continuous because f is differentiable (or equivalent), and
 - answer “yes” in some way.
- A response may reference either the Mean Value Theorem or Rolle’s Theorem.
- A response that references the Intermediate Value Theorem cannot earn the second point.

Total for part (b) 2 points

- (c) The rate of flow of gasoline, in gallons per second, can also be modeled by

$g(t) = \left(\frac{t}{500}\right) \cos\left(\left(\frac{t}{120}\right)^2\right)$ for $0 \leq t \leq 150$. Using this model, find the average rate of flow of gasoline over the time interval $0 \leq t \leq 150$. Show the setup for your calculations.

$\frac{1}{150 - 0} \int_0^{150} g(t) dt$	Average value formula	1 point
$= 0.0959967$	Answer	1 point

The average rate of flow of gasoline, in gallons per second, is 0.096 (or 0.095).

Scoring notes:

- The exact value of $\frac{1}{150} \int_0^{150} g(t) dt$ is $\frac{12}{125} \sin\left(\frac{25}{16}\right)$.
- A response may present the average value formula in single or multiple steps. For example, the following response earns both points: $\int_0^{150} g(t) dt = 14.399504$ so the average rate is 0.0959967.
- A response that presents the average value formula in multiple steps but provides incorrect or incomplete communication (e.g., $\int_0^{150} g(t) dt = \frac{14.399504}{150} = 0.0959967$) earns 1 out of 2 points.
- A response of $\int_0^{150} g(t) dt = 0.0959967$ does not earn either point.
- Degree mode: A response that presents answers obtained by using a calculator in degree mode does not earn the first point it would have otherwise earned. The response is generally eligible for all subsequent points (unless no answer is possible in degree mode or the question is made simpler by using degree mode). In degree mode, $\frac{1}{150} \int_0^{150} g(t) dt = 0.149981$ or 0.002618.

Total for part (c) 2 points

- (d) Using the model g defined in part (c), find the value of $g'(140)$. Interpret the meaning of your answer in the context of the problem.

$g'(140) \approx -0.004908$	$g'(140)$	1 point
$g'(140) = -0.005$ (or -0.004)		
The rate at which gasoline is flowing into the tank is decreasing at a rate of 0.005 (or 0.004) gallon per second per second at time $t = 140$ seconds.	Interpretation	1 point

Scoring notes:

- The exact value of $g'(140)$ is $\frac{1}{500} \cos\left(\frac{49}{36}\right) - \frac{49}{9000} \sin\left(\frac{49}{36}\right)$.
- The value of $g'(140)$ may appear only in the interpretation.
- To be eligible for the second point a response must present some numerical value for $g'(140)$.
- To earn the second point the interpretation must include “the rate of flow of gasoline is changing at a rate of [the declared value of $g'(140)$]” and “at $t = 140$ ” (or equivalent).
- An interpretation of “decreasing at a rate of -0.005 ” or “increasing at a rate of 0.005” does not earn the second point.
- Degree mode: In degree mode, $g'(140) = 0.001997$ or 0.00187.

Total for part (d) 2 points**Total for question 1 9 points****Part B (BC): Graphing calculator not allowed****Question 6****9 points****General Scoring Notes**

The model solution is presented using standard mathematical notation.

Answers (numeric or algebraic) need not be simplified. Answers given as a decimal approximation should be correct to three places after the decimal point. Within each individual free-response question, at most one point is not earned for inappropriate rounding.

The function f has derivatives of all orders for all real numbers. It is known that $f(0) = 2$, $f'(0) = 3$,

$$f''(x) = -f(x^2), \text{ and } f'''(x) = -2x \cdot f'(x^2).$$

Model Solution	Scoring	
(a) Find $f^{(4)}(x)$, the fourth derivative of f with respect to x . Write the fourth-degree Taylor polynomial for f about $x = 0$. Show the work that leads to your answer.	Form of product rule	1 point
	$f^{(4)}(x)$	1 point
$f^{(4)}(x) = -2 \cdot f'(x^2) + (-2x) \cdot f''(x^2) \cdot 2x$		
$f'''(0) = -f(0) = -2$	Two terms of polynomial	1 point
$f'''(0) = -2(0) \cdot f'(0) = 0$		
$f^{(4)}(0) = -2 \cdot f'(0) + 0 \cdot f''(0) \cdot 0 = -2 \cdot 3 + 0 = -6$	Remaining terms	1 point
The fourth-degree Taylor polynomial for f about $x = 0$ is		
$T_4(x) = 2 + 3x + \frac{-2}{2!}x^2 + \frac{0}{3!}x^3 + \frac{-6}{4!}x^4$		
$= 2 + 3x - x^2 - \frac{1}{4}x^4$		

Scoring notes:

- The first point is earned for a correct fourth derivative or for $f^{(4)}(x) = -2 \cdot f'(x^2) + (-2x) \cdot f''(x^2)$.
- The second point is earned only for a completely correct expression for $f^{(4)}(x)$.
- A response that earns the first point but not the second may evaluate the presented expression for $f^{(4)}(x)$ at $x = 0$ and use the consistent nonzero value in computing the coefficient of x^4 in the fourth-degree Taylor polynomial.
- A polynomial that includes a nonzero third-degree term, any terms of degree greater than four, or $+\dots$ does not earn the fourth point.

Total for part (a) 4 points

- (b) The fourth-degree Taylor polynomial for f about $x = 0$ is used to approximate $f(0.1)$. Given that $|f^{(5)}(x)| \leq 15$ for $0 \leq x \leq 0.5$, use the Lagrange error bound to show that this approximation is within $\frac{1}{10^5}$ of the exact value of $f(0.1)$.

By the Lagrange error bound,	Form of error bound	1 point
$ T_4(0.1) - f(0.1) \leq \frac{\max_{0 \leq x \leq 0.1} f^{(5)}(x) }{5!} \cdot (0.1)^5$	Shows $ \text{Error} \leq \frac{1}{10^5}$	1 point
$\leq \frac{15}{120} \cdot \frac{1}{10^5} \leq \frac{1}{10^5}$		

Scoring notes:

- The first point is earned for $\frac{\max_{0 \leq x \leq 0.1} |f^{(5)}(x)|}{5!} \cdot (0.1)^5$ or $\frac{15}{5!}(0.1)^5$. Subsequent errors in simplification will not earn the second point.
- To earn the second point a response must communicate the inequality $\text{Error} \leq \frac{15}{5!}(0.1)^5 \leq \frac{1}{10^5}$.
- A response that states $\text{Error} = \frac{15}{5!}(0.1)^5$ or $\text{Error} = \frac{1}{10^5}$ does not earn the second point.

Total for part (b) 2 points

- (c) Let g be the function such that $g(0) = 4$ and $g'(x) = e^x f(x)$. Write the second-degree Taylor polynomial for g about $x = 0$.

$g''(x) = e^x \cdot f(x) + e^x \cdot f'(x)$	$g''(x)$	1 point
$g'(0) = e^0 \cdot f(0) = 2$	First two terms of polynomial	1 point
$g''(0) = e^0 \cdot f(0) + e^0 \cdot f'(0) = 2 + 3 = 5$	Taylor polynomial	1 point
The second-degree Taylor polynomial for g about $x = 0$ is $T_2(x) = 4 + 2x + \frac{5}{2}x^2$.		

Scoring notes:

- The first point is earned for $g''(x) = e^x \cdot f(x) + e^x \cdot f'(x)$, $g''(0) = e^0 \cdot f(0) + e^0 \cdot f'(0)$, or $g''(0) = f(0) + f'(0)$.
- A presented polynomial of the form $4 + 2x + ax^2$ earns the second point with or without any supporting work for the first two terms.
- A response that earned neither the first nor the second point only earns the third point for a polynomial of the form $a + bx + \frac{c}{2}x^2$, where $c \neq 0$ is declared to be $g''(0)$.

- A polynomial that includes any terms of degree greater than two, or $+ \dots$, does not earn the third point.
- Alternate solution:

$$e^x = 1 + x + \frac{x^2}{2} + \dots$$

$$e^x f(x) = \left(1 + x + \frac{x^2}{2} + \dots\right)(2 + 3x - x^2 + \dots) = 2 + 5x + \dots$$

$$g(x) = \int e^x f(x) dx = C + 2x + \frac{5}{2}x^2 + \dots$$

$$g(0) = 4 \Rightarrow C = 4$$

$$g(x) \approx 4 + 2x + \frac{5}{2}x^2$$

- A response that is using this alternate solution method earns the first point for $e^x f(x) = 2 + 5x + \dots$, the second point for any two correct terms in a second-degree polynomial, and the third point for a completely correct second-degree Taylor polynomial with supporting work.
- Note: There is not enough information to conclude that $f(x)$ is equal to its Maclaurin series on its interval of convergence. The second and third lines of the alternate solution are being accepted as identifications of the Maclaurin series for $e^x f(x)$ and $g(x)$, respectively.

Total for part (c) 3 points**Total for question 6 9 points**