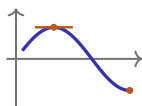


Analytical Applications of Differentiation

AP Calculus BC ■ Unit 5

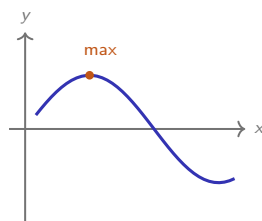
AP Calculus BC



Analytical Applications of Differentiation

What we will cover

- 1 The Mean Value Theorem
- 2 Critical points & extrema
- 3 The first derivative test
- 4 Concavity & inflection
- 5 Connecting f , f' and f''
- 6 Optimization



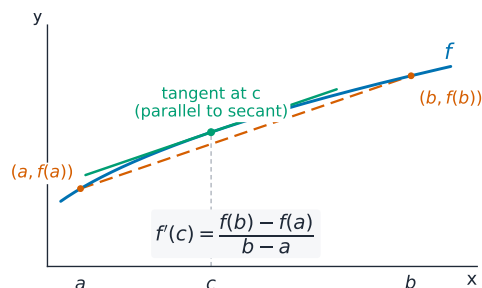
1 Mean Value Theorem

The Mean Value Theorem

If f is **continuous** on $[a, b]$ and **differentiable** on (a, b) , then for some c in (a, b) :

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Somewhere inside, the **instantaneous** rate equals the **average** rate – some tangent is **parallel to the secant**.



Exam skill – justifying the MVT

The **Mean Value Theorem** 中值定理 is an **existence** theorem, so the marks are in the justification:

- 1 state f is continuous on $[a, b]$ and differentiable on (a, b)
- 2 compute $\frac{f(b) - f(a)}{b - a}$
- 3 conclude “by the MVT there is a c in (a, b) with $f'(c)$ equal to that value”

Both hypotheses must be named – dropping one loses the mark even with the right c .

Verify the MVT for $f(x) = x^2$ on $[1, 3]$

A polynomial, so continuous and differentiable everywhere. Average rate $\frac{9 - 1}{2} = 4$; solve $f'(c) = 2c = 4$ to get $c = 2$, which lies in $(1, 3)$.

2 Critical points

Extreme values and critical points

The **Extreme Value Theorem** 极值定理: a function **continuous** on a **closed** interval attains **both** an absolute max and an absolute min. A **critical point** 临界点 is an interior point where $f'(x) = 0$ or f' does not exist.

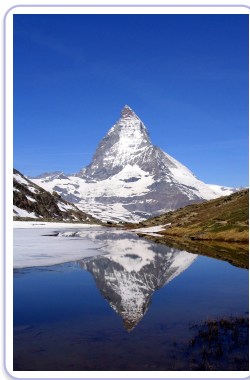


Garden fence

Increasing and decreasing intervals

The **first derivative** says where f rises or falls:

- $f'(x) > 0 \Rightarrow f$ is **increasing** 递增
- $f'(x) < 0 \Rightarrow f$ is **decreasing** 递减



Mountain

extrema

The first derivative test

Classify a critical point c by how f' **changes sign** there:

f' turns $+$ $\rightarrow$ $-$
local maximum
极大值

f' turns $-$ $\rightarrow$ $+$
local minimum
极小值

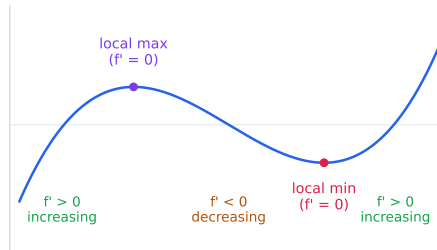
f' does **not** change sign
neither

Always state the **sign change** as your justification.

Worked example – classify the critical points

$$f(x) = x^3 - 3x^2$$

$f'(x) = 3x^2 - 6x = 3x(x - 2)$, zero at $x = 0, 2$. Signs: $f' > 0$ for $x < 0$; $f' < 0$ on $(0, 2)$; $f' > 0$ for $x > 2$.
 $+$ $\rightarrow$ $-$ at $x = 0$: **local max**, $f(0) = 0$.
 $-$ $\rightarrow$ $+$ at $x = 2$: **local min**, $f(2) = -4$.



The candidates test – absolute extrema

On a **closed** interval, absolute extrema occur only at **critical points** or **endpoints**:

- 1 list every critical point in $[a, b]$ and both endpoints
- 2 evaluate f at each candidate
- 3 largest is the absolute max; smallest the absolute min

Show the **table of values** – the comparison *is* the argument.

endpoints a, b
 $+$
 every critical point
 $=$ the full candidate list

3

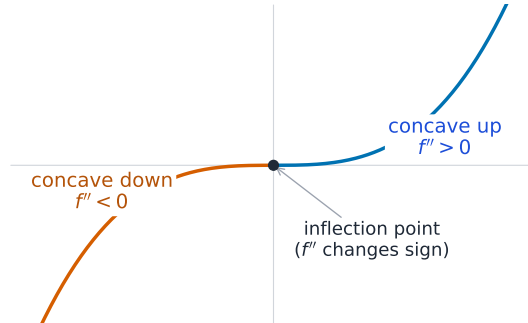
Concavity

Concavity and points of inflection

The **second derivative** describes bending:

- $f'' > 0$: **concave up** 上凹 (f increasing)
- $f'' < 0$: **concave down** 下凹 (f decreasing)

A **point of inflection** 拐点 is where f' **changes sign** – not merely where $f'' = 0$.



The second derivative test

An alternative at a critical point c with $f'(c) = 0$:

$$f''(c) > 0 \Rightarrow \text{local minimum}$$

$$f''(c) < 0 \Rightarrow \text{local maximum}$$

$$f''(c) = 0 \Rightarrow \text{inconclusive}$$

If $f''(c) = 0$, fall back on the **first derivative test**.

Special case: if a continuous function has only **one** critical point on an interval and it is a local extremum, it is the **absolute** extremum there.

Connecting f , f' and f''

A very common setup: you are given the graph of f' and asked about f .

$$f \text{ increasing} \Leftrightarrow f' \text{ above the axis}$$

$$f \text{ has a local max} \Leftrightarrow f' \text{ crosses } + \rightarrow -$$

$$f \text{ concave up} \Leftrightarrow f' \text{ increasing}$$

$$f \text{ has an inflection} \Leftrightarrow f' \text{ has a local extremum}$$

Answer questions about f using the **height** and **slope** of the f' graph.

4 Optimization

Solving an optimization problem

Optimization 最优化 is the same machinery aimed at a real goal:

- 1 write the quantity as a function of **one** variable (use the constraint to eliminate the rest)
- 2 state the allowed interval
- 3 find and **test** the critical points
- 4 answer with **units** and interpretation

constraint $\rightarrow$ one variable



differentiate, set $= 0$



test, then interpret

Worked example – the farmer's fence

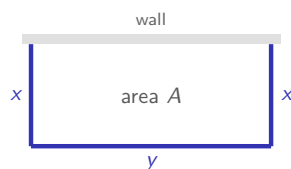
100 m of fence, three sides (a wall is the fourth). Maximize the area.

Ends x , side y : constraint $2x + y = 100$, so $y = 100 - 2x$.

$$A(x) = x(100 - 2x) = 100x - 2x^2$$

$$A'(x) = 100 - 4x = 0 \Rightarrow x = 25$$

$A'' = -4 < 0$, a maximum: $y = 50$ and $A = 1250 \text{ m}^2$.



Behaviour of implicit relations

It all extends to **implicitly** defined relations:

- horizontal tangent where $\frac{dy}{dx} = 0$
- vertical tangent where $\frac{dy}{dx}$ is **undefined**

$\frac{dy}{dx}$ usually involves **both** x and y – substitute it back in when finding $\frac{d^2y}{dx^2}$, then read concavity from the sign.

5

Recap

Unit 5 – key takeaways

1 MVT

some tangent is parallel to the secant – name both hypotheses

2 Critical points

$f' = 0$ or undefined: candidates only, so test each

3 The two tests

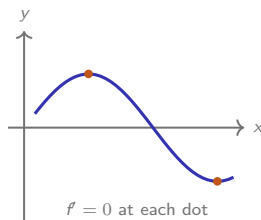
f' sign change, or $f'' > 0$ min / $f'' < 0$ max

4 Justify

cite the sign of f' or f'' – the reasoning earns the marks

Check yourself

- 1 f' turns $-$ to $+$ at c . What is c ?
- 2 $f''(c) = 0$ at a critical point. What does the test tell you?
- 3 Where can an absolute max on $[a, b]$ occur?
- 4 Is every critical point an extremum?



Answers 1. a local minimum 2. inconclusive – use the f' test 3. a critical point or an endpoint 4. no