

These cover the material BC adds to AB: series, parametric and polar, the extra integration techniques, and the error bounds. Each solution names the test or technique before using it, because on this exam the name is a scoring line of its own. 每题先写明所用判别法或方法名称, 这本身就是得分点。

Series – convergence

I Determine whether the series with terms $n^2/(2^n)$ converges.

It converges.

By the ratio test, the ratio of consecutive terms is $[(n+1)^2/2^{n+1}] \div [n^2/2^n] = \frac{1}{2} \cdot (n+1)^2/n^2$. As n grows, $(n+1)^2/n^2$ tends to 1, so the limit is $\frac{1}{2}$. Since $\frac{1}{2} < 1$, the series converges absolutely by the ratio test. Naming the test and stating the comparison of the limit with 1 are two separate scored statements; computing the limit alone is not a conclusion.

I Find the interval of convergence of the power series with terms $(x-2)^n/(n \cdot 3^n)$.

Converges for $-1 \leq x < 5$.

Ratio test: the limit of the absolute ratio is $|x-2|/3$, which is below 1 when $|x-2| < 3$, giving 3 as the radius and the open interval $-1 < x < 5$. Then test both endpoints separately, which is the step that carries the marks. At $x = 5$ the series becomes the harmonic series, which diverges. At $x = -1$ it becomes the alternating harmonic series, whose terms decrease to zero, so it converges by the alternating series test. The interval is therefore $-1 \leq x < 5$, closed on the left only.

I Use a Maclaurin series to find the first three non-zero terms of the series for $x \cdot \cos(x^2)$, and hence approximate the integral of that function from 0 to 1.

$x - x^5/2 + x^9/24$; the integral is approximately 0.4176.

Start from the known series $\cos u = 1 - u^2/2! + u^4/4! - \dots$, substitute $u = x^2$, giving $1 - x^4/2 + x^8/24$, then multiply by x : $x - x^5/2 + x^9/24$. Integrate term by term from 0 to 1: $1/2 - 1/12 + 1/240 = 0.5 - 0.0833 + 0.00417 = 0.4176$. Building from a memorised series by substitution is the intended route; differentiating $x \cdot \cos(x^2)$ repeatedly to build a Taylor table is legal but takes four times as long and rarely finishes.

Parametric and polar

I A curve is given by $x = t^2 - 2t$ and $y = t^3 - 3t$. Find dy/dx at $t = 2$ and determine where the tangent is vertical.

$dy/dx = 4.5$ at $t = 2$; the tangent is vertical at $t = 1$.

$dx/dt = 2t - 2$ and $dy/dt = 3t^2 - 3$, so $dy/dx = (3t^2 - 3)/(2t - 2)$. At $t = 2$ this is $(12 - 3)/(4 - 2) = 9/2 = 4.5$. The tangent is vertical where $dx/dt = 0$ while dy/dt is not, so $2t - 2 = 0$ gives $t = 1$; there $dy/dt = 3 - 3 = 0$ as well, so this point needs care – both derivatives vanish, and the limit of dy/dx as t approaches 1 is, by factoring, $3(t+1)/2 \rightarrow 3$. So the tangent is in fact not vertical: it has slope 3. That check is the whole point of the question.

I Find the area enclosed by one petal of the polar curve $r = 2 \sin(3\theta)$.

$\pi/3$.

First find the limits, which is where the credit is: the petal starts and ends where $r = 0$, so $\sin(3\theta) = 0$ at $\theta = 0$ and $\theta = \pi/3$. Then $A = \frac{1}{2} \int r^2 d\theta = \frac{1}{2} \int_0^{\pi/3} 4 \sin^2(3\theta) d\theta = 2 \int_0^{\pi/3} \sin^2(3\theta) d\theta$. Using $\sin^2 = (1 - \cos 6\theta)/2$ gives $\int_0^{\pi/3} (1 - \cos 6\theta) d\theta = [\theta - \sin(6\theta)/6]_0^{\pi/3} = \pi/3$. Integrating over 0 to 2π instead would traverse all three petals and, worse, overlap.

Integration techniques and differential equations

I Evaluate the integral of $x \cdot e^{-x}$ from 0 to infinity.

1.

This is improper, so write it as a limit: $\lim_{b \rightarrow \infty} \int_0^b x \cdot e^{-x} dx$ – the limit notation is a scored step. Integrate by parts with $u = x$ and $dv = e^{-x} dx$, giving $-xe^{-x} - e^{-x}$ evaluated from 0 to b , which is $(-be^{-b}) - e^{-(-b)}) - (0 - 1)$. As b grows, both terms with e^{-b} tend to zero (the exponential beats the linear factor), leaving 1. State that reasoning about the competing growth rates; simply writing “= 1” does not show it.

I Solve $dy/dx = 2xy$ with $y(0) = 3$, and state the interval on which the solution is valid.

$y = 3e^{x^2}$, valid for all real x .

Separate: $dy/y = 2x dx$. Integrate both sides: $\ln|y| = x^2 + C$. Exponentiate: $y = Ae^{x^2}$, where $A = e^C$. Apply the initial condition BEFORE simplifying further: $3 = Ae^0 = A$, so $y = 3e^{x^2}$. Since $y(0)$ is positive and the exponential never vanishes, the solution stays positive and the absolute value can be dropped. It is defined for every real x , so the interval is the whole real line.

I A population follows $dP/dt = 0.04P(1 - P/2000)$ with $P(0) = 300$. State the carrying capacity, the population when growth is fastest, and the long-run limit.

Carrying capacity 2000; fastest growth at $P = 1000$; the limit is 2000.

This is the logistic equation, so read the parameters directly: the carrying capacity is the value of P that makes the bracket vanish, 2000. Growth is fastest at half the carrying capacity, $P = 1000$, which you can confirm by maximising the quadratic $0.04P(1 - P/2000)$ – its vertex is at $P = 1000$. As t grows the population approaches the carrying capacity from below, since it started under it, so the limit is 2000. All three answers come from recognising the form; no solving is required, and attempting to solve it wastes most of the time available.