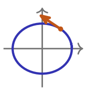
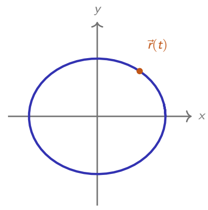
Parametric, Polar & Vector Functions

AP Calculus BC ▪ Unit 9

AP Calculus BC



What we will cover



1 Parametric

Differentiating a parametric curve

Parametric equations 参数方程 give x and y each as functions of a **parameter** 参数 t – so the curve need not be a function of x.

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} \qquad (dx/dt \neq 0).$$

> It is the chain rule: the dt cancels.

$x = t^2$, $y = t^3 - t$: slope at $t = 2$

$$\frac{dx}{dt} = 2t, \quad \frac{dy}{dt} = 3t^2 - 1:$$

$$\frac{dy}{dx} = \frac{3t^2 - 1}{2t} \xrightarrow{t=2} \frac{11}{4}.$$

The second derivative – a classic trap

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{dx/dt}.$$

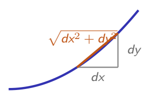
Differentiate the first derivative with respect to t, then divide by dx/dt again. Use it to test concavity of a parametric curve.

> The ISS in orbit: parametric equations describe position as a function of time

Arc length of a parametric curve

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt.$$

> This is Pythagoras: sum tiny hypotenuses $\sqrt{dx^2 + dy^2}$ over the parameter.



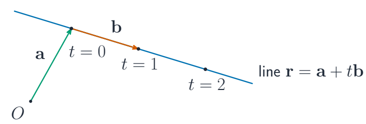
2 Vectors

Vector-valued functions

A **vector-valued function** 向量值函数 $\vec{r}(t) = \langle x(t), y(t) \rangle$ gives a position vector for each t. Differentiate component-wise:

$$\vec{v} = \langle x', y' \rangle, \qquad \vec{a} = \langle x'', y'' \rangle.$$

Speed is the magnitude $|\vec{v}| = \sqrt{x'^2 + y'^2}$ – a scalar.



Integrating a vector function, and plane motion

Integrate component by component; use the initial condition to fix each constant – recovering $\vec{v}$ from $\vec{a}$, or $\vec{r}$ from $\vec{v}$. Distance travelled over $[a, b]$:

$$\int_a^b \sqrt{x'(t)^2 + y'(t)^2}\, dt.$$

$\vec{r}(t) = \langle t^2,\ t^3 - t \rangle$

$\vec{v}(t) = \langle 2t,\ 3t^2 - 1 \rangle$, so at $t = 1$:

$$\vec{v}(1) = \langle 2, 2 \rangle, \quad |\vec{v}| = 2\sqrt{2}.$$

Position at $t = 2$: $\langle 4,\ 6 \rangle$.

Exam skill: plane-motion problems appear on the BC free-response nearly every year.

3 Polar

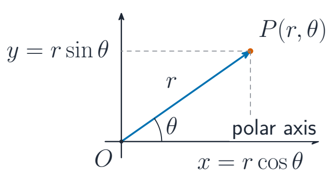
Polar coordinates

Polar coordinates 极坐标 locate a point by its distance r and angle θ:

$$x = r\cos\theta, \qquad y = r\sin\theta.$$

A polar curve $r = f(\theta)$ is a parametric curve in θ, so

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta}.$$

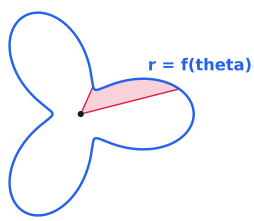


Area of a polar region

The region is a fan of thin triangular sectors, each of area $\frac{1}{2} r^2\, d\theta$:

$$A = \frac{1}{2}\int_\alpha^\beta \big(f(\theta)\big)^2\, d\theta.$$

Choosing the right θ-limits – where the curve starts and finishes tracing the region – is the main challenge.



area = 1/2 * integral of r^2 dtheta

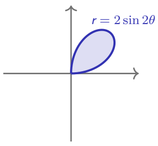
Worked example – one petal of a rose

Area of one petal of $r = 2\sin(2\theta)$, traced for $0 \le \theta \le \frac{\pi}{2}$

Using $\sin^2 u = \frac{1}{2}(1 - \cos 2u)$:

$$A = \frac{1}{2}\int_0^{\pi/2} (2\sin 2\theta)^2 \, d\theta = \int_0^{\pi/2} (1 - \cos 4\theta)\, d\theta$$

$$= \left[\theta - \tfrac{\sin 4\theta}{4}\right]_0^{\pi/2} = \frac{\pi}{2}.$$

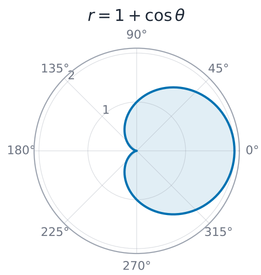


Between two polar curves

For the region between an outer $r_1(\theta)$ and an inner $r_2(\theta)$, subtract the fans:

$$A = \frac{1}{2}\int_\alpha^\beta \left(r_1^2 - r_2^2\right) d\theta.$$

Subtract the squares, not the radii – and find where the curves intersect to set α and β.



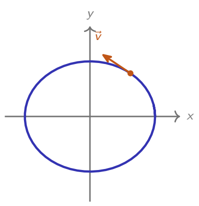
4 Recap

Unit 9 – key takeaways

1. **Parametric**
   $dy/dx = (dy/dt)/(dx/dt)$; the 2nd derivative divides by dx/dt again
2. **Arc length**
   $\int \sqrt{(dx/dt)^2 + (dy/dt)^2}\, dt$ – tiny hypotenuses
3. **Vectors**
   differentiate component-wise; speed is $|\vec{v}|$
4. **Polar area**
   $\frac{1}{2}\int r^2\, d\theta$; between curves subtract the squares

Check yourself

1. For $x = t^2$, $y = t^3 - t$, find $\frac{dy}{dx}$.
2. Is $\frac{d^2y}{dx^2}$ equal to $\frac{d^2y/dt^2}{d^2x/dt^2}$?
3. Give the speed of $\vec{r}(t) = \langle x(t), y(t)\rangle$.
4. Write the area swept by $r = f(\theta)$ from α to β.



Answers **1.** $\frac{3t^2-1}{2t}$ **2.** no **3.** $\sqrt{x'^2 + y'^2}$ **4.** $\frac{1}{2}\int_\alpha^\beta f(\theta)^2\, d\theta$