These cover the material BC adds to AB: series, parametric and polar, the extra integration techniques, and the error bounds. Each solution names the test or technique before using it, because on this exam the name is a scoring line of its own. 每题先写明所用判别法或方法名称，这本身就是得分点。

Series – convergence

Determine whether the series with terms $n^2/(2^n)$ converges.

It converges.

By the ratio test, the ratio of consecutive terms is $[(n+1)^2/2^{n+1}] \div [n^2/2^n] = ½ \cdot (n+1)^2/n^2$. As n grows, $(n+1)^2/n^2$ tends to 1, so the limit is ½. Since ½ < 1, the series converges absolutely by the ratio test. Naming the test and stating the comparison of the limit with 1 are two separate scored statements; computing the limit alone is not a conclusion.

Find the interval of convergence of the power series with terms $(x-2)^n/(n \cdot 3^n)$.

Converges for $-1 \quad x < 5$.

Ratio test: the limit of the absolute ratio is $|x-2|/3$, which is below 1 when $|x-2| < 3$, giving 3 as the radius and the open interval $-1 < x < 5$. Then test both endpoints separately, which is the step that carries the marks. At $x = 5$ the series becomes the harmonic series, which diverges. At $x = -1$ it becomes the alternating harmonic series, whose terms decrease to zero, so it converges by the alternating series test. The interval is therefore $-1 \quad x < 5$, closed on the left only.

Use a Maclaurin series to find the first three non-zero terms of the series for $x \cdot \cos(x^2)$, and hence approximate the integral of that function from 0 to 1.

$x - x^5/2 + x^9/24$; the integral is approximately 0.4176.

Start from the known series $\cos u = 1 - u^2/2! + u^4/4! - \ldots$, substitute $u = x^2$, giving $1 - x^4/2 + x^8/24$, then multiply by x: $x - x^5/2 + x^9/24$. Integrate term by term from 0 to 1: $1/2 - 1/12 + 1/240 = 0.5 - 0.0833 + 0.00417 = 0.4176$. Building from a memorised series by substitution is the intended route; differentiating $x \cdot \cos(x^2)$ repeatedly to build a Taylor table is legal but takes four times as long and rarely finishes.

Parametric and polar

A curve is given by $x = t^2 - 2t$ and $y = t^3 - 3t$. Find dy/dx at $t = 2$ and determine where the tangent is vertical.

$dy/dx = 4.5$ at $t = 2$; the tangent is vertical at $t = 1$.

$dx/dt = 2t - 2$ and $dy/dt = 3t^2 - 3$, so $dy/dx = (3t^2 - 3)/(2t - 2)$. At $t = 2$ this is $(12 - 3)/(4 - 2) = 9/2 = 4.5$. The tangent is vertical where $dx/dt = 0$ while dy/dt is not, so $2t - 2 = 0$ gives $t = 1$; there $dy/dt = 3 - 3 = 0$ as well, so this point needs care – both derivatives vanish, and the limit of dy/dx as t approaches 1 is, by factoring, $3(t+1)/2 \to 3$. So the tangent is in fact not vertical: it has slope 3. That check is the whole point of the question.

Find the area enclosed by one petal of the polar curve $r = 2\sin(3\theta)$.

$\pi/3$.

First find the limits, which is where the credit is: the petal starts and ends where $r = 0$, so $\sin(3\theta) = 0$ at $\theta = 0$ and $\theta = \pi/3$. Then A = ½ $r^2\,d\theta$ = ½ $_0$^($\pi/3$) 4$\sin^2(3\theta)d\theta$ = 2 $_0$^($\pi/3$)$\sin^2(3\theta)d\theta$. Using $\sin^2 = (1 - \cos 6\theta)/2$ gives $_0$^($\pi/3$)$(1 - \cos 6\theta)d\theta$ = [$\theta - \sin(6\theta)/6$]$_0$^($\pi/3$) = $\pi/3$. Integrating over 0 to 2π instead would traverse all three petals and, worse, overlap.

Integration techniques and differential equations

Evaluate the integral of x · e^(−x) from 0 to infinity.

1.

This is improper, so write it as a limit: lim(b→∞) $_0$^b x · e^(−x)dx – the limit notation is a scored step. Integrate by parts with $u = x$ and dv = e^(−x)dx, giving −xe^(−x) − e^(−x) evaluated from 0 to b, which is (−be^(−b) − e^(−b)) − (0 − 1). As b grows, both terms with e^(−b) tend to zero (the exponential beats the linear factor), leaving 1. State that reasoning about the competing growth rates; simply writing "= 1" does not show it.

Solve dy/dx = 2xy with y(0) = 3, and state the interval on which the solution is valid.

y = 3e^(x^2), valid for all real x.

Separate: dy/y = 2x dx. Integrate both sides: ln|y| = x^2 + C. Exponentiate: y = Ae^(x^2), where A = e^C. Apply the initial condition BEFORE simplifying further: 3 = Ae^0 = A, so y = 3e^(x^2). Since y(0) is positive and the exponential never vanishes, the solution stays positive and the absolute value can be dropped. It is defined for every real x, so the interval is the whole real line.

A population follows dP/dt = 0.04P(1 − P/2000) with P(0) = 300. State the carrying capacity, the population when growth is fastest, and the long-run limit.

Carrying capacity 2000; fastest growth at P = 1000; the limit is 2000.

This is the logistic equation, so read the parameters directly: the carrying capacity is the value of P that makes the bracket vanish, 2000. Growth is fastest at half the carrying capacity, P = 1000, which you can confirm by maximising the quadratic 0.04P(1 − P/2000) – its vertex is at P = 1000. As t grows the population approaches the carrying capacity from below, since it started under it, so the limit is 2000. All three answers come from recognising the form; no solving is required, and attempting to solve it wastes most of the time available.