Infinite Sequences & Series

AP Calculus BC ▪ Unit 10

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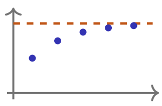
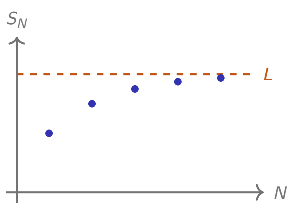


What we will cover

1. Convergence & partial sums
2. Geometric series
3. The convergence tests
4. Absolute vs conditional
5. Taylor polynomials & error
6. Power series & intervals

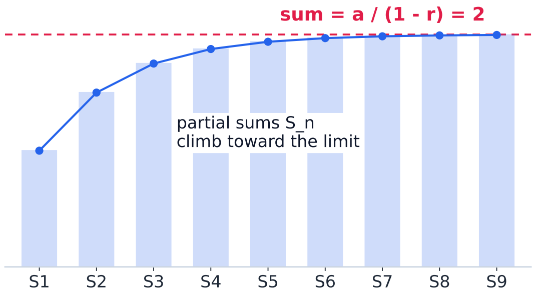


1 Convergence

Convergent and divergent series

An **infinite series** 无穷级数 $\sum a_n$ is defined as the limit of its **partial sums** 部分和 $S_N = a_1 + \cdots + a_N$.

> Every convergence question is really a question about the limit of the partial sums.



Geometric series – the one you can sum exactly

A **geometric series** 几何级数 has a constant ratio r. It converges exactly when $|r| < 1$:

$$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}.$$

> It underlies power series later in the unit.

Sum $3 + \frac{3}{2} + \frac{3}{4} + \frac{3}{8} + \cdots$

Here $a = 3$ and $r = \frac{1}{2}$, and $|r| < 1$:

$$\frac{a}{1-r} = \frac{3}{1-\frac{1}{2}} = 6.$$

2 The tests

Start here – the nth term test

If the terms do not shrink to zero, the sum cannot settle:

$$\lim_{n\to\infty} a_n \neq 0 \implies \text{diverges.}$$

Always check this quick test first.

It is **only** a test for divergence

If $a_n \to 0$ the test is **inconclusive** – the series may still diverge

p-series and the harmonic series

$$\sum \frac{1}{n^p} \begin{cases} \text{converges} & p > 1 \\ \text{diverges} & p \leq 1 \end{cases}$$

The **integral test** 积分判别法 proves this: for positive, decreasing, continuous f, $\sum a_n$ and $\int_1^\infty f$ both converge or both diverge.

The **harmonic series** 调和级数 $\sum \frac{1}{n}$ ($p = 1$) diverges – even though its terms go to zero. A famous, must-know fact.

The p-series family is the standard yardstick for the comparison tests.

Comparison and alternating series

Direct comparison 直接比较
$0 \leq a_n \leq b_n$ and $\sum b_n$ converges $\Rightarrow$ so does $\sum a_n$

Limit comparison 极限比较
$\lim \frac{a_n}{b_n}$ finite and positive $\Rightarrow$ both do the same thing

Alternating series 交错级数
b_n positive, decreasing, and $b_n \to 0 \Rightarrow$ converges

Ratio test 比值判别法
$L = \lim \left|\frac{a_{n+1}}{a_n}\right|$: $L < 1$ converges, $L > 1$ diverges, $L = 1$ inconclusive

Worked example – the ratio test

Test $\sum \frac{n}{2^n}$

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{n+1}{2^{n+1}} \cdot \frac{2^n}{n} = \frac{n+1}{2n} \to \frac{1}{2} < 1$$

so the series converges.

The ratio test is the go-to for factorials or nth powers – and it is exactly how you find a radius of convergence.

Absolute vs conditional convergence

Absolutely 绝对收敛
$\sum |a_n|$ converges –
the stronger property

Conditionally 条件收敛
$\sum a_n$ converges but
$\sum |a_n|$ does not

The classic: $\sum \frac{(-1)^n}{n}$ converges conditionally – it relies on the cancellation of signs, since $\sum \frac{1}{n}$ diverges.

Alternating error bound

$$|S - S_N| \leq b_{N+1}$$

The error is no larger than the first omitted term.

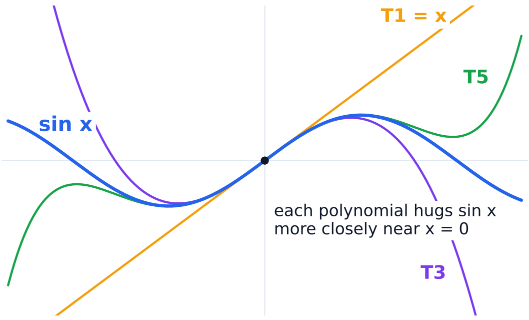
3 Taylor

Taylor polynomials

A **Taylor polynomial** 泰勒多项式 approximates f near a centre a using its derivatives there:

$$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k.$$

> Each added term matches one more derivative. Centred at $a = 0$ it is a Maclaurin polynomial.



The Lagrange error bound

How far can a Taylor polynomial be from the truth?

$$|R_n(x)| \le \frac{\max\left|f^{(n+1)}(z)\right|}{(n+1)!}|x-a|^{n+1}.$$

> Bound the $(n+1)$th derivative on the interval, then compute – the standard way to prove an estimate is accurate enough.

Radius and interval of convergence

A **power series** 幂级数 $\sum c_n(x-a)^n$ converges within a **radius** 收敛半径 R of its centre. Find R with the ratio test, then test the endpoints separately.

> The ratio test is inconclusive at the endpoints – that is why they need their own check.

Interval of $\sum \frac{x^n}{n}$

Ratio: $|x|\frac{n}{n+1} \to |x| < 1$, so $R = 1$. $x = -1$: alternating harmonic – converges.
$x = 1$: harmonic – diverges.

Interval $= [-1, 1)$.

The four Maclaurin series to memorise

$$e^x = \sum \frac{x^n}{n!}$$

$$\frac{1}{1-x} = \sum x^n$$

$$\sin x = \sum \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

$$\cos x = \sum \frac{(-1)^n x^{2n}}{(2n)!}$$

> New series come from manipulating these – substituting, differentiating, integrating, multiplying.

Worked example – the substitution trick

Maclaurin series for e^{x^2}

Substitute x^2 for x in $e^x = \sum \frac{x^n}{n!}$:

$$e^{x^2} = \sum_{n=0}^{\infty} \frac{(x^2)^n}{n!} = 1 + x^2 + \frac{x^4}{2!} + \frac{x^6}{3!} + \cdots$$

which converges for all x.

> This substitution is far faster than differentiating e^{x^2} six times – which is exactly what the exam is testing.

4 Recap

Unit 10 – key takeaways

1 Convergence

the limit of the partial sums; check $a_n \to 0$ first

2 Geometric

$|r| < 1 \Rightarrow a/(1-r)$ – the one exact sum

3 The tests

p-series yardstick; comparison, alternating, ratio

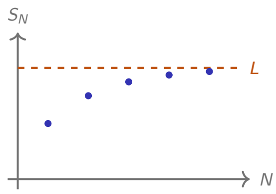
4 Taylor

P_n matches derivatives at a; ratio test gives R

Infinite Sequences & Series
└ Recap

Check yourself

1. Does $\sum \frac{1}{n}$ converge?
2. Sum $3 + \frac{3}{2} + \frac{3}{4} + \cdots$
3. The ratio test gives $L = 1$. What follows?
4. Give the Maclaurin series for e^{x^2}.



partial sums

Answers **1.** no – it diverges **2.** 6 **3.** inconclusive – use another test **4.** $\sum x^{2n}/n!$