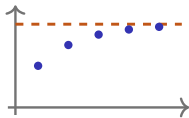


Infinite Sequences & Series

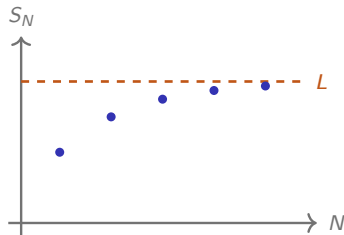
AP Calculus BC ■ Unit 10

AP Calculus BC



What we will cover

- 1 Convergence & partial sums
- 2 Geometric series
- 3 The convergence tests
- 4 Absolute vs conditional
- 5 Taylor polynomials & error
- 6 Power series & intervals



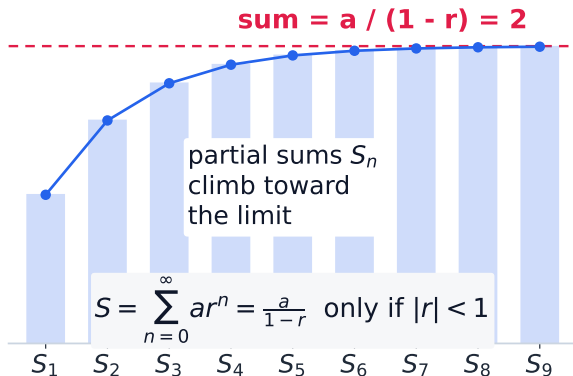
1

Convergence

Convergent and divergent series

An **infinite series** 无穷级数 $\sum a_n$ is defined as the limit of its **partial sums** 部分和 $S_N = a_1 + \cdots + a_N$.

Every convergence question is really a question about **the limit of the partial sums**.



Geometric series – the one you can sum exactly

A **geometric series** 几何级数 has a constant ratio r . It converges **exactly** when $|r| < 1$:

$$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$$

first term a , ratio r ,
only when $|r| < 1$

It underlies **power series** later in the unit.

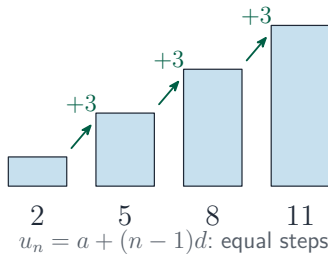
$$\text{Sum } 3 + \frac{3}{2} + \frac{3}{4} + \frac{3}{8} + \dots$$

Here $a = 3$ and $r = \frac{1}{2}$, and $|r| < 1$:

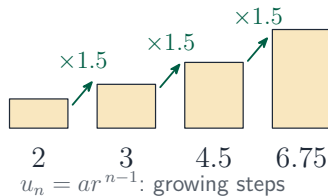
$$\frac{a}{1-r} = \frac{3}{1-\frac{1}{2}} = 6.$$

Working with Geometric Series

arithmetic: add d each step



geometric: multiply by r each step



A geometric sequence multiplies by the same ratio at each step

2

The tests

Start here – the n th term test

If the terms do not shrink to zero, the sum cannot settle:

$$\lim_{n \rightarrow \infty} a_n \neq 0 \implies \text{diverges.}$$

Always check this **quick test first**.

✗ It is **only** a test for **divergence**

If $a_n \rightarrow 0$ the test is **inconclusive** – the series may **still** diverge

p -series and the harmonic series

$$\sum \frac{1}{n^p} \begin{cases} \text{converges} & p > 1 \\ \text{diverges} & p \leq 1 \end{cases}$$

The **integral test** 积分判别法 proves this: for positive, decreasing, continuous f , $\sum a_n$ and $\int_1^\infty f$ **both** converge or **both** diverge.

The **harmonic series** 调和级数 $\sum \frac{1}{n}$ ($p = 1$) **diverges** – even though its terms go to zero. A famous, must-know fact.

The p -series family is the standard **yardstick** for the comparison tests.

Comparison and alternating series

Direct comparison 直接比较

$0 \leq a_n \leq b_n$ and $\sum b_n$
converges $\Rightarrow$ so does $\sum a_n$

Limit comparison 极限比较

$\lim \frac{a_n}{b_n}$ finite and positive
 $\Rightarrow$ both do the **same** thing

Alternating series 交错级数

b_n positive, **decreasing**,
and $b_n \rightarrow 0 \Rightarrow$ converges

Ratio test 比值判别法

$L = \lim \left| \frac{a_{n+1}}{a_n} \right|$: $L < 1$ converges,
 $L > 1$ diverges, $L = 1$ **inconclusive**

Worked example – the ratio test

Test $\sum \frac{n}{2^n}$

$$\left| \frac{a_{n+1}}{a_n} \right| = \frac{n+1}{2^{n+1}} \cdot \frac{2^n}{n} = \frac{n+1}{2n} \rightarrow \frac{1}{2} < 1$$

so the series **converges**.

The ratio test is the go-to for **factorials** or **n th powers** – and it is exactly how you find a **radius of convergence**.

Absolute vs conditional convergence

Absolutely 绝对收敛

$\sum |a_n|$ converges –
the **stronger** property

Conditionally 条件收敛

$\sum a_n$ converges but
 $\sum |a_n|$ does not

The classic: $\sum \frac{(-1)^n}{n}$ converges **conditionally** – it relies on the **cancellation of signs**, since $\sum \frac{1}{n}$ diverges.

Alternating error bound

$$|S - S_N| \leq b_{N+1}$$

The error is **no larger than the first omitted term**.

Absolute against conditional convergence

Converges absolutely

A series **converges absolutely** when $\sum |a_n|$ converges – and then $\sum a_n$ converges too, whatever the signs do.

Converges conditionally

It **converges conditionally** when $\sum a_n$ converges but $\sum |a_n|$ does not – the alternating harmonic series is the standard case.

Why it matters

An absolutely convergent series can be rearranged freely; a conditionally convergent one can be rearranged to give *any* sum at all.

How to test

Test $\sum |a_n|$ first. If it converges you are finished; if not, try the alternating series test on $\sum a_n$.

“Converges” is not one property. Say which kind, because the error bound and the rearrangement both depend on it.

3

Taylor

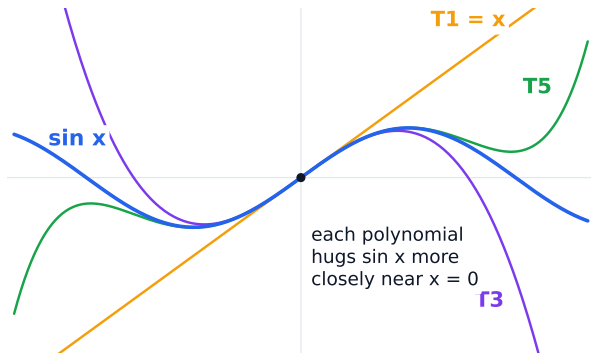
Taylor polynomials

A **Taylor polynomial** 泰勒多项式 approximates f near a centre a using its derivatives there:

$$P_n(x) = \sum_{k=0}^n \frac{f^{(k)}(a)}{k!} (x - a)^k.$$

Each added term matches **one more derivative**. Centred at $a = 0$ it is a **Maclaurin** polynomial.

Taylor polynomials, in detail



$$T_n(x) = \sum_{k=0}^n \frac{f^{(k)}(0)}{k!} x^k$$

centre $a = 0$: a

Maclaurin polynomial

For sin x: $x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots$

Each added term matches **one more derivative** at the centre: a closer fit, but only near a .

The Lagrange error bound

How far can a Taylor polynomial of degree n , centred at a , be from the truth?

$$|R_n(x)| \leq \frac{\max |f^{(n+1)}(z)|}{(n+1)!} |x - a|^{n+1}$$

max taken over z between a and x

Bound the $(n+1)$ th derivative on the interval, then compute – the standard way to **prove** an estimate is accurate enough.

Radius and interval of convergence

A **power series** 幂级数 $\sum c_n(x-a)^n$ converges within a **radius** 收敛半径 R of its centre. Find R with the **ratio test**, then test the **endpoints separately**.

The ratio test is **inconclusive at the endpoints** – that is why they need their own check.

Interval of $\sum \frac{x^n}{n}$

Ratio: $|x| \frac{n}{n+1} \rightarrow |x| < 1$, so $R = 1$.

$x = -1$: alternating harmonic – **converges**.

$x = 1$: harmonic – **diverges**.

Interval = $[-1, 1)$.

The four Maclaurin series to memorise

$$e^x = \sum \frac{x^n}{n!}$$

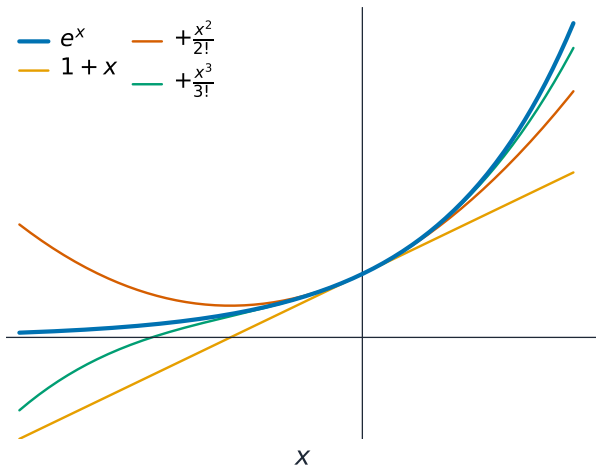
$$\frac{1}{1-x} = \sum x^n$$

$$\sin x = \sum \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

$$\cos x = \sum \frac{(-1)^n x^{2n}}{(2n)!}$$

New series come from **manipulating** these – substituting, differentiating, integrating, multiplying.

The four Maclaurin series to memorise, in detail



Each extra Maclaurin term hugs the function over a wider range

Worked example – the substitution trick

Maclaurin series for e^{x^2}

Substitute x^2 for x in $e^x = \sum \frac{x^n}{n!}$:

$$e^{x^2} = \sum_{n=0}^{\infty} \frac{(x^2)^n}{n!} = 1 + x^2 + \frac{x^4}{2!} + \frac{x^6}{3!} + \dots$$

which converges for all x .

This substitution is **far faster** than differentiating e^{x^2} six times – which is exactly what the exam is testing.

Defining Convergent and Divergent Infinite Series



Nested matryoshka dolls: an infinite series keeps adding terms – some converge, some diverge

Exam tips

- Test a series for convergence with the right tool: **geometric** ($|r| < 1$, $\sum \frac{a}{1-r}$), n th-term, ratio, integral, comparison, or alternating-series test.
- A **geometric** infinite sum converges only when $|r| < 1$; otherwise it diverges.
- Build a **Taylor/Maclaurin** series to approximate a function; more terms give a better fit **near the centre**.
- Know the standard Maclaurin series for e^x , $\sin x$, $\cos x$, and $\frac{1}{1-x}$.
- Find the **radius/interval of convergence** with the ratio test, then check the endpoints separately.

4

Recap

Unit 10 – key takeaways

1

Convergence

the limit of the partial sums;
check $a_n \rightarrow 0$ first

2

Geometric

$|r| < 1 \Rightarrow a/(1 - r)$ – the one
exact sum

3

The tests

p -series yardstick; comparison,
alternating, ratio

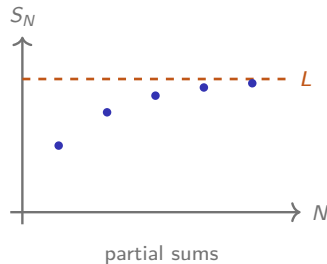
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Taylor

P_n matches derivatives at a ; ratio
test gives R

Check yourself

- 1 Does $\sum \frac{1}{n}$ converge?
- 2 Sum $3 + \frac{3}{2} + \frac{3}{4} + \dots$
- 3 The ratio test gives $L = 1$. What follows?
- 4 Give the Maclaurin series for e^{x^2} .



Answers 1. no – it diverges 2. 6 3. inconclusive – use another test 4. $\sum x^{2n}/n!$