# Differentiation: Definition & Properties

AP Calculus BC ▪ Unit 2

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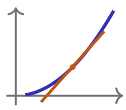
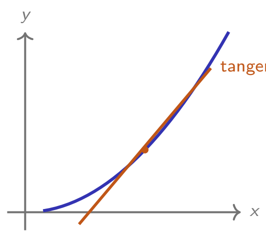




## What we will cover

1. Average vs instantaneous rate
2. The derivative as a limit
3. When a derivative exists
4. Basic derivative rules
5. Product & quotient rules
6. Trig, exp & log derivatives



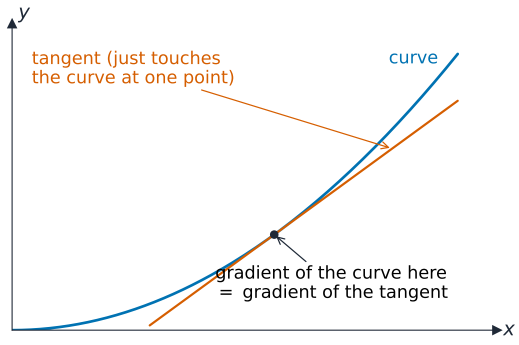

# 1 The derivative

## Average vs instantaneous rate

Over an interval, the average rate is a **difference quotient** 差商:

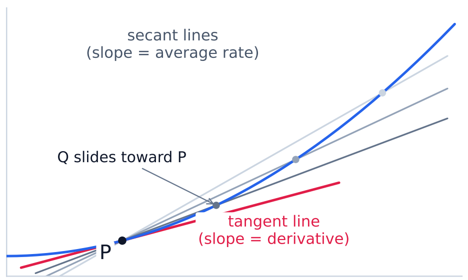
$$\frac{f(a+h)-f(a)}{h} = \frac{f(x)-f(a)}{x-a}.$$

At a point, the **instantaneous** 瞬时 rate is the gradient of the **tangent** 切线.






## From secant to tangent




The remaining trigonometric derivatives are not memorized separately – you rewrite them with **identities** 恒等式 and apply the quotient (or product) rule.

Rewrite $\tan x = \sin x / \cos x$; the quotient rule gives $\frac{d}{dx}\tan x = \sec^2 x$.



## The derivative as a function, and its notation

Let $a$ vary and the derivative becomes a new function:

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}.$$

This is differentiating **by first principles** 用定义求导.

$\frac{dy}{dx}$ $f'(x)$ $y'$

three names for the same thing

Be ready to move between graph, table, formula and words.

## The tangent line – a routine exam task

The tangent at $x = a$ passes through $(a, f(a))$ with slope $f'(a)$:

$$y - f(a) = f'(a)\,(x - a).$$

Keep point-slope form ready – you need a point and a slope, nothing else.

**$f(x) = x^2$ at $x = 3$**

Point: $f(3) = 9$. Slope: $f'(x) = 2x$, so $f'(3) = 6$.

$$y - 9 = 6(x - 3).$$



## Exam skill – estimating $f'$ from a table

With no formula, use a difference quotient over a small interval around $a$:

$$f'(a) \approx \frac{f(b) - f(c)}{b - c}, \qquad c < a < b.$$

Full credit needs the numbers and the **units** 单位 – output per input.

**"Approximate $M'(7.5)$ over $5 \le t \le 10$"**

$$M'(7.5) \approx \frac{M(10) - M(5)}{10 - 5}$$

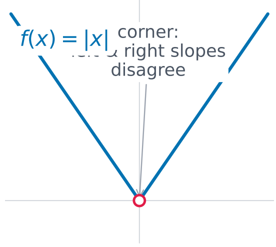
Values on both sides of 7.5 give the best estimate. Answer in units of $M$ per unit of $t$.

# 2 When a derivative exists

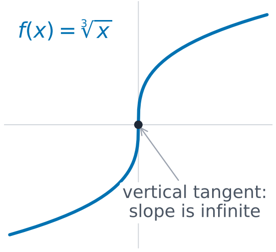
## Differentiable $\Rightarrow$ continuous (not the reverse)

If $f$ is **differentiable** 可导 at a point, then $f$ is **continuous** 连续 there. A continuous function can still fail to be differentiable:

continuous, not differentiable




continuous, not differentiable




> Use the contrapositive: if $f$ is not continuous at $a$, it is not differentiable at $a$.

# 3 Rules



## The power rule

The **power rule** 幂法则 handles any real power:

$$\frac{d}{dx} x^r = r x^{r-1}.$$

> Rewrite as a power first: $\frac{1}{x} = x^{-1}$, $\sqrt{x} = x^{1/2}$.

$$x^5 \to 5x^4$$

$$\frac{1}{x} = x^{-1} \to -x^{-2}$$

$$\sqrt{x} = x^{1/2} \to \tfrac{1}{2}x^{-1/2}$$

## Constant, multiple, sum & difference

| Constant | Constant multiple | Sum / difference |
|---|---|---|
| $\frac{d}{dx}k = 0$ | $\frac{d}{dx}(kf) = kf'$ | $(f \pm g)' = f' \pm g'$ |

**Differentiate a polynomial term by term**

$$\frac{d}{dx}\left(4x^3 - 5x + 7\right) = 12x^2 - 5.$$

The constant 7 contributes nothing – it does not change.



## Four building blocks to know by heart

| | | | |
|---|---|---|---|
| $\frac{d}{dx}\sin x = \cos x$ | $\frac{d}{dx}\cos x = -\sin x$ | $\frac{d}{dx}e^x = e^x$ | $\frac{d}{dx}\ln x = \frac{1}{x}$ |

Note the minus sign on $\cos$, and that $e^x$ is its own derivative.



## A limit that is really a derivative

If a limit has the shape of the definition

$$\lim_{h\to 0}\frac{f(a+h) - f(a)}{h},$$

do not expand it – just evaluate $f'(a)$.

**Evaluate the limit**

$$\lim_{h\to 0}\frac{\sin\left(\frac{\pi}{2} + h\right) - 1}{h}$$

This is $\displaystyle \frac{d}{dx}\sin x\Big|_{x=\pi/2} = \cos\frac{\pi}{2} = 0.$

## The product rule

A product is not the product of the derivatives. Use the **product rule** 乘积法则:

$$(uv)' = u'v + uv'.$$

> "Derivative of the first times the second, plus the first times the derivative of the second."

**Differentiate $f(x) = x^2 \sin x$**

$u = x^2$ ($u' = 2x$), $v = \sin x$ ($v' = \cos x$):

$$f'(x) = 2x\sin x + x^2 \cos x.$$

Read the structure first – a product needs the product rule.



## The quotient rule

**Quotient rule** 商法则:

$$\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}.$$

> The order matters – there is a minus sign. Write the numerator carefully.

**Differentiate $\dfrac{x}{\cos x}$**

$$\frac{1 \cdot \cos x - x \cdot (-\sin x)}{\cos^2 x} = \frac{\cos x + x\sin x}{\cos^2 x}.$$



## $\tan x$, $\cot x$, $\sec x$, $\csc x$ – derive, do not memorise

Rewrite with **identities** 恒等式, then use the quotient rule:

**$\tan x = \dfrac{\sin x}{\cos x}$**

$$\frac{\cos x \cos x - \sin x(-\sin x)}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x.$$

$\tan x \to \sec^2 x$

$\cot x \to -\csc^2 x$

$\sec x \to \sec x \tan x$

$\csc x \to -\csc x \cot x$

## Higher-order derivatives

Differentiating $f'$ again gives the **second derivative** 二阶导数:

$$f''(x) = \frac{d^2y}{dx^2}.$$

It is the rate of change of the rate of change.

> "Find $k''(3)$" just means differentiate twice, then substitute.

$f(x)$
↓
$f'(x)$ slope
↓
$f''(x)$ slope of slope

# 4 Recap



## Unit 2 – key takeaways

**1 The derivative**
the limit of the secant slope – the tangent's gradient

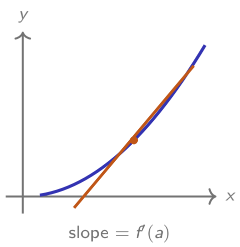
**2 Existence**
differentiable $\Rightarrow$ continuous; corners and vertical tangents fail

**3 The rules**
power, constant multiple, sum; product and quotient

**4 Building blocks**
$\sin$, $\cos$, $e^x$, $\ln x$ – know them by heart

## Check yourself

1. Differentiate $4x^3 - 5x + 7$.
2. $f(x) = |x|$ is continuous at $0$. Is it differentiable there?
3. Differentiate $x^2 \sin x$.
4. Write the tangent to $y = x^2$ at $x = 3$.




**Answers** **1.** $12x^2 - 5$ **2.** no – a corner **3.** $2x \sin x + x^2 \cos x$ **4.** $y - 9 = 6(x - 3)$