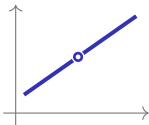


Limits & Continuity

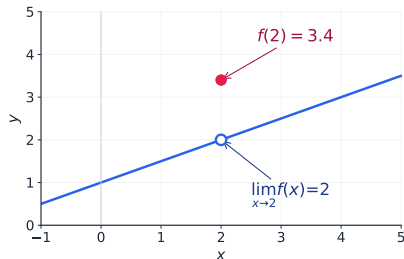
AP Calculus BC ■ Unit 1

AP Calculus BC



What we will cover

- 1 Change at an instant
- 2 Reading a limit
- 3 Computing limits
- 4 The squeeze theorem
- 5 Discontinuity & continuity
- 6 Asymptotes
- 7 The Intermediate Value Theorem



Can change occur at an instant?

A speedometer reads 60 km/h. But **at a single instant** no time passes and no distance is covered:

$$\text{speed} = \frac{0}{0} - \text{undefined.}$$

The clever move: do **not** plug in zero. Watch what the average rate **approaches** 趋近 as the interval shrinks.



bridge limit

That approaching value is a **limit** 极限 – it powers the **derivative** (Unit 2) and, in reverse, the **integral** (Unit 6).

Defining a limit

$\lim_{x \rightarrow c} f(x) = R$ means $f(x)$ can be made
arbitrarily 任意地 close to R by taking x
sufficiently 足够 close to c – but not equal
to c .

A limit describes the **behaviour near** c ,
not the value at c .

graphically 用图象

numerically 用数值

analytically 用解析式

Moving between the three is a core skill.

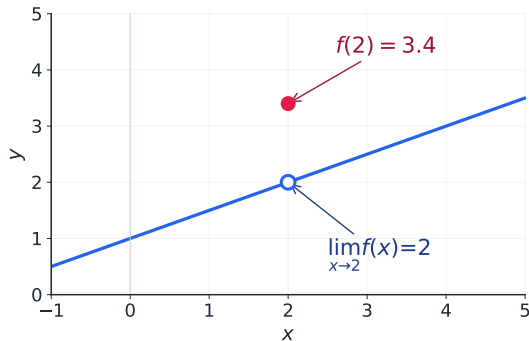
Reading a limit from a graph

Trace toward c from each side:

- from the left: **left-hand limit** 左极限
- from the right: **right-hand limit** 右极限

If both head to the **same** height, the two-sided limit exists.

Open circle 空心圆: a height approached but **not reached**. **Closed circle** 实心圆: the actual value.



When a limit does not exist

Jump 跳跃

the two sides **disagree**

$$\lim_{x \rightarrow 0} \frac{|x|}{x} \text{ DNE}$$

Unbounded 无界

grows without limit

$$\lim_{x \rightarrow 0} \frac{1}{x^2} = \infty$$

Oscillates 振荡

forever near c

$$\lim_{x \rightarrow 0} \sin \frac{1}{x} \text{ DNE}$$

Watch the **scale** 比例: a zoomed-out graph can hide behaviour near a point – confirm with algebra.

Estimating a limit from a table

Estimate $\lim_{x \rightarrow 2} \frac{x^2 - 4}{x - 2}$ (it is $\frac{0}{0}$ at $x = 2$)

x	1.9	1.99	1.999	$\rightarrow 2 \leftarrow$	2.001	2.01	2.1
$f(x)$	3.9	3.99	3.999	?	4.001	4.01	4.1

Both sides march toward 4.

A table only **suggests** a value – it is a numerical **estimate**, not a proof.

The limit theorems

If both limits exist, the limit of a combination is that combination of the limits:

$$\lim(f \pm g) = \lim f \pm \lim g$$

$$\lim(fg) = \lim f \cdot \lim g$$

$$\lim \frac{f}{g} = \frac{\lim f}{\lim g} \quad (\text{bottom} \neq 0)$$

$$\lim g(f(x)) = g(\lim f(x)) \quad (g \text{ continuous})$$

The practical rule: try **direct substitution** 直接代入 **first**. A real number out means you are done.

When substitution gives $\frac{0}{0}$

$\frac{0}{0}$ is an **indeterminate form** 未定式 – it does **not** mean the limit fails.

Rewrite into an **equivalent form** 等价形式, then substitute.

The cancelled factor is **why the graph had a hole**: the two functions agree everywhere except at c .

Three standard moves

Factor and cancel 因式分解并约分:

$$\lim_{x \rightarrow 2} \frac{(x-2)(x+2)}{x-2} = \lim_{x \rightarrow 2} (x+2) = 4$$

Multiply by the **conjugate** 共轭:

$$\lim_{x \rightarrow 0} \frac{\sqrt{x+1} - 1}{x} = \frac{1}{2}$$

Or use a **trig identity**.

Choosing a procedure

1. substitute – a real answer means **done**

2. $\frac{0}{0}$? **rewrite**, then substitute

3. $\frac{\infty}{0}$? infinite or DNE
– **check each side**

4. $x \rightarrow \pm\infty$? compare the **dominant** 主导 terms

5. trapped between two functions? **squeeze theorem**

The squeeze theorem

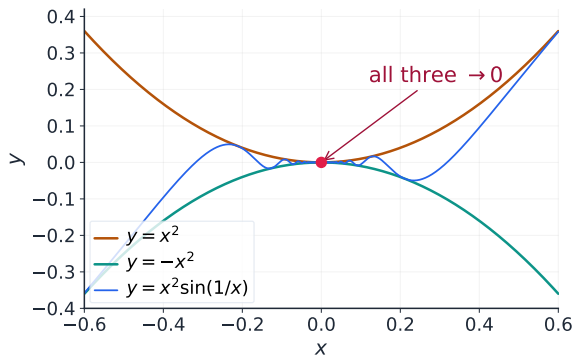
If $g(x) \leq f(x) \leq h(x)$ near c and both bounds meet at L , then f has nowhere to go:

$$\lim_{x \rightarrow c} f(x) = L.$$

The two famous results proved

this way: $\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$ and

$$\lim_{x \rightarrow 0} \frac{1 - \cos x}{x} = 0.$$



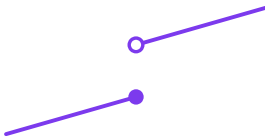
Three types of discontinuity

A function is **discontinuous** 间断 at c when its graph **breaks** there:

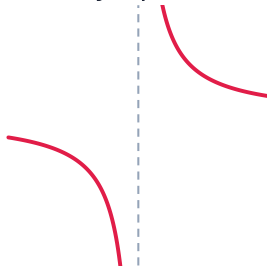
Removable
(hole)



Jump



Infinite
(asymptote)



Removable 可去间断 (a hole) ▪ **jump** 跳跃间断 (sides disagree) ▪ **infinite** 无穷间断 (an asymptote).

Continuity at a point – the three-part test

f is **continuous** 连续 at $x = c$ exactly when **all three** hold:

$f(c)$ **exists**

$\lim_{x \rightarrow c} f(x)$ **exists**

they **agree**: $\lim_{x \rightarrow c} f(x) = f(c)$

The point is there, the limit is there, and the two **agree**. If any one fails, f is discontinuous.

Learn it as a **checklist** – it is the backbone of nearly every continuity question.

Continuity over an interval, and repairing a hole

You rarely check point by point – whole families are continuous **on their domains**:

Polynomial, rational, power, **exponential** 指数, **logarithmic** 对数 and **trigonometric** 三角 functions are continuous at every point of their domains.

So a rational function is continuous everywhere **except** where its denominator is zero.

Removing a discontinuity

If the limit exists at a hole, **redefine** the function there to equal the limit – the graph is repaired.

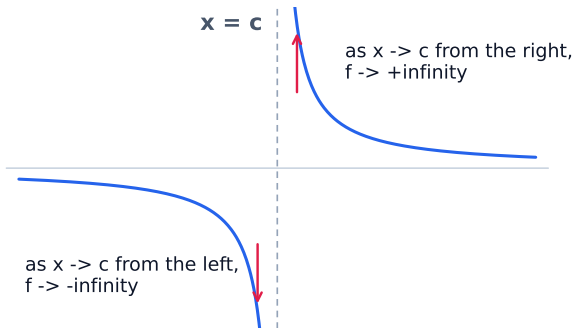
For a **piecewise** 分段函数 function at a boundary, solve for the **parameter** 参数 that makes the pieces **meet**:

$$\lim_{x \rightarrow c^-} f = \lim_{x \rightarrow c^+} f = f(c).$$

Infinite limits and vertical asymptotes

Where a **non-zero number** is divided by something approaching 0, the graph hugs the line $x = c$ – a **vertical asymptote** 垂直渐近线.

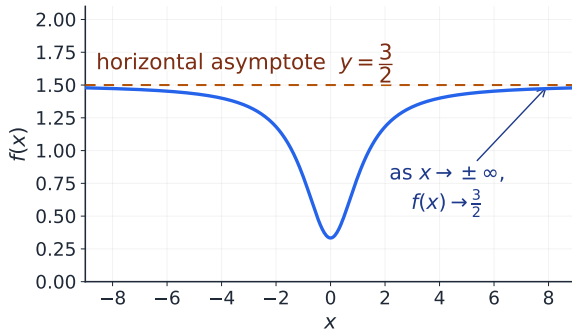
Always check **each side**: the two sides can shoot **opposite** ways.



Limits at infinity and horizontal asymptotes

Let the **input** grow: **end**
behaviour 末端行为 as $x \rightarrow \pm\infty$. If
outputs settle toward L , then $y = L$ is
a **horizontal asymptote** 水平渐近线.

Far out, an **exponential** beats any
polynomial, and a polynomial beats
any logarithm.



Rational functions – compare the degrees

top < bottom
limit is 0
(asymptote $y = 0$)

top = bottom
the **ratio of
leading coefficients** 首项系数之比

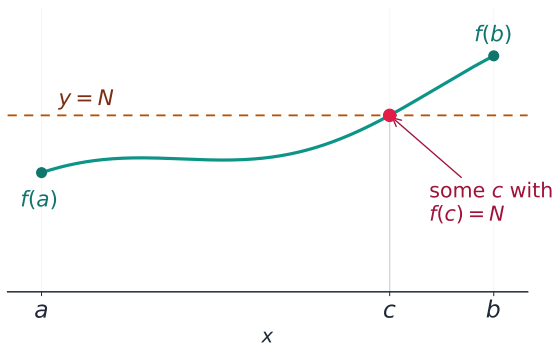
top > bottom
unbounded –
no horizontal
asymptote

“As $t \rightarrow \infty$, where does the rate settle?” is answered with a **limit at infinity**.

The Intermediate Value Theorem

If f is **continuous** on $[a, b]$ and d lies **between** $f(a)$ and $f(b)$, then $f(c) = d$ for at least one c in (a, b) .

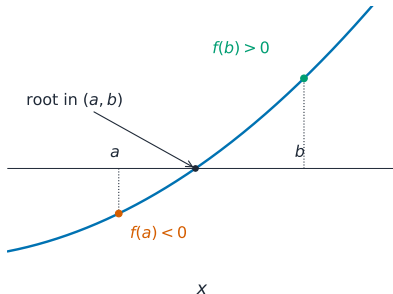
An unbroken curve **cannot skip** a height between its endpoints.



Exam skill – justifying with the IVT

These appear almost every year. A full-credit justification has **three moves**:

- 1 state continuity on $[a, b]$
- 2 show d is trapped: $f(a) < d < f(b)$
- 3 conclude by name: “by the IVT there is a c in (a, b) with $f(c) = d$ ”



Skipping the continuity statement – or not showing d is between the endpoints – **loses the point**.

Worked example – a hole at $x = 3$

Evaluate $\lim_{x \rightarrow 3} \frac{x^2 - 9}{x - 3}$

Direct substitution gives $\frac{0}{0}$, so factor:

$$\frac{(x - 3)(x + 3)}{x - 3} = x + 3 \quad (x \neq 3)$$

$$\lim_{x \rightarrow 3} (x + 3) = 6.$$

The graph has a **removable discontinuity** at $x = 3$ – the limit exists even though the function is **undefined** there.

Unit 1 – key takeaways

1 A limit

what f approaches near c – not the value at c

2 Computing

substitute first; on $0/0$ factor, cancel or rationalise

3 Continuity

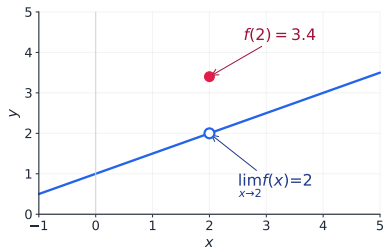
value exists, limit exists, and the two agree

4 The IVT

continuous + d trapped $\Rightarrow$ some c hits d

Check yourself

- 1 Evaluate $\lim_{x \rightarrow 3} \frac{x^2 - 9}{x - 3}$.
- 2 Must $\lim_{x \rightarrow c} f(x)$ equal $f(c)$?
- 3 Name the three parts of the continuity test.
- 4 Which two conditions must an IVT justification state?



Answers 1. 6 2. no – only if f is continuous at c 3. $f(c)$ exists, the limit exists, they agree 4. continuity, and d between $f(a)$ and $f(b)$