Limits & Continuity

AP Calculus BC ▪ Unit 1

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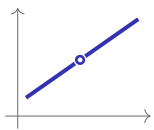


What we will cover

1. Change at an instant
2. Reading a limit
3. Computing limits
4. The squeeze theorem
5. Discontinuity & continuity
6. Asymptotes
7. The Intermediate Value Theorem

1 Change at an instant

Can change occur at an instant?

A speedometer reads 60 km/h. But at a single instant no time passes and no distance is covered:

$$\text{speed} = \frac{0}{0} \text{ – undefined.}$$

The clever move: do not plug in zero. Watch what the average rate **approaches** 趋近 as the interval shrinks.

That approaching value is a **limit** 极限 – it powers the derivative (Unit 2) and, in reverse, the integral (Unit 6).

What calculus is for, and the words limits are described in

calculus	the mathematics of **change** 变化 and of **accumulation** 累积
the two questions	how fast is something changing at an instant, and how much accumulates over an interval
a limit	the value $f(x)$ approaches as x approaches a – and it ignores the point itself
the graph's distinction	a hole means the limit exists and the value does not; a filled dot elsewhere means they differ
average rate of change 平均变化率	the slope of a secant; the derivative is the limit of it
not tested	the epsilon–delta definition is not on the AP exam, so it is not used here

"Ignore the point itself" is the whole idea. A limit describes the approach, and a function can be undefined exactly where its limit is clearest.

2 Reading a limit

Defining a limit

$\lim_{x \to c} f(x) = R$ means $f(x)$ can be made **arbitrarily** 任意地 close to R by taking x **sufficiently** 足够 close to c – but not equal to c.

> A limit describes the behaviour near c, not the value at c.

graphically 用图象

numerically 用数值

analytically 用解析式

Moving between the three is a core skill.

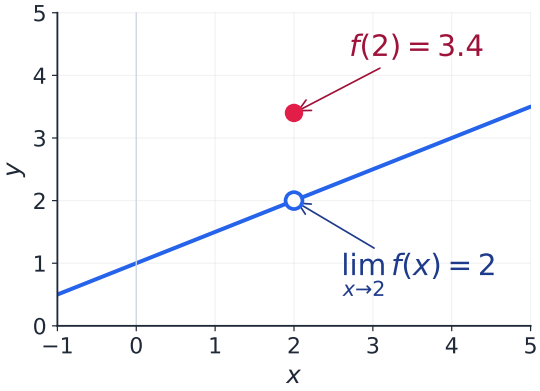
Reading a limit from a graph

Trace toward c from each side:

- from the left: **left-hand limit** 左极限
- from the right: **right-hand limit** 右极限

If both head to the same height, the two-sided limit exists.

> **Open circle** 空心圆: a height approached but not reached. **Closed circle** 实心圆: the actual value.



When a limit does not exist

Jump 跳跃

the two sides disagree

$$\lim_{x\to 0}\frac{|x|}{x} \text{ DNE}$$

Unbounded 无界

grows without limit

$$\lim_{x\to 0}\frac{1}{x^2} = \infty$$

Oscillates 振荡

forever near c

$$\lim_{x\to 0}\sin\frac{1}{x} \text{ DNE}$$

Watch the **scale** 比例: a zoomed-out graph can hide behaviour near a point – confirm with algebra.

Estimating a limit from a table

Estimate $\lim_{x \to 2} \dfrac{x^2 - 4}{x - 2}$ **(it is** $\frac{0}{0}$ **at** $x = 2$**)**

x	1.9	1.99	1.999	$\to 2 \leftarrow$	2.001	2.01	2.1
$f(x)$	3.9	3.99	3.999	?	4.001	4.01	4.1

From the left *and* from the right, $f(x)$ approaches **4**.

A table only suggests a value: it is a numerical estimate, not a proof.

3 Computing limits

The limit theorems

If both limits exist, the limit of a combination is that combination of the limits:

$$\lim(f \pm g) = \lim f \pm \lim g$$

$$\lim(fg) = \lim f \cdot \lim g$$

$$\lim \frac{f}{g} = \frac{\lim f}{\lim g} \quad \text{(bottom} \neq 0\text{)}$$

$$\lim g(f(x)) = g(\lim f(x)) \quad (g \text{ continuous})$$

The practical rule: try **direct substitution** 直接代入 first. A real number out means you are done.

The vocabulary a limit question uses

Piecewise-defined function	A **piecewise-defined function** uses different rules on different intervals – so the one-sided limits at a join must be checked separately.
Composite	For a **composite** $f(g(x))$, the limit passes inside when f is continuous there: $\lim f(g(x)) = f(\lim g(x))$.
Radical	A **radical** expression with an indeterminate form usually needs the conjugate – multiply top and bottom by it and the difference of squares clears the root.
Continuous on an interval	A function is **continuous on an interval** when it is continuous at every point of it – and one-sided at the endpoints.
Existence theorem	An **existence theorem** – the Intermediate Value Theorem, the Extreme Value Theorem – guarantees that something exists without saying where.

The **instantaneous rate of change** is the limit of an average rate as the interval shrinks.

When substitution gives $\frac{0}{0}$

$\frac{0}{0}$ is an **indeterminate form** 未定式 – it does not mean the limit fails. Rewrite into an **equivalent form** 等价形式, then substitute.

> The cancelled factor is why the graph had a hole: the two functions agree everywhere except at c.

Three standard moves

Factor and cancel 因式分解并约分:

$$\lim_{x\to 2}\frac{(x-2)(x+2)}{x-2}=\lim_{x\to 2}(x+2)=4$$

Multiply by the **conjugate** 共轭:

$$\lim_{x\to 0}\frac{\sqrt{x+1}-1}{x}=\frac{1}{2}$$

Or use a trig identity.

Choosing a procedure

1. substitute – a real answer means done

2. $\frac{0}{0}$? rewrite, then substitute

3. $\frac{5}{0}$? infinite or DNE – check each side

4. $x \to \pm\infty$? compare the **dominant** 主导 terms

5. trapped between two functions? squeeze theorem

Choosing a method for a limit

1. substitute	if it gives a number, that is the limit – for a continuous function
2. factor and cancel	for $0/0$ from a common factor
3. rationalise	multiply by the conjugate when a surd causes the $0/0$
4. use identities	alternate forms of trigonometric functions often simplify what algebra cannot
5. squeeze	trap the function between two with the same limit
6. compare degrees	for a limit at infinity, the leading terms decide

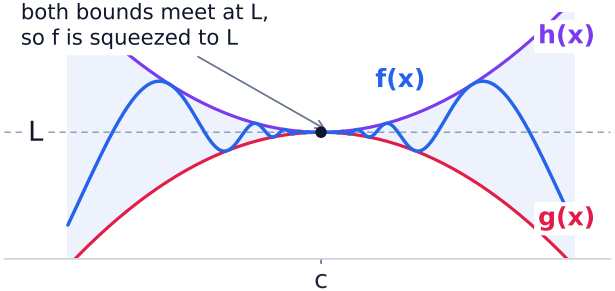
This is a skill topic, not new content: choose the right tool for the limit in front of you. And the same limit lives in a graph, a table and an algebraic form – move between them.

The squeeze theorem

If $g(x) \leq f(x) \leq h(x)$ near c and both bounds meet at L, then f has nowhere to go:

$$\lim_{x \to c} f(x) = L$$

The two famous results proved this way: $\lim_{x \to 0} \frac{\sin x}{x} = 1$ and $\lim_{x \to 0} \frac{1 - \cos x}{x} = 0$.

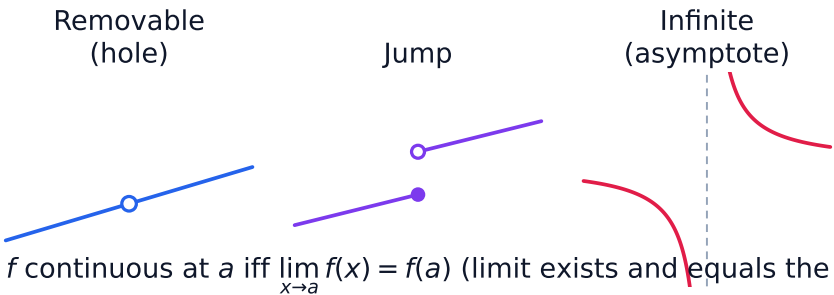


if $g \leq f \leq h$ near c and $\lim g = \lim h = L$, then $\lim f = L$

4 Continuity

Three types of discontinuity

A function is **discontinuous** 间断 at c when its graph breaks there:



f continuous at a iff $\lim_{x \to a} f(x) = f(a)$ (limit exists and equals the value)

Removable 可去间断 (a hole) ▪ **jump** 跳跃间断 (sides disagree) ▪ **infinite** 无穷间断 (an asymptote).

Continuity at a point – the three-part test

f is **continuous** 连续 at $x = c$ exactly when all three hold:

$f(c)$ **exists**

$\lim_{x \to c} f(x)$ **exists**

they **agree**: $\lim_{x \to c} f(x) = f(c)$

> The point is there, the limit is there, and the two agree. If any one fails, f is discontinuous.

Learn it as a checklist – it is the backbone of nearly every continuity question.

Continuity over an interval, and repairing a hole

You rarely check point by point – whole families are continuous on their domains:

> Polynomial, rational, power, **exponential** 指数, **logarithmic** 对数 and **trigonometric** 三角 functions are continuous at every point of their domains.

So a rational function is continuous everywhere except where its denominator is zero.

Removing a discontinuity

If the limit exists at a hole, redefine the function there to equal the limit – the graph is repaired.

For a **piecewise** 分段函数 function at a boundary, solve for the **parameter** 参数 that makes the pieces meet:

$$\lim_{x \to c^-} f = \lim_{x \to c^+} f = f(c).$$

5 Asymptotes

Infinite limits and vertical asymptotes

Where a non-zero number is divided by something approaching 0, the graph hugs the line $x = c$ – a **vertical asymptote** 垂直渐近线.

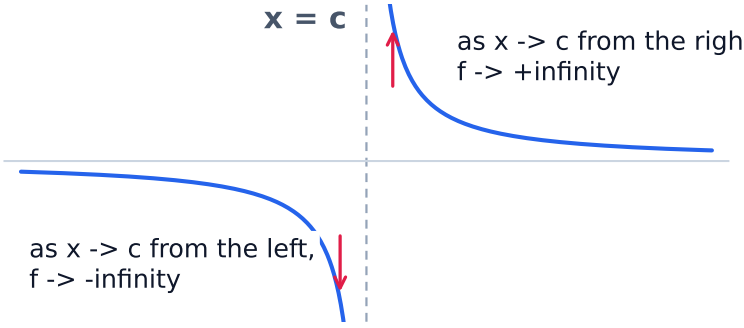
> Always check each side: the two sides can shoot opposite ways.

End behaviour, from the leading terms

Relative magnitudes	At infinity only the **relative magnitudes** matter: exponentials beat any polynomial, and a polynomial beats any logarithm.
Equal degrees	When numerator and denominator have the same degree, the limit is the **ratio of the leading coefficients** – and that ratio is the horizontal asymptote.
Unequal degrees	Numerator smaller: the limit is 0. Numerator larger: it grows without bound, and polynomial division gives the slant asymptote.
Naming it	That long-run **behavior** – the **end behavior** – is what a limit at infinity reports.

Divide every term by the highest power in the denominator. Every rational end-behaviour limit falls out of that one step.

Infinite limits and vertical asymptotes, in detail



vertical asymptote $x = c$ when $\lim_{x \to c^+} f(x) = \pm\infty$ or $\lim_{x \to c^-} f(x) = \pm\infty$

Where a non-zero number is divided by something approaching 0, the graph hugs the line $x = c$ – a **vertical asymptote** 垂直渐近线.

Exploring Types of Discontinuities

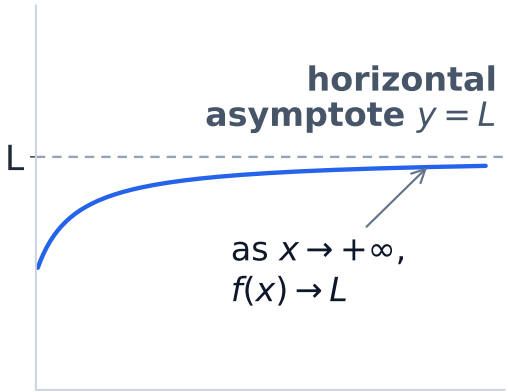
A function is **discontinuous** 间断 at c when its graph "breaks" there.

- **Removable discontinuity** 可去间断 – a single hole. The two-sided limit exists, but the point is missing or misplaced.
- **Jump discontinuity** 跳跃间断 – the two one-sided limits exist but disagree, so the curve jumps.
- **Infinite discontinuity** 无穷间断 – the function blows up to $\pm\infty$ at a **vertical asymptote** 垂直渐近线.

Limits at infinity and horizontal asymptotes

Let the input grow: **end behaviour** 末端行为 as $x \to \pm\infty$. If outputs settle toward L, then $y = L$ is a **horizontal asymptote** 水平渐近线.

> Far out, an exponential beats any polynomial, and a polynomial beats any logarithm.



Rational functions – compare the degrees

top $<$ bottom
limit is 0
(asymptote $y = 0$)

top $=$ bottom
the **ratio of leading coefficients** 首项系数之比

top $>$ bottom
unbounded – no horizontal asymptote

"As $t \to \infty$, where does the rate settle?" is answered with a limit at infinity.

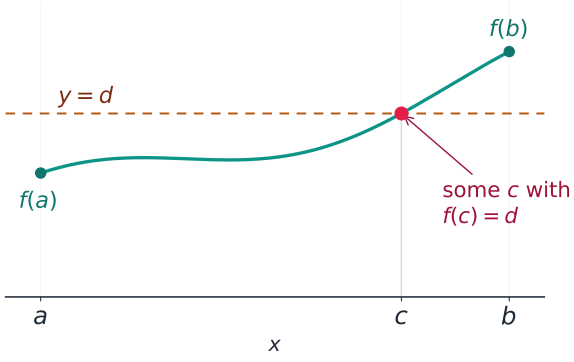
6

IVT

The Intermediate Value Theorem

If f is continuous on $[a, b]$ and d lies between $f(a)$ and $f(b)$, then $f(c) = d$ for at least one c in (a, b).

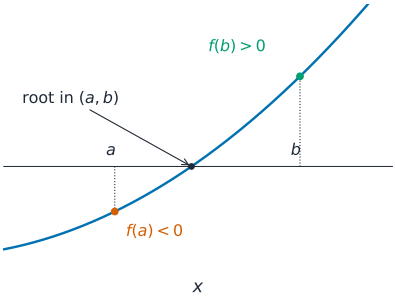
> An unbroken curve cannot skip a height between its endpoints.



Exam skill – justifying with the IVT

These appear almost every year. A full-credit justification has three moves:

1. state continuity on $[a, b]$
2. show d is trapped: $f(a) < d < f(b)$
3. conclude by name: "by the IVT there is a c in (a, b) with $f(c) = d$"



Skipping the continuity statement – or not showing d is between the endpoints – loses the point.

Worked example – a hole at $x = 3$

Evaluate $\lim_{x \to 3} \dfrac{x^2 - 9}{x - 3}$

Direct substitution gives $\frac{0}{0}$, so factor:

$$\frac{(x-3)(x+3)}{x-3} = x + 3 \quad (x \neq 3)$$

$$\lim_{x \to 3} (x + 3) = 6.$$

The graph has a removable discontinuity at $x = 3$ – the limit exists even though the function is undefined there.

Defining Limits and Using Limit Notation

A smooth bridge curve: limits describe the value a graph approaches as we zoom in

Check yourself, in detail

Q1 – the $\frac{0}{0}$ form	Substitution gives $\frac{0}{0}$, which is a signal to rewrite, not a failure. Factor and cancel: $\frac{(x-3)(x+3)}{x-3} = x+3$ for $x \neq 3$, so the limit is $3+3=6$.
Q2 – limit versus value	No. The two agree only when f is continuous at c. The hole graph is the counter-example: the curve approaches one height while the filled dot sits at another.
Q3 – the three parts	$f(c)$ exists; $\lim_{x\to c} f(x)$ exists; the two are equal. Each way of failing names a discontinuity – a missing or misplaced value, one-sided limits that disagree, or an unbounded side.
Q4 – the IVT	Continuity on the closed interval $[a, b]$, and the target d trapped between $f(a)$ and $f(b)$. Then conclude by name – an existence claim with no theorem behind it earns nothing.

All four turn on one distinction: a limit is about the approach, a value is about the point. Continuity is the sentence that says the two coincide.

Exam tips

- A limit describes what $f(x)$ **approaches**, which need not equal $f(a)$ – the two-sided limit exists only if both sides agree.
- Try direct substitution first; for a $\frac{0}{0}$ form, **factor and cancel** or rationalise before substituting.
- A function is **continuous** at a when the limit exists, $f(a)$ is defined, and they are equal.
- Use the **Intermediate Value Theorem** to guarantee a root: a continuous function that changes sign on $[a, b]$ takes every value between.
- Read horizontal asymptotes from end behaviour (limits at $\pm\infty$) and vertical asymptotes where the denominator (not the numerator) is zero.

7 Recap

Unit 1 – key takeaways

1 A limit

what f approaches near c – not the value at c

2 Computing

substitute first; on $0/0$ factor, cancel or rationalise

3 Continuity

value exists, limit exists, and the two agree

4 The IVT

continuous + d trapped $\Rightarrow$ some c hits d

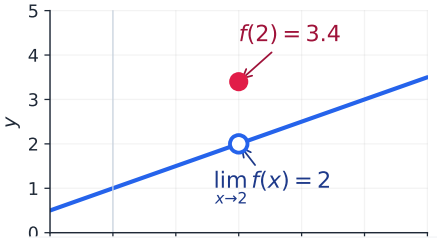
Check yourself

1. Evaluate $\lim_{x\to 3} \frac{x^2-9}{x-3}$.

2. Must $\lim_{x\to c} f(x)$ equal $f(c)$?

3. Name the three parts of the continuity test.

4. Which two conditions must an IVT justification state?



limit exists when nearby values approach L
even if $f(a) \neq L$ (or $f(a)$ is undefined)

Answers **1.** 6 **2.** no – only if f is continuous at c **3.** $f(c)$ exists, the limit exists, they agree **4.** continuity, and d between $f(a)$ and $f(b)$