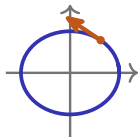


Parametric, Polar & Vector Functions

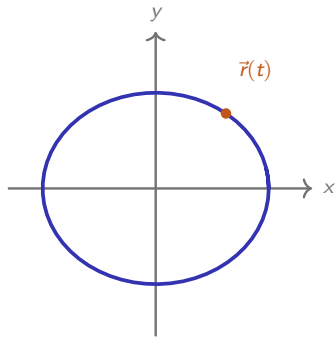
AP Calculus BC ■ Unit 9

AP Calculus BC



What we will cover

- 1 Differentiating parametric curves
- 2 Second derivatives & arc length
- 3 Vector-valued functions
- 4 Plane motion
- 5 Polar coordinates
- 6 Area of a polar region



1

Parametric

Differentiating a parametric curve

Parametric equations 参数方程 give x and y each as functions of a **parameter** 参数 t – so the curve need not be a function of x .

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} \quad (dx/dt \neq 0)$$

It is the chain rule: the dt cancels.

$x = t^2$, $y = t^3 - t$: slope at $t = 2$

$$\frac{dx}{dt} = 2t, \quad \frac{dy}{dt} = 3t^2 - 1:$$

$$\frac{dy}{dx} = \frac{3t^2 - 1}{2t} \xrightarrow{t=2} \frac{11}{4}.$$

The second derivative – a classic trap

$$\frac{d^2 y}{dx^2} = \frac{\frac{d}{dt} \left(\frac{dy}{dx} \right)}{dx/dt}$$

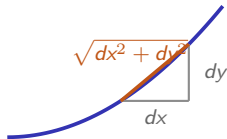
Differentiate the **first derivative** with respect to t , then divide by dx/dt **again**. Use it to test **concavity** of a parametric curve.

The trap: $\frac{d^2 y}{dx^2}$ is **not** $\frac{d^2 y/dt^2}{d^2 x/dt^2}$. Differentiate $\frac{dy}{dx}$ in t , then divide by $\frac{dx}{dt}$.

Arc length of a parametric curve

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$$

This is Pythagoras: sum tiny hypotenuses $\sqrt{dx^2 + dy^2}$ over the parameter.



2

Vectors

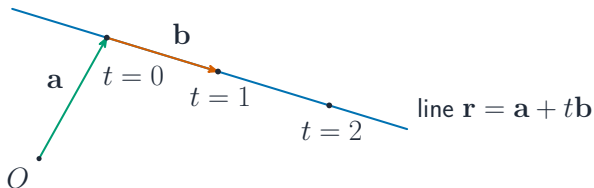
Vector-valued functions

A **vector-valued function** 向量值函数

$\vec{r}(t) = \langle x(t), y(t) \rangle$ gives a position vector for each t . Differentiate component-wise:

$$\vec{v} = \langle x', y' \rangle, \quad \vec{a} = \langle x'', y'' \rangle$$

Speed is the magnitude $|\vec{v}| = \sqrt{x'^2 + y'^2}$ – a scalar.



Integrating a vector function, and plane motion

Integrate **component by component**; use the initial condition to fix each constant – recovering $\vec{v}$ from $\vec{a}$, or $\vec{r}$ from $\vec{v}$.

Distance travelled over $[a, b]$:

$$\int_a^b \sqrt{x'(t)^2 + y'(t)^2} dt$$

$$\vec{r}(t) = \langle t^2, t^3 - t \rangle$$

$$\vec{v}(t) = \langle 2t, 3t^2 - 1 \rangle, \text{ so at } t = 1:$$

$$\vec{v}(1) = \langle 2, 2 \rangle, \quad |\vec{v}| = 2\sqrt{2}.$$

$$\text{Position at } t = 2: \langle 4, 6 \rangle.$$

Exam skill: plane-motion problems appear on the BC free-response nearly every year.

3

Polar

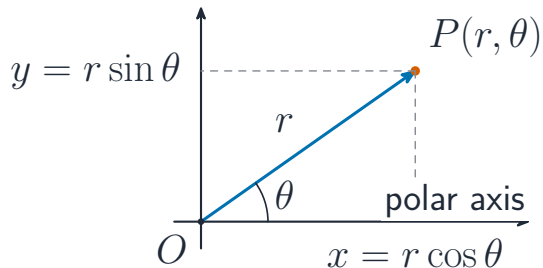
Polar coordinates

Polar coordinates 极坐标 locate a point by its distance r and angle θ :

$$x = r \cos \theta, \quad y = r \sin \theta$$

A polar curve $r = f(\theta)$ is a **parametric curve in θ** , so

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta}.$$

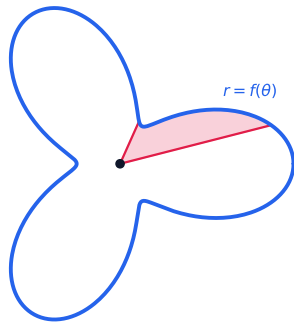


Area of a polar region

The region is a **fan** of thin triangular sectors, each of area $\frac{1}{2}r^2 d\theta$:

$$A = \frac{1}{2} \int_{\alpha}^{\beta} (f(\theta))^2 d\theta.$$

Choosing the right θ -limits – where the curve **starts and finishes tracing** the region – is the main challenge.



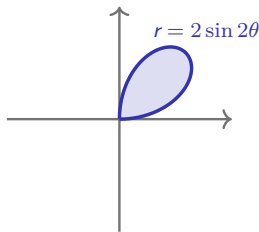
$$A = \frac{1}{2} \int_{\alpha}^{\beta} r^2 d\theta$$

Worked example – one petal of a rose

Area of one petal of $r = 2 \sin(2\theta)$, traced for $0 \leq \theta \leq \frac{\pi}{2}$

Using $\sin^2 u = \frac{1}{2}(1 - \cos 2u)$:

$$\begin{aligned} A &= \frac{1}{2} \int_0^{\pi/2} (2 \sin 2\theta)^2 d\theta = \int_0^{\pi/2} (1 - \cos 4\theta) d\theta \\ &= \left[\theta - \frac{\sin 4\theta}{4} \right]_0^{\pi/2} = \frac{\pi}{2}. \end{aligned}$$

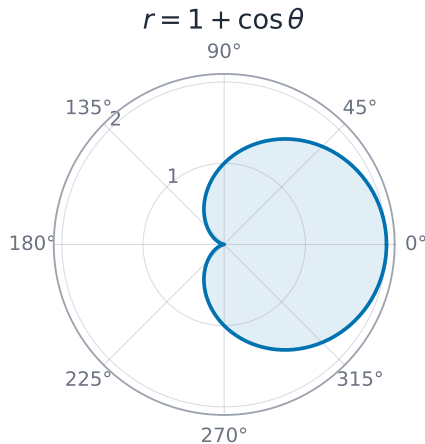


Between two polar curves

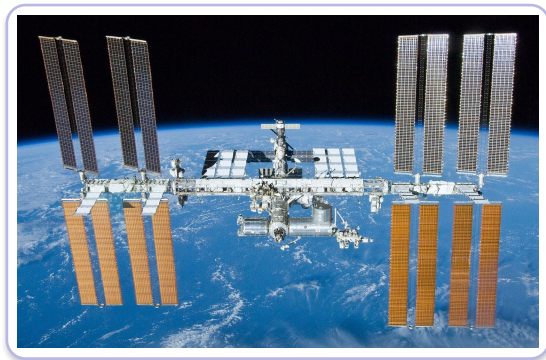
For the region between an outer $r_1(\theta)$ and an inner $r_2(\theta)$, subtract the fans:

$$A = \frac{1}{2} \int_{\alpha}^{\beta} (r_1^2 - r_2^2) d\theta.$$

Subtract the **squares**, not the radii – and find where the curves **intersect** to set α and β .



Defining and Differentiating Parametric Equations



The ISS in orbit: parametric equations describe position as a function of time

Exam tips

- For **parametric** curves, $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$; speed is $\sqrt{(dx/dt)^2 + (dy/dt)^2}$.
- A **vector-valued** function carries the same information – differentiate/integrate it component by component.
- In **polar**, convert with $x = r \cos \theta$, $y = r \sin \theta$; area swept is $\frac{1}{2} \int r^2 d\theta$.
- Watch the direction of tracing (the sign of dx/dt) and set correct θ -limits for polar area.
- Eliminate the parameter to recover the ordinary y -vs- x shape when it helps.

4

Recap

Unit 9 – key takeaways

1

Parametric

$dy/dx = (dy/dt)/(dx/dt)$; the 2nd derivative divides by dx/dt again

2

Arc length

$\int \sqrt{(dx/dt)^2 + (dy/dt)^2} dt$ – tiny hypotenuses

3

Vectors

differentiate component-wise;
speed is $|\vec{v}|$

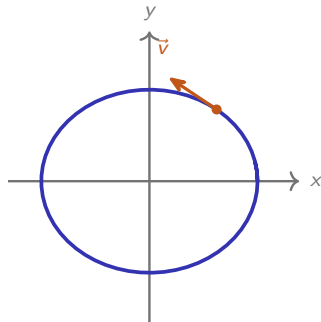
4

Polar area

$\frac{1}{2} \int r^2 d\theta$; between curves subtract the squares

Check yourself

- 1 For $x = t^2$, $y = t^3 - t$, find $\frac{dy}{dx}$.
- 2 Is $\frac{d^2y}{dx^2}$ equal to $\frac{d^2y/dt^2}{d^2x/dt^2}$?
- 3 Give the speed of $\vec{r}(t) = \langle x(t), y(t) \rangle$.
- 4 Write the area swept by $r = f(\theta)$ from α to β .



Answers 1. $\frac{3t^2-1}{2t}$ 2. no 3. $\sqrt{x'^2 + y'^2}$ 4. $\frac{1}{2} \int_{\alpha}^{\beta} f(\theta)^2 d\theta$