

Differential Equations

AP Calculus BC • Unit 7

AP Calculus BC



Differential Equations

What we will cover

- 1 Modelling with differential equations
- 2 Verifying a solution
- 3 Slope fields
- 4 Separation of variables
- 5 Particular solutions
- 6 Exponential growth & decay
- 7 Logistic growth

$$\frac{dy}{dt} = ky$$

$$y = y_0 e^{kt}$$

1 Modelling

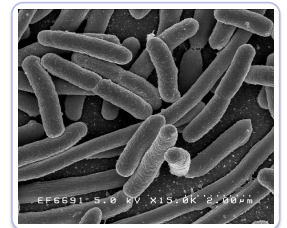
Differential Equations

Modelling

Modelling a situation with a rate

A **differential equation** 微分方程 relates a function to its **own derivatives** – it describes a situation by its **rate of change**. “The rate of change of a quantity is **proportional to its size**” becomes

$$\frac{dy}{dt} = ky.$$



Bacteria growth

Differential Equations

Modelling

Verifying a proposed solution

A **solution** 解 is a **function** that makes the equation true. To verify one: **differentiate** it and substitute – if both sides match, it is a solution.

A differential equation usually has **infinitely many** solutions – a whole family, differing by a constant.

Check $y = 2e^{x^2/2}$ solves $\frac{dy}{dx} = xy$

$$\frac{dy}{dx} = 2e^{x^2/2} \cdot x = x(2e^{x^2/2}) = xy.$$

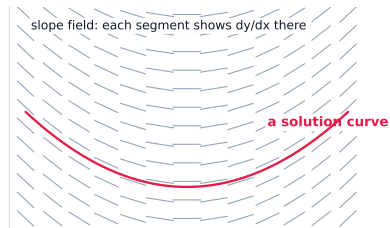
True.

2 Slope fields

Slope fields

A **slope field** 斜率场 draws the equation as a grid of short segments: at each (x, y) the segment has slope $\frac{dy}{dx}$ there.

It shows the **shape** of the solution curves **without solving**.



Reading a slope field

segments **flat** $\Rightarrow \frac{dy}{dx} = 0$: the solution is momentarily **level**

segments **steepen** $\Rightarrow$ the solution rises or falls **faster**

a **horizontal row** of equal slopes $\Rightarrow$ the rate depends only on y

Exam skill: start at the given point and **follow** the segments

A solution curve stays **tangent** to the field everywhere.

3 Separation of variables

Separation of variables

If $\frac{dy}{dx}$ factors into a function of x times a function of y , split them and integrate:

$$\frac{dy}{dx} = g(x)h(y) \Rightarrow \int \frac{dy}{h(y)} = \int g(x) dx.$$

Add the constant of integration **once** – on the x -side.

all y one side,
all x the other

integrate both sides

the **general solution** 通解

Worked example – separate, integrate, then pin it down

Solve $\frac{dy}{dx} = xy$ with $y(0) = 2$

$$\int \frac{dy}{y} = \int x dx \Rightarrow \ln |y| = \frac{x^2}{2} + C$$

$$\Rightarrow y = Ae^{x^2/2}$$

Applying $y(0) = 2$ gives $A = 2$:

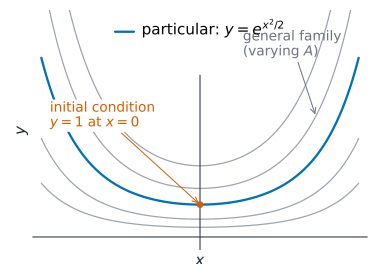
$$y = 2e^{x^2/2}.$$

An **initial condition** 初始条件 (x_0, y_0) picks **one** curve from the family – there is exactly **one** solution through a given point.

From the family to the particular solution

The constant gives a **family** of curves; the initial condition selects the **particular solution** 特解.

Watch **domain restrictions** 定义域限制: keep the **branch containing the initial point** (e.g. the right sign of a square root).



4 Exponential models

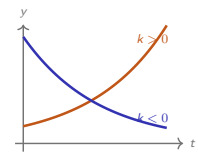
Exponential growth and decay

The most important model: rate proportional to size.

$$\frac{dy}{dt} = ky \implies y = y_0 e^{kt}$$

where y_0 is the value at $t = 0$.

$k > 0$ gives **growth** 增长; $k < 0$ gives **decay** 衰减.



Worked example – radioactive decay and half-life

$$\frac{dy}{dt} = -0.10y \text{ (years)}, y_0 = 50 \text{ g}$$

The solution is $y = 50e^{-0.10t}$, so after 10 years

$$y = 50e^{-1} = 18.4 \text{ g.}$$

Half-life 半衰期: solve $25 = 50e^{-0.10t}$, i.e. $e^{-0.10t} = \frac{1}{2}$:

$$t = \frac{\ln 2}{0.10} = 6.9 \text{ years.}$$

The half-life is **independent** of the starting amount – that is what makes it a useful constant.

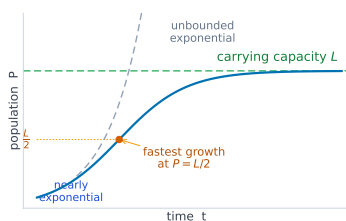
5 Logistic growth

Logistic growth (BC only)

Real populations cannot grow forever. The **logistic** 逻辑斯蒂 model adds a ceiling, the **carrying capacity** 环境容纳量 L :

$$\frac{dP}{dt} = kP \left(1 - \frac{P}{L} \right).$$

Growth is fastest at $P = \frac{L}{2}$, and $P \rightarrow L$ as $t \rightarrow \infty$.



6 Recap

Unit 7 – key takeaways

1 Modelling

a sentence about a rate becomes an equation in dy/dt

2 Slope fields

segments show the slope; a solution follows them

3 Separation

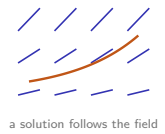
split the variables, integrate, add $+C$ once

4 Exponentials

$dy/dt = ky \Rightarrow y = y_0 e^{kt}$ – growth or decay

Check yourself

- 1 Write "the rate of decay is proportional to the amount".
- 2 Solve $\frac{dy}{dx} = xy$ in general form.
- 3 How many times do you add $+C$ when separating?
- 4 Does half-life depend on the starting amount?



Answers 1. $\frac{dy}{dt} = -ky$ 2. $y = Ae^{x^2/2}$ 3. once, on the x-side 4. no