Course Companion

AP Calculus BC

Every topic handout in one volume

全部专题讲义合集

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Introducing Calculus: Can Change Occur at an Instant?

Why it matters. About 2,400 years ago the Greek thinker Zeno argued that a runner can never finish a race: first she must cover half the distance, then half of what is left, then half of that, for ever. Infinitely many steps, he said, cannot be finished. The answer took more than two thousand years: the distances $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots$ get as close to 1 as we like, so their limit is exactly 1. Newton and Leibniz built calculus on this idea in the 1660s-1680s, and Augustin-Louis Cauchy gave it a careful definition in 1821. Every derivative and integral in this course is a limit.



Zeno's halves add up to a limit of 1.

Calculus is the mathematics of **change** 变化 and of **accumulation** 累积. It answers two big questions: how fast is something changing *right now*, and how much has piled up *so far*? Unit 1 builds the one tool both questions rest on – the **limit** 极限.

Start with a puzzle. A car's speedometer reads 60 km/h. What does that mean *at a single instant* 瞬间? Speed is distance over time. But at one instant no time passes and no distance is covered, so the fraction looks like $\frac{0}{0}$ – undefined.

- The **average rate of change** 平均变化率 uses a whole *interval* 区间: the change in one quantity divided by the change in another. It divides by zero, and so is undefined, when the change in the input would be zero.
- The **instantaneous rate of change** 瞬时变化率 is what we want *at a point*. It is the value the average rate **approaches** 趋近 as the interval shrinks toward zero length.

The clever move is not to plug in zero (undefined), but to watch what the average rate *approaches* as the interval gets smaller and smaller. That approaching value is a **limit**. So calculus lets us describe change at an instant – as a limit of average rates over ever-shorter intervals. This one idea powers the **derivative** 导数 (Unit 2) and, run in reverse, the **integral** 积分 (Unit 6). Everything else in this unit defines limits carefully and computes them reliably.

Defining Limits and Using Limit Notation

Given a function f, the limit of $f(x)$ as x approaches c is a real number R if $f(x)$ can be made **arbitrarily** 任意地 close to R by taking x **sufficiently** 足够 close to c – **but not equal to** c. We write

$$\lim_{x \to c} f(x) = R$$

and read it: "the limit of $f(x)$, as x approaches c, equals R."



A smooth bridge curve: limits describe the value a graph approaches as we zoom in

The last words are the heart of a limit: it describes the **behavior** 行为 of f *near* c, not the value *at* c. The function may be undefined at c, or defined but equal to something else – the limit does not care.

A limit can be shown in three ways: **graphically** 用图象, **numerically** 用数值 (a table), and **analytically** 用解析式 (algebra). Learning to move between these representations is a core skill.

*(Note: the epsilon-delta definition of a limit is **not** tested on the AP Exam, so this handout does not use it.)*

Estimating Limit Values from Graphs

A graph is often the fastest way to read a limit. To find $\lim_{x \to c} f(x)$, run your finger along the curve toward $x = c$ from each side and ask: what height is the curve heading for?

- Trace from the **left** (inputs smaller than c): this gives the **left-hand limit** 左极限, $\lim_{x \to c^-} f(x)$.
- Trace from the **right** (inputs larger than c): this gives the **right-hand limit** 右极限, $\lim_{x \to c^+} f(x)$.

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- These are the **one-sided limits** 单侧极限. If both head to the *same* height R, then the two-sided limit exists and $\lim_{x \to c} f(x) = R$.

Crucially, **ignore the point itself**. Graphs mark the difference between the limit and the value:

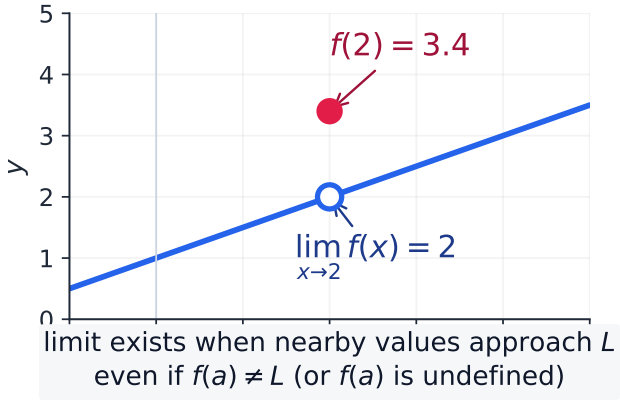
- An **open circle** 空心圆 marks a height the curve *approaches* but does not reach – a "hole" 空洞.
- A **closed circle** 实心圆 marks the actual value $f(c)$.

So a curve may approach $R = 3$ from both sides (limit is 3) while a filled dot sits at height 5 (value $f(c) = 5$). The limit is 3; the two need not match.

A limit **does not exist** (often written **DNE**) when the two sides disagree (a **jump** 跳跃), when the function is **unbounded** 无界 (grows without limit), or when it **oscillates** 振荡 near c with fluctuations that do not shrink toward one value. Oscillation alone does not prevent a limit: $x \sin(1/x) \to 0$ as $x \to 0$. For example:

$$\lim_{x \to 0} \frac{1}{x^2} = \infty, \qquad \lim_{x \to 0} \frac{|x|}{x} \text{ DNE}, \qquad \lim_{x \to 0} \sin\left(\frac{1}{x}\right) \text{ DNE}.$$

Watch the **scale** 比例 of a graph: a zoomed-out picture can hide important behavior near a point, so confirm with algebra when you can.



The open circle is the height the curve approaches (the limit); the filled dot is the actual value $f(2)$ – they need not agree.

Estimating Limit Values from Tables

When you have data or a formula but no picture, a **table** 表格 of values estimates a limit numerically. Choose inputs that creep toward c from both sides and watch the outputs.

For example, to estimate $\lim_{x\to 2}\dfrac{x^2-4}{x-2}$ (which is $\frac{0}{0}$ at $x=2$):

x	1.9	1.99	1.999	$\to 2\leftarrow$	2.001	2.01	2.1
$f(x)$	3.9	3.99	3.999	?	4.001	4.01	4.1

Both sides march toward 4, so we estimate the limit is 4. A table only *suggests* a value – it is a numerical estimate, not a proof.

Determining Limits Using Algebraic Properties of Limits

Most limits are found **analytically** using **limit theorems** 极限定理. If $\lim_{x\to c} f(x)$ and $\lim_{x\to c} g(x)$ both exist, the limit of a combination is the same combination of the limits:

- **Sum / difference:** $\lim\limits_{x\to c}\big[f(x)\pm g(x)\big]=\lim\limits_{x\to c} f(x)\pm\lim\limits_{x\to c} g(x)$
- **Product:** $\lim\limits_{x\to c}\big[f(x)\,g(x)\big]=\lim\limits_{x\to c} f(x)\cdot\lim\limits_{x\to c} g(x)$
- **Quotient:** $\lim\limits_{x\to c}\dfrac{f(x)}{g(x)}=\dfrac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)}$, provided the bottom limit is not 0.
- **Composite** 复合函数: if g is continuous at $\lim_{x\to c} f(x)$, then $\lim\limits_{x\to c} g\big(f(x)\big)=g\Big(\lim\limits_{x\to c} f(x)\Big)$.

The practical rule: for a function built from polynomials, roots, and the like, first try **direct substitution** 直接代入 – put $x=c$ in. If you get a real number, that is the limit. One-sided limits obey the same theorems, read from one direction only.

Determining Limits Using Algebraic Manipulation

Direct substitution sometimes gives the **indeterminate form** 未定式 $\frac{0}{0}$. This does **not** mean the limit fails – it means you must rewrite the function into an **equivalent form** 等价形式 that removes the trouble, then substitute. Three standard moves:

- **Factor and cancel** 因式分解并约分 a **rational function** 有理函数. Example: $\lim\limits_{x\to 2}\dfrac{x^2-4}{x-2}=\lim\limits_{x\to 2}\dfrac{(x-2)(x+2)}{x-2}=\lim\limits_{x\to 2}(x+2)=4$.
- Multiply by the **conjugate** 共轭 to simplify a **radical** 根式. Example: $\lim\limits_{x\to 0}\dfrac{\sqrt{x+1}-1}{x}=\lim\limits_{x\to 0}\dfrac{x}{x\left(\sqrt{x+1}+1\right)}=\dfrac{1}{2}$.
- **Use alternate forms** of trigonometric functions (identities) to simplify.

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The cancelled factor is why the original graph had a hole: the two functions agree everywhere except at $x=c$, so they share the same limit there.

Selecting Procedures for Determining Limits

This is a skill topic, not new content: choose the right tool for the limit in front of you.

1. **Try direct substitution first.** A real answer means you are done.
2. Getting $\frac{0}{0}$? **Rewrite** – factor and cancel, or use the conjugate, or a trig identity – then substitute.
3. A **non-zero number over** 0 (like $\frac{5}{0}$)? The limit is infinite or DNE – check the sign from each side (see vertical asymptotes below).
4. As $x\to\pm\infty$? Compare the **dominant** 主导 terms (see limits at infinity).
5. Trapped between two functions? The **squeeze theorem** may apply.

Determining Limits Using the Squeeze Theorem

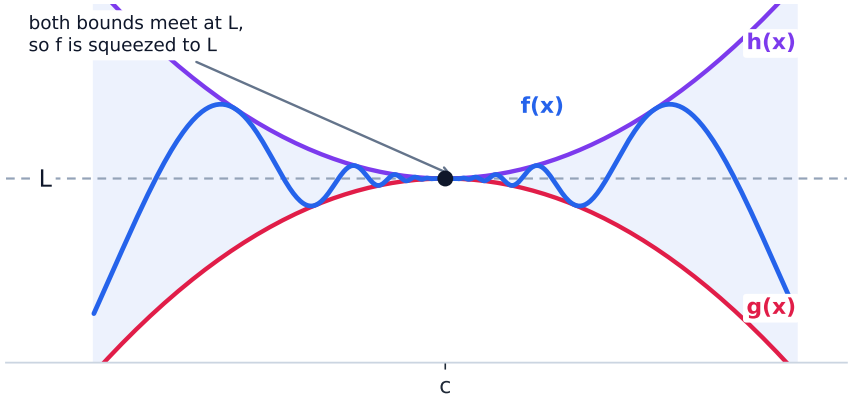
The **squeeze theorem** 夹逼定理 (also called the sandwich theorem) finds a limit by trapping the function between two others. If $g(x)\le f(x)\le h(x)$ near c, and

$$\lim_{x\to c} g(x)=\lim_{x\to c} h(x)=L,$$

then f is squeezed to the same place: $\lim\limits_{x\to c} f(x)=L$.

The two famous results proved this way, both used throughout calculus, are:

$$\lim_{x\to 0}\frac{\sin x}{x}=1 \qquad \text{and} \qquad \lim_{x\to 0}\frac{1-\cos x}{x}=0.$$



if $g \le f \le h$ near c and $\lim g = \lim h = L$, then $\lim f = L$

The Squeeze Theorem traps x squared sin(1/x) between minus x squared and x squared

Connecting Multiple Representations of Limits

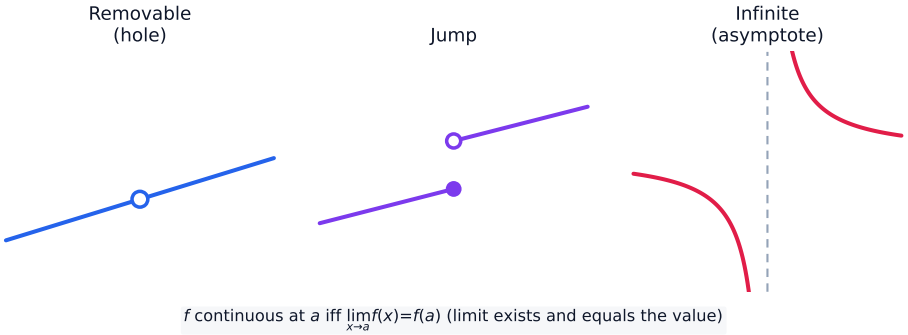
Another skill topic: the same limit lives in a graph, a table, and an algebraic form, and you should be able to translate between them. A graph shows the shape and any holes or jumps; a table gives numerical evidence; algebra gives an exact value and a reason. Strong answers use one representation to *confirm* another.

Exploring Types of Discontinuities

A function is **discontinuous** 间断 at c when its graph "breaks" there. Three common types are:

- **Removable discontinuity** 可去间断 – a single hole. The two-sided limit exists, but the point is missing or misplaced.
- **Jump discontinuity** 跳跃间断 – the two one-sided limits exist but disagree, so the curve jumps.
- **Infinite discontinuity** 无穷间断 – the function blows up to $\pm\infty$ at a **vertical asymptote** 垂直渐近线.

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The three types of discontinuity: removable, jump, and infinite

Defining Continuity at a Point

Continuity is defined by a three-part test. A function f is **continuous** 连续 at $x = c$ exactly when all three hold:

$$\boxed{f(c) \text{ exists} \quad \text{and} \quad \lim_{x\to c} f(x) \text{ exists} \quad \text{and} \quad \lim_{x\to c} f(x) = f(c)}$$

In words: the point is there, the limit is there, and the two agree. If any one fails, f is discontinuous at c. This test is the backbone of nearly every continuity question, so learn it as a checklist.

Confirming Continuity over an Interval

A function is **continuous on an interval** 在区间上连续 if it is continuous at *every* point of that interval. You rarely check point by point, because whole families are continuous on their domains:

> **Polynomial, rational, power, exponential 指数, logarithmic 对数, and trigonometric 三角 functions are continuous at every point of their domains.**

So a rational function is continuous everywhere *except* where its denominator is zero; $\ln x$ is continuous for $x > 0$; and so on. Knowing this lets you declare continuity quickly and correctly.

Removing Discontinuities

If the **limit exists** at a hole, the discontinuity is **removable**: redefine the function at that one point to equal the limit, and the graph is repaired. Formally, set the missing value to $\lim_{x\to c} f(x)$.

For a **piecewise-defined function** 分段函数, continuity at a boundary $x = c$ needs

the two pieces to *meet*: the left piece's value, the right piece's value, and $f(c)$ must all be equal. This is a common exam setup – you solve for a **parameter** 参数 (an unknown constant) that makes the pieces match:

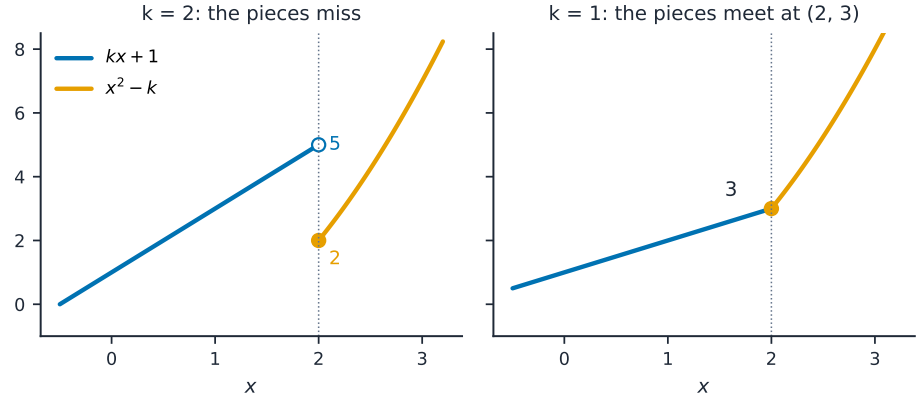
$$\lim_{x\to c^-} f(x) = \lim_{x\to c^+} f(x) = f(c).$$

Worked example. Find the constant k that makes this function continuous at $x = 2$:

$$f(x) = \begin{cases} kx + 1, & x < 2 \\ x^2 - k, & x \geq 2 \end{cases}$$

- Left-hand limit: $\lim_{x\to 2^-} (kx + 1) = 2k + 1$.
- Right-hand limit and value: $\lim_{x\to 2^+} (x^2 - k) = f(2) = 4 - k$.
- Set them equal: $2k + 1 = 4 - k$, so $3k = 3$ and $k = 1$.

Check: with $k = 1$ both pieces give 3 at $x = 2$, so all three conditions of the continuity test hold.



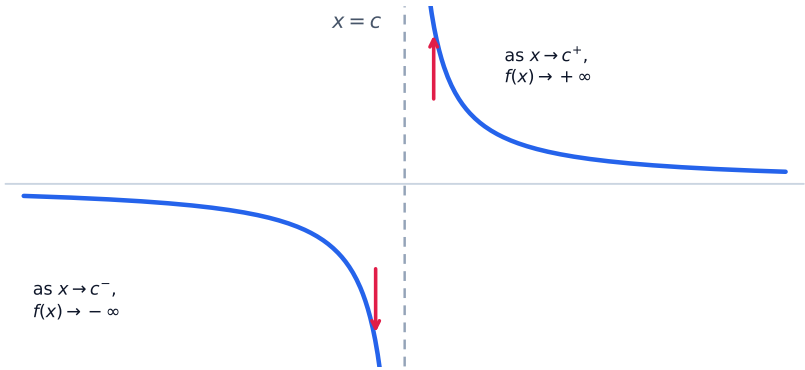
Solve for k so the pieces meet: k = 1 joins them at (2, 3)

Connecting Infinite Limits and Vertical Asymptotes

The idea of a limit extends to **infinite limits** 无穷极限. When a function grows without bound near $x = c$, we write $\lim_{x\to c} f(x) = \pm\infty$. This describes a **vertical asymptote** at $x = c$: the graph hugs the vertical line $x = c$ and shoots off toward $\pm\infty$.

This happens where a **non-zero number is divided by something approaching** 0, such as at a zero of a denominator that does not cancel. Always check each side separately – the two sides can shoot **opposite** ways (one to $+\infty$, one to $-\infty$).

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vertical asymptote $x = c$ when $\lim_{x\to c^+} f(x) = \pm\infty$ or $\lim_{x\to c^-} f(x) = \pm\infty$

Near a vertical asymptote the graph hugs the line $x = c$, and the two sides can shoot to opposite infinities.

Worked example. Find $\lim_{x\to 1^-} \dfrac{x+2}{x-1}$ and $\lim_{x\to 1^+} \dfrac{x+2}{x-1}$.

Substitution gives $\frac{3}{0}$: a non-zero number over 0, so each one-sided limit is infinite. Decide the sign from each side:

- For x just below 1 (try $x = 0.99$): top ≈ 3, bottom $= -0.01$, a small negative number. So $\lim_{x\to 1^-} \dfrac{x+2}{x-1} = -\infty$.
- For x just above 1 (try $x = 1.01$): top ≈ 3, bottom $= +0.01$. So $\lim_{x\to 1^+} \dfrac{x+2}{x-1} = +\infty$.

The line $x = 1$ is a vertical asymptote, and the two-sided limit does not exist.

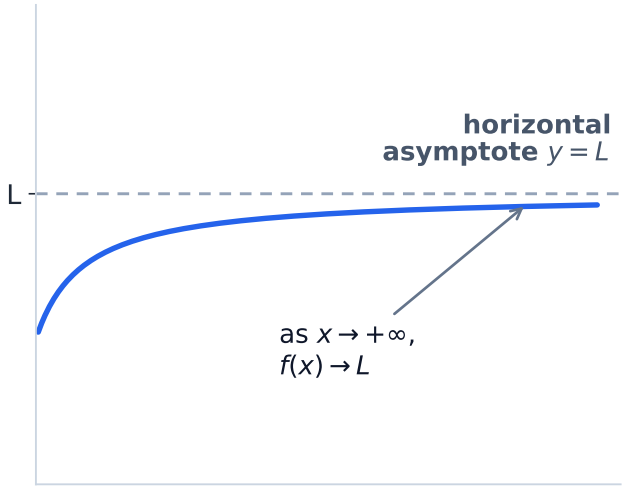
Connecting Limits at Infinity and Horizontal Asymptotes

We can also let the **input** grow: **limits at infinity** 无穷远处的极限 describe the **end behavior** 末端行为 of a function as $x \to \pm\infty$. If the outputs settle toward a finite value L, then $y = L$ is a **horizontal asymptote** 水平渐近线.

For a rational function, compare the **degrees** 次数 of the top and bottom:

- top degree $<$ bottom degree $\Rightarrow$ limit is 0 (asymptote $y = 0$);
- top degree $=$ bottom degree $\Rightarrow$ limit is the **ratio of the leading coefficients** 首项系数之比;
- top degree $>$ bottom degree $\Rightarrow$ the function is unbounded (no horizontal asymptote).

More generally, we compare the **relative magnitudes** 相对大小 (relative growth rates) of functions: far out, an exponential beats any polynomial, and a polynomial beats any logarithm. On the exam, "as $t \to \infty$, which quantity is larger/where does the rate settle?" is answered with a limit at infinity.



A limit at infinity produces a horizontal asymptote

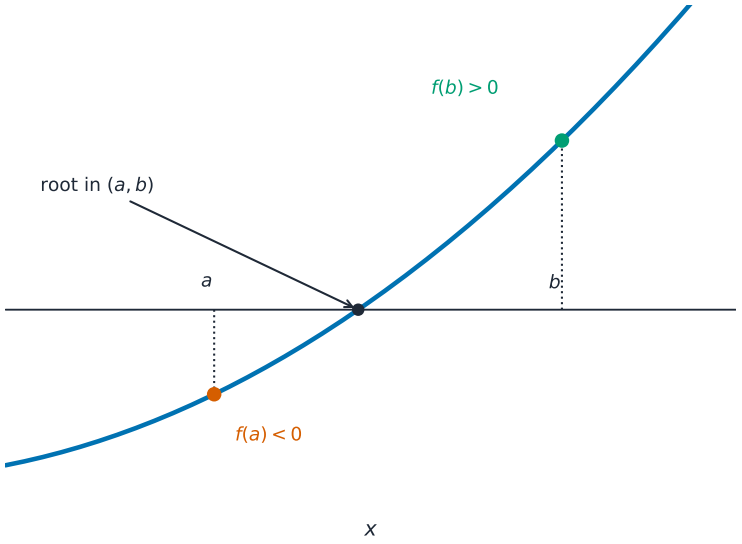
Worked example. Evaluate $\lim_{x\to\infty} \dfrac{3x^2-5}{2x^2+x}$. Divide top and bottom by the highest power, x^2: $\dfrac{3-5/x^2}{2+1/x} \to \dfrac{3-0}{2+0} = \dfrac{3}{2}$. Because the limit is a finite number, the line $y = \frac{3}{2}$ is a **horizontal asymptote** of the graph.

Working with the Intermediate Value Theorem (IVT)

The **Intermediate Value Theorem** 介值定理 is an **existence theorem** 存在性定理 – it guarantees a value exists without telling you where:

> If f is **continuous** on the closed interval $[a, b]$, and d is any number **between** $f(a)$ and $f(b)$, then there is **at least one** number c in (a, b) with $f(c) = d$.

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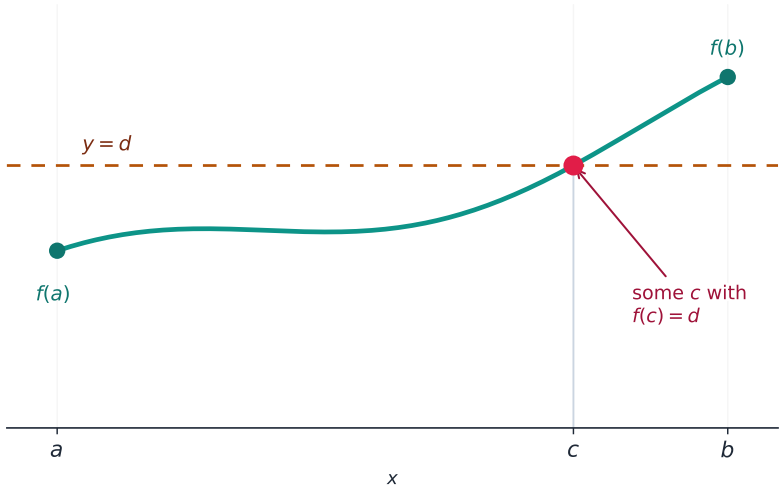
Opposite signs of f(a) and f(b) trap a root between a and b

An unbroken curve cannot skip a height between its endpoints – it must pass through every one.

Exam skill – how to justify with the IVT. These questions appear almost every year (for example, "Must there be a value c with $R(c) = 155$?" or "Is there a time when $r'(t) = -6$?"). A full-credit justification has three moves:

1. **State continuity.** Say the function is continuous on $[a, b]$ (often because it is differentiable, or given continuous).
2. **Show d is trapped.** Compute the two endpoint values and show the target d lies **between** them, e.g. $f(a) < d < f(b)$.
3. **Conclude by name.** "By the Intermediate Value Theorem, there is a c in (a, b) with $f(c) = d$."

Skipping the continuity statement, or not showing d is between the endpoints, loses the point – the theorem *requires* both conditions.



A continuous curve from $(a, f(a))$ to $(b, f(b))$ must cross every height d in between at least once.

Worked example. Show that $f(x) = x^3 - x - 1$ has a zero between $x = 1$ and $x = 2$.

1. f is a polynomial, so it is continuous on $[1, 2]$.
2. $f(1) = 1 - 1 - 1 = -1$ and $f(2) = 8 - 2 - 1 = 5$, so $f(1) < 0 < f(2)$.
3. By the Intermediate Value Theorem, there is a c in $(1, 2)$ with $f(c) = 0$.

The theorem says nothing about *where* c is or how many such values there are (here $c \approx 1.32$, found with a calculator).

Common mistakes with the IVT.

- Using it on a function that is not continuous on the whole interval: $g(x) = \frac{1}{x}$ has $g(-1) = -1$ and $g(1) = 1$, but g is never 0, because g breaks at $x = 0$.
- Reading it backwards: a continuous f can reach a height *outside* $[f(a), f(b)]$ too, and can reach d more than once. The theorem only promises *at least one* c for each d between the endpoint values.

Exam tips

- A limit describes what $f(x)$ **approaches**, which need not equal $f(a)$ – the two-sided limit exists only if both sides agree.
- Try direct substitution first; for a $\frac{0}{0}$ form, **factor and cancel** or rationalise before substituting.
- A function is **continuous** at a when the limit exists, $f(a)$ is defined, and they are equal.

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- Use the **Intermediate Value Theorem** to guarantee a root: a continuous function that changes sign on $[a, b]$ takes every value between.
- Read horizontal asymptotes from end behaviour (limits at $\pm\infty$) and vertical asymptotes at denominator zeros that remain after common factors are cancelled; check the one-sided limits.

Common mistakes.

- Saying a limit does not exist because $f(a)$ is undefined. A hole in the graph does not stop the limit from existing.
- Stopping at $\frac{0}{0}$ and writing "0" or "undefined". $\frac{0}{0}$ is a signal to factor, rationalise or simplify, not an answer.
- Using the IVT without stating that the function is **continuous** on the closed interval - on the exam, the justification earns no credit without it.
- Writing $\lim_{x \to a} f(x) = \infty$ and then calling the limit a number. An infinite limit describes unbounded behaviour (a vertical asymptote); the limit does not exist as a real number.
- Finding a horizontal asymptote by substituting a large number instead of comparing the leading terms.

Practice. Questions 3 and 4 follow the style of recent free-response parts (IVT from a table; the squeeze theorem with two bounding functions).

1. Find $\lim_{x \to 3} \dfrac{x^2 - 9}{x - 3}$.
2. Find $\lim_{x \to 0} \dfrac{\sin 3x}{x}$.
3. A continuous function f has $f(1) = 4$, $f(3) = -2$ and $f(5) = 6$. What is the least number of solutions that $f(x) = 0$ must have on $1 \le x \le 5$? Justify your answer.
4. For all x near 1, $2x + 1 \le k(x) \le x^2 + 2$. Find $\lim_{x \to 1} k(x)$, and say what else you need to know to conclude that k is continuous at $x = 1$.
5. Let $f(x) = \dfrac{2x^2 - 8}{x^2 - x - 2}$. Find every vertical asymptote, horizontal asymptote and removable discontinuity.
6. Find k so that $f(x) = x^2 + k$ for $x < 2$ and $f(x) = 3x - 1$ for $x \ge 2$ is continuous at $x = 2$.

Answers.

1. Factor: $\dfrac{(x-3)(x+3)}{x-3} = x + 3$ for $x \neq 3$, so the limit is 6.
2. $\dfrac{\sin 3x}{x} = 3 \cdot \dfrac{\sin 3x}{3x} \to 3 \cdot 1 = 3$.
3. At least two. f is continuous on $[1, 3]$ and $[3, 5]$; $f(1) = 4 > 0 > -2 = f(3)$, and $f(3) = -2 < 0 < 6 = f(5)$, so by the IVT there is a zero in $(1, 3)$ and another in $(3, 5)$.

4. Both bounds equal 3 at $x = 1$ (and $x^2 + 2 - (2x + 1) = (x - 1)^2 \geq 0$), so by the squeeze theorem $\lim_{x \to 1} k(x) = 3$. k is continuous at 1 only if also $k(1) = 3$.
5. $f(x) = \dfrac{2(x-2)(x+2)}{(x-2)(x+1)}$. Removable discontinuity (hole) at $x = 2$, where the limit is $\frac{2 \cdot 4}{3} = \frac{8}{3}$; vertical asymptote $x = -1$; horizontal asymptote $y = 2$ (ratio of leading coefficients).
6. Left limit $4 + k$ must equal $f(2) = 5$, so $k = 1$.

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Differentiation: Definition and Fundamental Properties

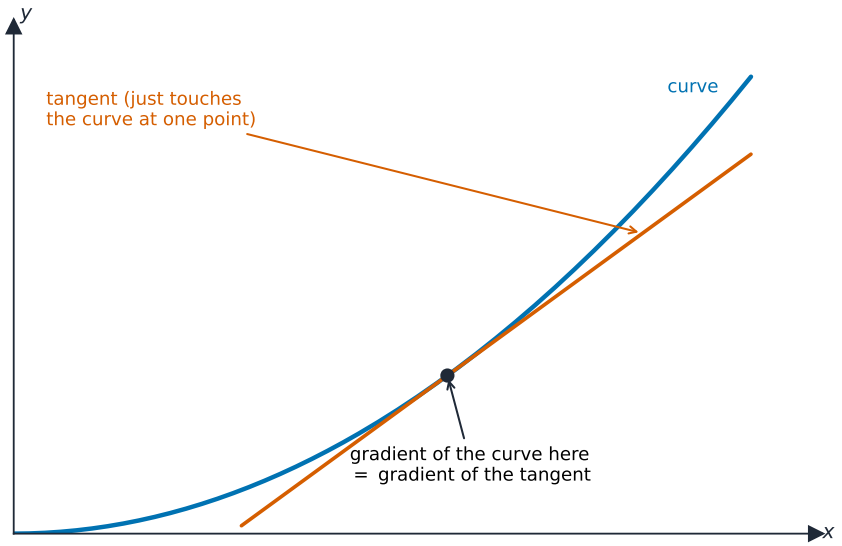
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Average and Instantaneous Rates of Change at a Point



A roller coaster on a lift hill: the derivative measures instantaneous rate of change

Unit 1 built the limit. Unit 2 uses it to define the **derivative** 导数 – the exact rate of change at a point.



The instantaneous rate of change is the gradient of the tangent at a point

Over an interval, the average rate of change is a **difference quotient** 差商. Two equivalent forms appear:

$$\frac{f(a+h)-f(a)}{h} \qquad \text{and} \qquad \frac{f(x)-f(a)}{x-a}.$$

The first uses a step of size h from a; the second uses two points x and a. Both compute $\dfrac{\text{change in output}}{\text{change in input}}$ over the interval.

The **instantaneous** 瞬时 rate of change at $x=a$ is what the difference quotient approaches as the interval shrinks to zero. This limit *is* the derivative at a, written $f'(a)$:

$$f'(a)=\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}=\lim_{x\to a}\frac{f(x)-f(a)}{x-a},$$

provided the limit exists.

Defining the Derivative and Reading Its Notation

Let the point a vary and the derivative becomes a **new function**:

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}.$$

This is the **definition of the derivative** (sometimes called differentiating "by first principles" 用定义求导). Its value at each x is the instantaneous rate of change there.

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Worked example. Use the definition to find $f'(x)$ for $f(x)=x^2+3x$.

Write $f(x+h)$ in full, subtract $f(x)$, and simplify *before* letting $h\to 0$:

$$f(x+h)-f(x)=\big(x^2+2xh+h^2+3x+3h\big)-\big(x^2+3x\big)=2xh+h^2+3h.$$

$$f'(x)=\lim_{h\to 0}\frac{2xh+h^2+3h}{h}=\lim_{h\to 0}(2x+h+3)=2x+3.$$

Every term without h cancels, so you can divide by h; that is what removes the $\frac{0}{0}$.

Common **notations** 记号 for the derivative of $y=f(x)$ are:

$$\frac{dy}{dx}, \qquad f'(x), \qquad y'.$$

The derivative can be represented graphically, numerically, analytically, and verbally – be ready to move between them.

Geometric meaning. The derivative at a point is the **slope** 斜率 of the **tangent line** 切线 to the graph there. So the tangent line at $x=a$ passes through $\big(a,f(a)\big)$ with slope $f'(a)$:

$$y-f(a)=f'(a)\,(x-a).$$

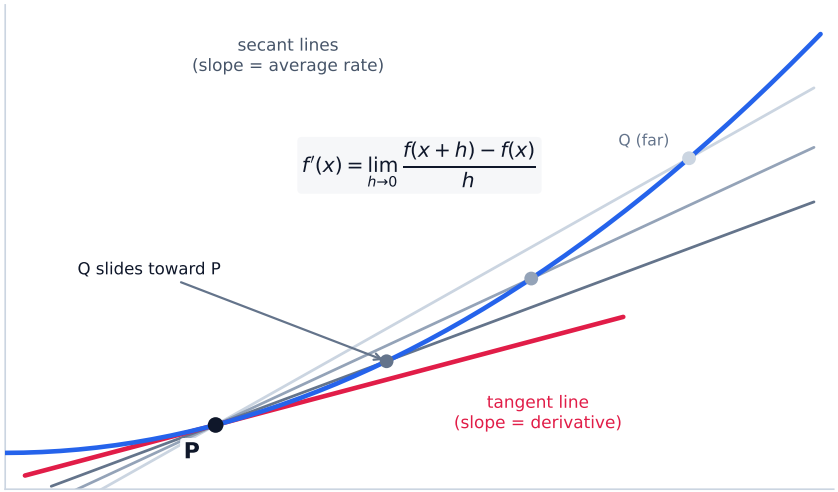
Writing this line is a routine exam task, so keep the point-slope form ready.

Worked example. Find the tangent line to $f(x)=x^3-2x$ at $x=2$.

- The point: $f(2)=2^3-2(2)=8-4=4$, so the line passes through $(2,4)$.
- The slope: $f'(x)=3x^2-2$, so $f'(2)=3(4)-2=10$.
- The line: $y-4=10(x-2)$, or $y=10x-16$.

Use the derivative only for the slope; the point comes from f itself.

Common mistakes. Using $f'(2)=10$ as the y-coordinate of the point, and leaving x in the slope: $y-4=(3x^2-2)(x-2)$ is not a line. The slope of a tangent line is a **number**, $f'(a)$.



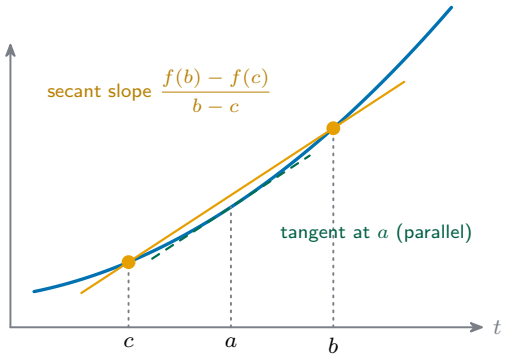
Secant slopes approach the tangent slope: the derivative is the limit of average rates

Estimating a Derivative at a Point

You do not always have a formula. When a function is given by a **table** 表格 or a graph, estimate the derivative $f'(a)$ with a difference quotient over a **small interval around** a. A table with values on both sides of a gives the best estimate:

$$f'(a) \approx \frac{f(b)-f(c)}{b-c}, \qquad \text{where } c < a < b \text{ are the closest table inputs.}$$

Technology (a calculator) can also estimate a derivative at a point.



Bracket the point with the closest table inputs.

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Exam skill (appears almost every year). Questions such as "Approximate $M'(7.5)$ using the average rate of change of M over the interval $5 \le t \le 10$" ask for exactly this difference quotient. Show the setup:

$$M'(7.5) \approx \frac{M(10)-M(5)}{10-5}.$$

Full credit needs the numbers plugged in **and** the correct **units** 单位 (output units per input unit), since these come from real-world models.

> **Worked example.** Water is pumped into a tank. $M(t)$ is the volume in the tank, in litres, t minutes after the pump starts.

t **(minutes)**	0	5	10	15
$M(t)$ (litres)	12	20	26	29

Estimate $M'(7.5)$. The closest table inputs on each side of 7.5 are 5 and 10:

$$M'(7.5) \approx \frac{M(10)-M(5)}{10-5} = \frac{26-20}{5} = 1.2 \text{ litres per minute.}$$

Meaning: about 7.5 minutes after the start, the volume is rising at about 1.2 litres per minute.

Common mistakes. Writing $M(10)-M(5)=6$ and forgetting to divide by $10-5$; dividing by the wrong difference; leaving out the units (litres **per minute**); and using a wider interval such as $5 \le t \le 15$ when the table has the closest values on both sides of 7.5.

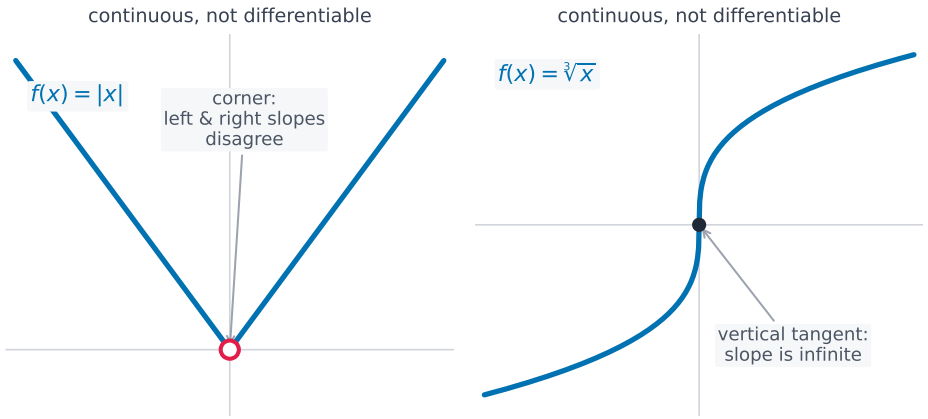
Differentiability and Continuity: When a Derivative Exists

Differentiability is stronger than continuity. The key relationship:

> **If f is differentiable 可导 at a point, then f is continuous 连续 there.**

So differentiability *implies* continuity. The reverse is false: a continuous function can fail to be differentiable. Two ways this happens:

- A **corner** 尖点: the left and right difference-quotient limits disagree, as with $f(x) = |x|$ at $x = 0$.
- A **vertical tangent** 垂直切线: the slope is infinite (no real number), as with $f(x) = \sqrt[3]{x}$ at $x = 0$.



f differentiable at *a* ⇒ *f* continuous at *a*; converse fails at corners and vertical tangents

Two ways a continuous function is not differentiable: a corner and a vertical tangent

Also, a point outside the domain of f cannot be in the domain of f'. Use the contrapositive on the exam: **if f is not continuous at a, then f is not differentiable at a.**

Common mistake. Arguing "continuous, so differentiable". The function $|x|$ is continuous at 0 but has a corner there. Continuity is **needed** for differentiability, but it is not enough.

> **Worked example (making a piecewise function differentiable).** Find k and m so that
>
> $$f(x) = \begin{cases} x^2 + k, & x \le 2 \\ mx - 1, & x > 2 \end{cases}$$
>
> is differentiable at $x = 2$. Two conditions are needed.
>
> 1. *Continuous:* the pieces meet, $2^2 + k = 2m - 1$, so $k = 2m - 5$.
> 2. *Same slope from both sides:* the left piece has slope $2x = 4$ at $x = 2$; the right piece has slope m. So $m = 4$, and then $k = 3$.
>
> Check: both pieces give $f(2) = 7$ and slope 4. Matching the slopes alone is not enough: without condition 1, f would jump at $x = 2$ and could not be differentiable.

The Power Rule

From here we use **rules** instead of the limit definition each time. The **power rule** 幂法则 handles any power of x:

$$\frac{d}{dx}\, x^r = r\, x^{r-1} \qquad \text{for any real } r.$$

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For arbitrary real powers this formula applies on $x > 0$; integer powers and some rational powers have larger real domains. Check the domain and differentiability at the point, especially at $x = 0$. Rewrite roots as powers before applying the rule.

> **Worked example.** Differentiate $y = 6\sqrt{x} - \dfrac{2}{x^3} + \dfrac{1}{3x}$. Rewrite every term as a constant times a power of x first:
>
> $$y = 6x^{1/2} - 2x^{-3} + \tfrac{1}{3}x^{-1} \quad \Rightarrow \quad y' = 3x^{-1/2} + 6x^{-4} - \tfrac{1}{3}x^{-2} = \frac{3}{\sqrt{x}} + \frac{6}{x^4} - \frac{1}{3x^2}.$$

Common mistakes. Writing $\dfrac{d}{dx}\dfrac{1}{x^3} = \dfrac{1}{3x^2}$ (differentiating the denominator), and treating $\dfrac{1}{3x}$ as $3x^{-1}$: the 3 stays in the denominator, so the constant is $\frac{1}{3}$.

Constant, Sum, Difference, and Constant Multiple Rules

These rules let you differentiate term by term:

- **Constant:** $\dfrac{d}{dx}\, k = 0$ (a constant does not change).
- **Constant multiple** 常数倍: $\dfrac{d}{dx}\big[k\, f(x)\big] = k\, f'(x)$.
- **Sum / difference:** $\dfrac{d}{dx}\big[f(x) \pm g(x)\big] = f'(x) \pm g'(x)$.

Combined with the power rule, they differentiate any **polynomial** 多项式 term by term. Example:

$$\frac{d}{dx}\big(4x^3 - 5x + 7\big) = 12x^2 - 5.$$

> **Worked example (horizontal tangents).** Where does $y = x^3 - 6x^2 + 9x + 1$ have a horizontal tangent? Set the derivative to zero: $y' = 3x^2 - 12x + 9 = 3(x-1)(x-3) = 0$, so $x = 1$ or $x = 3$. The points are $(1, 5)$ and $(3, 1)$. Give the **points**, not just the x-values, when the question asks "where on the curve".

Derivatives of cos x, sin x, e^x, and ln x

Learn these four building-block derivatives by heart:

$$\frac{d}{dx}\sin x = \cos x, \qquad \frac{d}{dx}\cos x = -\sin x,$$

$$\frac{d}{dx}e^x = e^x, \qquad \frac{d}{dx}\ln x = \frac{1}{x} \quad (x > 0).$$

Note the minus sign on the derivative of cosine, and that e^x is its own derivative.

A limit that is really a derivative (LIM-3.A.1). Sometimes a limit is secretly the definition of a known derivative. If you recognize

$$\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$$

for a function f whose derivative you know, just evaluate $f'(a)$. For example, $\lim_{h\to 0}\dfrac{\sin\left(\frac{\pi}{2}+h\right)-1}{h}=\dfrac{d}{dx}\sin x\Big|_{x=\pi/2}=\cos\frac{\pi}{2}=0.$

The Product Rule

A product of two functions is **not** differentiated by multiplying the derivatives. Use the **product rule** 乘积法则:

$$\frac{d}{dx}\big[u\,v\big]=u'v+uv'.$$

"Derivative of the first times the second, plus the first times the derivative of the second." Example:

$$\frac{d}{dx}\left(x^2e^x\right)=2x\,e^x+x^2e^x.$$

Exam questions often build a new function from given pieces, e.g. $k(x)=\big(f(x)\big)^2g(x)$, and ask for $k'(x)$ at a point, you to combine rules while reading values from a table.

The Quotient Rule

For a quotient, use the **quotient rule** 商法则:

$$\frac{d}{dx}\left[\frac{u}{v}\right]=\frac{u'v-uv'}{v^2}.$$

"Bottom times derivative of top, minus top times derivative of bottom, all over bottom squared." The order matters because of the minus sign, so write the numerator carefully. Example:

$$\frac{d}{dx}\left(\frac{x}{\cos x}\right)=\frac{1\cdot\cos x-x\cdot(-\sin x)}{\cos^2 x}=\frac{\cos x+x\sin x}{\cos^2 x}.$$

Worked example: rules from a table. You are given $f(3)=2$, $f'(3)=-1$, $g(3)=5$ and $g'(3)=4$.

- For $p(x)=f(x)\,g(x)$: $p'(3)=f'(3)g(3)+f(3)g'(3)=(-1)(5)+(2)(4)=3$.
- For $q(x)=\dfrac{f(x)}{g(x)}$:

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$$q'(3)=\frac{f'(3)g(3)-f(3)g'(3)}{\big(g(3)\big)^2}=\frac{(-1)(5)-(2)(4)}{5^2}=-\frac{13}{25}.$$

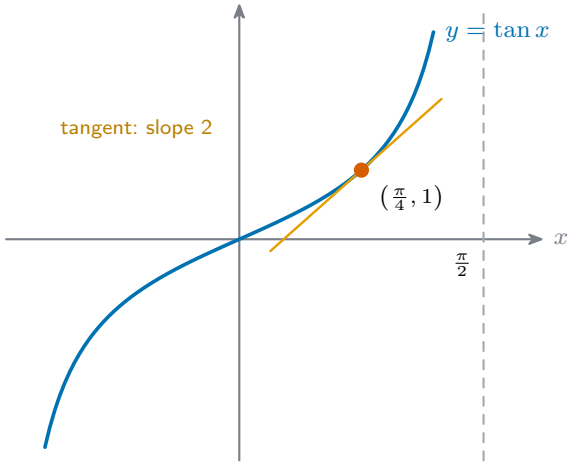
Common mistakes. Multiplying the derivatives: $p'(3)\neq f'(3)\,g'(3)=-4$. Writing the quotient-rule numerator in the wrong order ($fg'-f'g$ gives $+\frac{13}{25}$, the wrong sign). Forgetting to square the denominator.

Derivatives of Tangent, Cotangent, Secant, and Cosecant

The remaining trigonometric derivatives are not memorized separately – you rewrite them with **identities** 恒等式 and apply the quotient (or product) rule. For instance, $\tan x=\dfrac{\sin x}{\cos x}$, so the quotient rule gives

$$\frac{d}{dx}\tan x=\frac{\cos x\cos x-\sin x(-\sin x)}{\cos^2 x}=\frac{1}{\cos^2 x}=\sec^2 x.$$

The same method (writing $\cot x=\frac{\cos x}{\sin x}$, $\sec x=\frac{1}{\cos x}$, $\csc x=\frac{1}{\sin x}$) gives $-\csc^2 x$, $\sec x\tan x$, and $-\csc x\cot x$.



At pi/4 the slope is sec squared, which is 2.

Worked example. Find the tangent line to $y=\tan x$ at $x=\frac{\pi}{4}$. The point is $\left(\frac{\pi}{4},1\right)$ and the slope is $\sec^2\frac{\pi}{4}=\left(\sqrt{2}\right)^2=2$, so $y-1=2\left(x-\frac{\pi}{4}\right)$.

A memory aid: the three "co" functions (cot, csc, and cos) all have a **minus sign** in their derivatives.

Higher-order derivatives. Differentiating f' again gives the **second derivative** 二阶导数 $f''(x)$ (or $\frac{d^2y}{dx^2}$) – the rate of change of the rate of change. An exam part like "Find $k''(3)$" just means differentiate twice, then substitute. You can also estimate a second derivative from a table by applying the average-rate-of-change method to the f' values.

> **Worked example.** Differentiate $g(x) = \dfrac{\sin x}{x}$ with the **quotient rule** $\left(\frac{u}{v}\right)' = \dfrac{u'v - uv'}{v^2}$: take $u = \sin x$, $v = x$, giving $g'(x) = \dfrac{x\cos x - \sin x}{x^2}$. Keep the order $u'v - uv'$ in the numerator – swapping the terms flips the sign and loses the mark.

Exam tips

- The derivative is the **slope of the tangent** – the limit of the secant slope $\frac{f(x+h)-f(x)}{h}$ as $h \to 0$.
- Memorise the rules: power, product, quotient, and the derivatives of sin, cos, e^x, and $\ln x$.
- **Differentiability implies continuity**, but not the reverse (a corner or cusp is continuous yet not differentiable).
- Distinguish average rate of change (secant slope over an interval) from instantaneous rate (the derivative at a point).
- Give a tangent-line equation as $y - f(a) = f'(a)(x - a)$.

Practice.

1. Use the definition of the derivative to find $f'(x)$ for $f(x) = 3x^2 - x$.
2. A table gives $h(4) = 10$ and $h(7) = 19$. Estimate $h'(5.5)$ and state its units if h is in metres and t in seconds.
3. Differentiate $y = \dfrac{4}{\sqrt{x}} + x^5 - \pi^2$.
4. Find k and m so that $g(x) = kx^2$ for $x \le 1$ and $g(x) = 4x + m$ for $x > 1$ is differentiable at $x = 1$.
5. Differentiate $y = e^x \cos x$ and $y = \dfrac{\ln x}{x}$.
6. Find the equation of the tangent line to $y = \sec x$ at $x = 0$.

Answers.

1. $\dfrac{3(x+h)^2 - (x+h) - 3x^2 + x}{h} = \dfrac{6xh + 3h^2 - h}{h} = 6x + 3h - 1 \to 6x - 1.$
2. $\dfrac{19 - 10}{7 - 4} = 3$ metres per second.
3. $y = 4x^{-1/2} + x^5 - \pi^2$, so $y' = -2x^{-3/2} + 5x^4$ (π^2 is a constant).
4. Slopes: $2k = 4$, so $k = 2$. Continuity: $k = 4 + m$, so $m = -2$.
5. $e^x \cos x - e^x \sin x$; $\dfrac{\frac{1}{x}\cdot x - \ln x}{x^2} = \dfrac{1 - \ln x}{x^2}$.

6. $y' = \sec x \tan x = 0$ at $x = 0$, and $y(0) = 1$, so $y = 1$.

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Differentiation: Composite, Implicit, and Inverse Functions

AP Calculus BC

The Chain Rule



A mountain path: composite functions nest rates of change (chain rule)

Unit 2 differentiated single functions. Unit 3 differentiates functions built *inside* other functions. The **chain rule** 链式法则 differentiates a **composite function** 复合函数 $f\big(g(x)\big)$:

$$\frac{d}{dx}\,f\big(g(x)\big) = f'\big(g(x)\big)\cdot g'(x).$$

"Derivative of the outer function (leaving the inside alone), times the derivative of the inside." The inner derivative $g'(x)$ is the piece students forget, so always ask "what is the inside, and what is its derivative?" Example:

$$\frac{d}{dx}\sin(x^2) = \cos(x^2)\cdot 2x.$$

In **Leibniz notation** 莱布尼茨记号, with $y = f(u)$ and $u = g(x)$, the rule reads $\dfrac{dy}{dx} = \dfrac{dy}{du}\cdot\dfrac{du}{dx}$ – the intermediate du appears to "cancel." Exam questions often give a table for f, g, f', g' and ask for $h'(a)$ where $h(x) = f\big(g(x)\big)$; evaluate $f'\big(g(a)\big)\cdot g'(a)$ by reading values.

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Worked example. Differentiate $h(x) = (2x^2+1)^5$. The **outer function** 外函数 is "(something)^5" and the **inner function** 内函数 is $2x^2+1$:

$$h'(x) = 5(2x^2+1)^4\cdot 4x = 20x(2x^2+1)^4.$$

Worked example (from a table). The table gives values of f, f', g and g'.

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
-1	3	4	2	-3
2	-1	5	0	1
3	2	-2	-1	6

(a) $p(x) = f(x^2-x)$. Find the slope of the tangent line at $x=-1$. The inside is $x^2-x = (-1)^2-(-1) = 2$, and its derivative is $2x-1 = -3$. So $p'(-1) = f'(2)\cdot(-3) = 5(-3) = -15$.

(b) $k(x) = g\big(f(x)\big)$. Find $k'(3)$. The inside value is $f(3) = 2$, so $k'(3) = g'\big(f(3)\big)\,f'(3) = g'(2)\cdot(-2) = 1(-2) = -2$.

(c) $m(x) = \big(g(x)\big)^3$. Find $m'(-1)$. The outer function is "cube": $m'(x) = 3\big(g(x)\big)^2 g'(x)$, so $m'(-1) = 3(2)^2(-3) = -36$.

Parts (a) and (b) read **two** rows of the table: the row of the input x, and the row of the inside value. Part (c) uses only the row at $x=-1$, because the outer function is a cube rather than a table function.

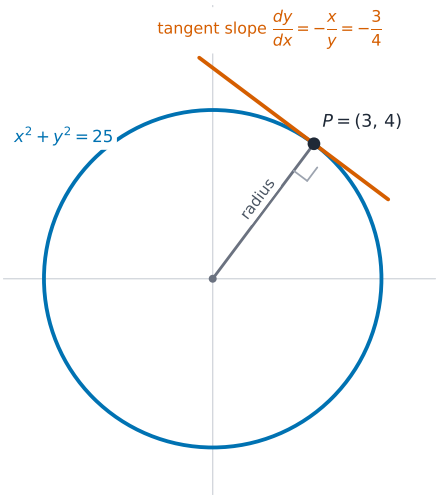
Implicit Differentiation

Some curves are defined **implicitly** – by an equation in x and y that is not solved for y, such as $x^2+y^2=25$. **Implicit differentiation** 隐函数求导 finds $\dfrac{dy}{dx}$ without solving for y first. It is just the chain rule, treating y as a function of x.

The method: differentiate **both sides** with respect to x; every time you differentiate a y-term, multiply by $\dfrac{dy}{dx}$ (the chain rule); then solve algebraically for $\dfrac{dy}{dx}$. For $x^2+y^2=25$:

$$2x + 2y\frac{dy}{dx} = 0 \;\Longrightarrow\; \frac{dy}{dx} = -\frac{x}{y}.$$

Worked example. Find the tangent to $x^2+y^2=25$ at $(3,4)$. Here $\dfrac{dy}{dx} = -\dfrac{3}{4}$, so the **tangent line** 切线 is $y-4 = -\frac{3}{4}(x-3)$ – **perpendicular** 垂直 to the **radius** 半径, as geometry predicts.



Implicit differentiation gives the tangent to a circle, perpendicular to the radius

Exam skill – "Show that $\frac{dy}{dx} = \ldots$". This exact prompt appears most years (e.g. "Show that $\frac{dy}{dx} = \frac{2y}{y^2 - 2x}$"). Because the target is given, you must show **every algebra step** cleanly: differentiate both sides, use the product/chain rules on mixed xy terms, collect all $\frac{dy}{dx}$ terms on one side, factor, and divide. A correct final line that skips the algebra earns little. Follow-up parts then ask for a tangent line, or where the tangent is horizontal ($\frac{dy}{dx} = 0$, so the numerator is 0) or vertical (the denominator is 0).

> **Worked example.** The curve $x^2 + xy + y^2 = 7$ passes through $(1, 2)$. Show that $\frac{dy}{dx} = -\frac{2x + y}{x + 2y}$, and find the slope at $(1, 2)$.

The middle term xy is a product, so it needs the product rule: $\frac{d}{dx}(xy) = 1 \cdot y + x\frac{dy}{dx}$. Differentiate every term:

$$2x + \left(y + x\frac{dy}{dx}\right) + 2y\frac{dy}{dx} = 0.$$

Collect the $\frac{dy}{dx}$ terms on one side and factor:

$$(x + 2y)\frac{dy}{dx} = -(2x + y) \quad \Longrightarrow \quad \frac{dy}{dx} = -\frac{2x + y}{x + 2y}.$$

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At $(1, 2)$ (check: $1 + 2 + 4 = 7$), the slope is $-\frac{2(1) + 2}{1 + 2(2)} = -\frac{4}{5}$.

Common mistakes. Writing $\frac{d}{dx}(xy) = \frac{dy}{dx}$ (the product rule is missing), and writing $\frac{d}{dx}(y^2) = 2y$ (the factor $\frac{dy}{dx}$ is missing).

> **Worked example (horizontal and vertical tangents).** On the same curve $x^2 + xy + y^2 = 7$, where is the tangent horizontal, and where is it vertical?

- **Horizontal:** the numerator is zero, $2x + y = 0$, so $y = -2x$. Substitute into the **original** equation: $x^2 - 2x^2 + 4x^2 = 7$, so $3x^2 = 7$ and $x = \pm\sqrt{7/3}$. The points are $\left(\sqrt{7/3}, -2\sqrt{7/3}\right)$ and $\left(-\sqrt{7/3}, 2\sqrt{7/3}\right)$.
- **Vertical:** the denominator is zero, $x + 2y = 0$, so $x = -2y$. Substitute: $4y^2 - 2y^2 + y^2 = 7$, so $y = \pm\sqrt{7/3}$, at $\left(-2\sqrt{7/3}, \sqrt{7/3}\right)$ and $\left(2\sqrt{7/3}, -\sqrt{7/3}\right)$.

At each point check that the other part of the fraction is **not** also zero; $\frac{0}{0}$ needs more work.

Differentiating Inverse Functions

If g is the **inverse function** 反函数 of f (so $f(g(x)) = x$), the chain rule links their derivatives:

$$g'(x) = \frac{1}{f'\big(g(x)\big)}, \qquad \text{provided } f'\big(g(x)\big) \neq 0.$$

In words: the derivative of the inverse at a point is the **reciprocal** 倒数 of the derivative of the original function at the *matching* point. A common exam setup gives a table and a point (a, b) on f (so (b, a) is on g), then asks for $g'(b) = \frac{1}{f'(a)}$.

> **Worked example.** If $f(2) = 5$ and $f'(2) = 3$, and g is the inverse of f, then $(5, 2)$ lies on g and $g'(5) = \frac{1}{f'(2)} = \frac{1}{3}$.

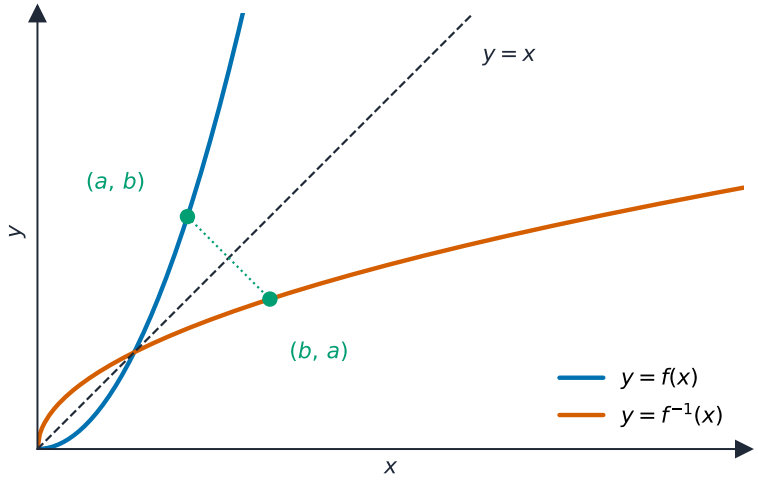
> **Worked example (no formula for the inverse).** $f(x) = x^5 + 2x + 1$ and $g = f^{-1}$. Find $g'(4)$.

You cannot solve $y = x^5 + 2x + 1$ for x, and you do not need to. First find the matching point: $f(a) = 4$ means $a^5 + 2a + 1 = 4$, and $a = 1$ works. So $g(4) = 1$, and

$$g'(4) = \frac{1}{f'(1)} = \frac{1}{5(1)^4 + 2} = \frac{1}{7}.$$

Why the formula is true. Differentiate $f\big(g(x)\big) = x$ with the chain rule: $f'\big(g(x)\big)\, g'(x) = 1$. Divide by $f'\big(g(x)\big)$.

Common mistake. Writing $g'(4) = \dfrac{1}{f'(4)}$. The derivative of f must be taken at the matching x-value $g(4) = 1$, not at 4.



The inverse function is the mirror image of the function in the line $y = x$

Differentiating Inverse Trigonometric Functions

The same idea gives the derivatives of the **inverse trigonometric functions** 反三角函数. The three you should know:

$$\frac{d}{dx}\arcsin x = \frac{1}{\sqrt{1-x^2}}, \quad \frac{d}{dx}\arctan x = \frac{1}{1+x^2}, \quad \frac{d}{dx}\operatorname{arcsec} x = \frac{1}{|x|\sqrt{x^2-1}}.$$

Combine these with the chain rule when the input is itself a function, e.g. $\dfrac{d}{dx}\arctan(3x) = \dfrac{3}{1+9x^2}$.

Where the formulas come from. Let $y = \arctan x$, so $\tan y = x$ with $-\frac{\pi}{2} < y < \frac{\pi}{2}$. Differentiate implicitly: $\sec^2 y \dfrac{dy}{dx} = 1$, so

$$\frac{dy}{dx} = \frac{1}{\sec^2 y} = \frac{1}{1+\tan^2 y} = \frac{1}{1+x^2}.$$

The same steps with $\sin y = x$ give $\dfrac{1}{\cos y} = \dfrac{1}{\sqrt{1-x^2}}$. The co-functions are the negatives: $\dfrac{d}{dx}\arccos x = -\dfrac{1}{\sqrt{1-x^2}}$.

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Worked example. $\dfrac{d}{dx}\Big(x\arcsin x\Big) = \arcsin x + \dfrac{x}{\sqrt{1-x^2}}$ (product rule). And $\dfrac{d}{dx}\arctan\left(e^x\right) = \dfrac{e^x}{1+e^{2x}}$ (chain rule: the inside is e^x).

Selecting Procedures for Calculating Derivatives

A skill topic: real derivatives mix several rules, so read the structure of the expression first, from the outside in.

- Is it a **sum**? Differentiate term by term.
- A **product** or **quotient**? Apply the **product rule** 乘积法则 or the **quotient rule** 商法则, and expect to use the chain rule inside.
- A **composite** (something inside something)? Chain rule.
- Given **implicitly**? Implicit differentiation.

Name the outermost operation, apply its rule, and recurse inward. Neatness prevents the sign and bookkeeping errors that cost marks.

Worked example. Differentiate $y = \dfrac{\ln(x^2+1)}{x}$. Outermost: a quotient. Inside the numerator: a chain (ln of x^2+1).

$$y' = \frac{\dfrac{2x}{x^2+1}\cdot x - \ln(x^2+1)\cdot 1}{x^2} = \frac{2}{x^2+1} - \frac{\ln(x^2+1)}{x^2}.$$

Sometimes a rewrite saves a rule: $\dfrac{5}{x^3} = 5x^{-3}$ needs only the power rule, $-15x^{-4}$, not the quotient rule.

The **laws of logarithms** 对数运算法则 save a rule in the same way. For $x > 0$, $y = \ln\left(x^2e^x\right) = 2\ln x + x$, so $\dfrac{dy}{dx} = \dfrac{2}{x} + 1$. The chain-rule route gives $\dfrac{2xe^x + x^2e^x}{x^2e^x}$, the same answer only after cancelling. Simplifying first is allowed whenever the new form is equal for every x in the domain.

Calculating Higher-Order Derivatives

Differentiating f' produces the **second derivative** 二阶导数 f''; repeating gives **higher-order derivatives** 高阶导数. The notations:

$$f''(x) = \frac{d^2y}{dx^2} = y'', \qquad \text{and in general} \qquad f^{(n)}(x) = \frac{d^ny}{dx^n}.$$

The second derivative measures how the *slope* is changing; it drives **concavity** 凹凸性 and **acceleration** 加速度 in later units. To find f'' implicitly, differentiate the

expression for $\dfrac{dy}{dx}$ again (with the quotient and chain rules), then substitute $\dfrac{dy}{dx}$ back in.

> **Worked example.** For the circle $x^2 + y^2 = 25$, find $\dfrac{d^2y}{dx^2}$ in terms of y.

Start from $\dfrac{dy}{dx} = -\dfrac{x}{y}$ and use the quotient rule, remembering that y depends on x:

$$\frac{d^2y}{dx^2} = -\frac{1\cdot y - x\dfrac{dy}{dx}}{y^2} = -\frac{y - x\left(-\dfrac{x}{y}\right)}{y^2} = -\frac{y^2 + x^2}{y^3} = -\frac{25}{y^3}.$$

The last step uses the curve itself: $x^2 + y^2 = 25$. On the top half ($y > 0$) the second derivative is negative, so the curve is concave down there, as a circle's top should be.

> **Worked example (implicit, in terms of x and y).** For $x^2 + xy + y^2 = 7$ we found $\dfrac{dy}{dx} = -\dfrac{2x + y}{x + 2y}$. The quotient rule gives

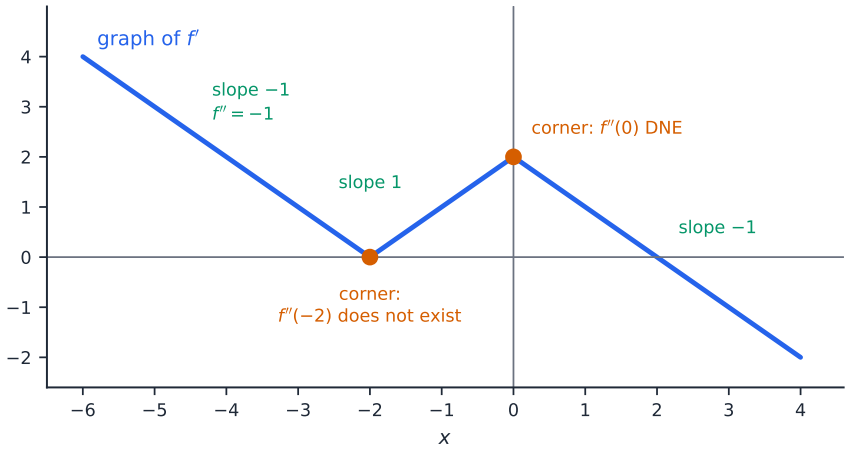
$$\frac{d^2y}{dx^2} = -\frac{\left(2 + \dfrac{dy}{dx}\right)(x + 2y) - (2x + y)\left(1 + 2\dfrac{dy}{dx}\right)}{(x + 2y)^2} = -\frac{3\left(y - x\dfrac{dy}{dx}\right)}{(x + 2y)^2}.$$

Substitute $\dfrac{dy}{dx}$: $y - x\dfrac{dy}{dx} = \dfrac{y(x + 2y) + x(2x + y)}{x + 2y} = \dfrac{2(x^2 + xy + y^2)}{x + 2y} = \dfrac{14}{x + 2y}$, using the curve. So

$$\frac{d^2y}{dx^2} = -\frac{42}{(x + 2y)^3}, \qquad \text{and at } (1, 2): \ -\frac{42}{125}.$$

Reading f'' from the graph of f'. f'' is the slope of the graph of f'. Where that graph has a corner, the left and right slopes differ, so f'' does not exist there.

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f'' is the slope of the graph of f': $f''(-4) = -1$, $f''(-1) = 1$, $f''(3) = -1$; at the corners $x = -2$ and $x = 0$, f'' does not exist.

A pattern of derivatives. For $f(x) = \ln(1 + x)$:

$$f' = (1 + x)^{-1}, \quad f'' = -(1 + x)^{-2}, \quad f''' = 2(1 + x)^{-3}, \quad f^{(4)} = -6(1 + x)^{-4}.$$

Each step brings down the power as a factor and the sign alternates, so $f^{(n)}(x) = (-1)^{n+1}\dfrac{(n-1)!}{(1 + x)^n}$. Patterns like this build Taylor polynomials in Unit 10: $f^{(4)}(0) = -6$.

Exam tips

- Use the **chain rule** for composite functions – differentiate the outside, then multiply by the derivative of the inside (the most-forgotten factor).
- For **implicit** differentiation, differentiate both sides with respect to x and attach $\frac{dy}{dx}$ each time y is differentiated, then solve.
- Get the **second derivative** by differentiating twice (velocity → acceleration).
- Combine rules carefully in layered expressions (chain inside product, etc.).
- The inverse function's graph is the reflection in $y = x$; its slope is the reciprocal of the original's at the matching point.
- For an inverse, find the **matching point** first: $g'(b) = \dfrac{1}{f'(a)}$ where $f(a) = b$.

Practice.

1. Differentiate $y = \sin^3(4x)$.
2. Use the table in the chain-rule section: find $\dfrac{d}{dx} f\big(g(x)\big)$ at $x = 3$.

3. For $y^2 - x^3 = xy + 1$, find $\dfrac{dy}{dx}$ at $(0, 1)$.
4. $f(x) = x^3 + 4x - 2$ and $g = f^{-1}$. Find $g'(3)$.
5. Find $\dfrac{d}{dx}\arcsin(3x)$ and $\dfrac{d}{dx}\arctan\left(\dfrac{1}{x}\right)$.
6. For $f(x) = e^{-2x}$, find $f^{(4)}(x)$ and a formula for $f^{(n)}(x)$.

Answers.

1. $3\sin^2(4x)\cdot\cos(4x)\cdot 4 = 12\sin^2(4x)\cos(4x)$.
2. $f'\big(g(3)\big)\,g'(3) = f'(-1)\cdot 6 = 4\cdot 6 = 24$.
3. $2y\dfrac{dy}{dx} - 3x^2 = y + x\dfrac{dy}{dx}$, so $\dfrac{dy}{dx} = \dfrac{3x^2+y}{2y-x} = \dfrac{0+1}{2-0} = \dfrac{1}{2}$.
4. $f(1) = 3$, so $g(3) = 1$ and $g'(3) = \dfrac{1}{f'(1)} = \dfrac{1}{3+4} = \dfrac{1}{7}$.
5. $\dfrac{3}{\sqrt{1-9x^2}}$; $\dfrac{1}{1+x^{-2}}\cdot\left(-\dfrac{1}{x^2}\right) = -\dfrac{1}{x^2+1}$.
6. $f^{(4)}(x) = 16e^{-2x}$; $f^{(n)}(x) = (-2)^n e^{-2x}$.

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Contextual Applications of Differentiation

AP Calculus BC

Interpreting the Meaning of the Derivative in Context

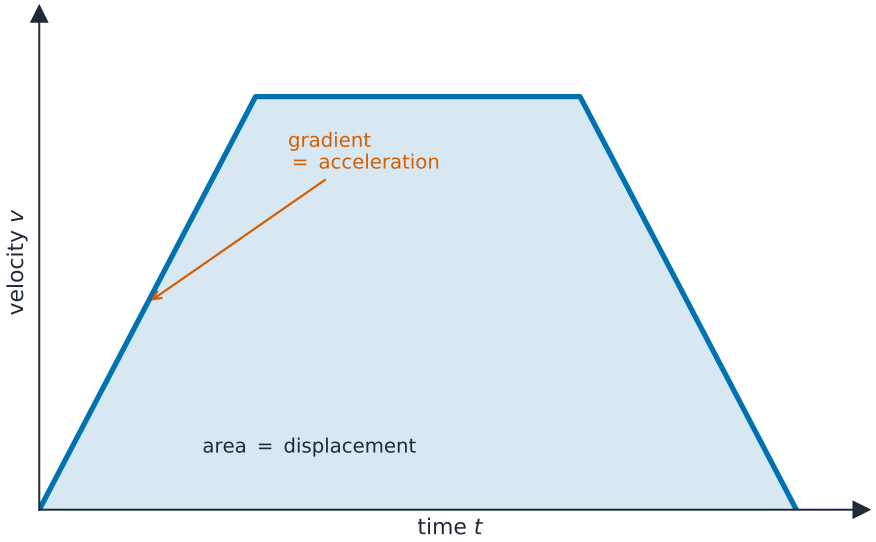
Once you can compute derivatives, you use them to describe the real world. The derivative $f'(x)$ **is** the **instantaneous rate of change** 瞬时变化率 of f **with respect to** 关于 its input. Reading and reporting this rate correctly is a graded skill.

Units matter. The unit of $f'(x)$ is the unit of f divided by the unit of x. If $C(t)$ is a number of acres and t is in weeks, then $C'(t)$ is in acres per week. On the exam, "Using correct units, interpret the meaning of $g'(140)$" wants a full sentence: the *value*, the *quantity*, the *rate word* "per", and the *moment*. For example: "$g'(140) = 2.3$ means that at $x = 140$, the quantity is increasing at about 2.3 units per unit of x."

Common mistakes. Reading $g'(140)$ as an amount ("the quantity is 2.3") instead of a rate; leaving out the units or the moment ($x = 140$); and calling a negative derivative "increasing".

Straight-Line Motion: Position, Velocity, and Acceleration

For a **particle** 质点 moving on a line, three functions of time are linked by **differentiation** 求导:



On a velocity-time graph, the area is displacement and the gradient is acceleration

- **position** 位置 $s(t)$;

- **velocity** 速度 $v(t) = s'(t)$ – signed; its sign gives direction;
- **acceleration** 加速度 $a(t) = v'(t) = s''(t)$.

Key readings (frequent exam parts):

- The particle is **at rest** 静止 when $v(t) = 0$.
- It moves **right/up** when $v(t) > 0$ and **left/down** when $v(t) < 0$; it **changes direction** 改变方向 where v changes sign.
- **Speed** 速率 is $|v(t)|$. Speed is *increasing* when v and a have the **same sign** (the particle is **speeding up** 加速), and *decreasing* when they have **opposite signs**.

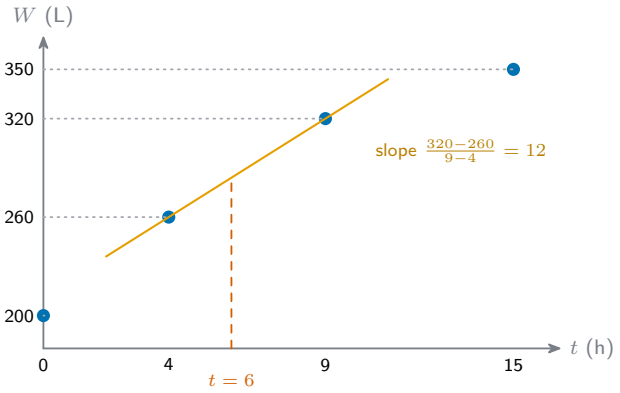
Distinguish carefully between velocity (has direction) and speed (does not) – the exam tests this exact difference.

Common mistakes. Saying the particle speeds up because $a > 0$: compare the signs of **both** v and a. Saying it changes direction wherever $v = 0$: v must change **sign**. With $v(t) = (t-2)^2$ the particle stops for an instant at $t = 2$, then keeps moving the same way.

> **Worked example.** A particle moves with $s(t) = t^3 - 6t^2 + 9t$. Then $v(t) = 3(t-1)(t-3)$, so it is at rest at $t = 1$ and $t = 3$ and changes direction at each. At $t = 2$, $v = -3 < 0$ and $a(2) = 6(2) - 12 = 0$; just after, $a > 0$ while $v < 0$, so the particle is **slowing down** there.

Rates of Change in Applied Contexts Other Than Motion

The same derivative idea models any changing quantity: a draining tank, a spreading population, a cooling cup. Whenever a problem says "the rate at which...", it is describing a derivative. Read the units to know which quantity's rate you have, then interpret in context.



Use the two measurements that bracket $t = 6$.

> **Worked example (estimating a rate from a table).** Water is pumped into a tank. The volume $W(t)$, in litres, is measured at times t in hours.

t (hours)	0	4	9	15
$W(t)$ (litres)	200	260	320	350

Estimate $W'(6)$. Use the two measurements that bracket $t = 6$:

$$W'(6) \approx \frac{W(9) - W(4)}{9 - 4} = \frac{320 - 260}{5} = 12 \text{ litres per hour.}$$

Interpretation: at $t = 6$ hours, the volume of water in the tank is increasing at about 12 litres per hour. A full answer names the quantity, the time, increasing or decreasing, the size and the units.

A rate of a rate. "Is the rate at which water enters increasing at $t = 6$?" asks about the sign of $W''(6)$, not $W'(6)$. Here the rate is 15 L/h on $[0, 4]$, 12 L/h on $[4, 9]$ and 5 L/h on $[9, 15]$, so the data suggest the rate is **decreasing**.

Introduction to Related Rates



A roller coaster: related rates connect how fast linked quantities change together

In a **related rates** 相关变化率 problem, several quantities change together over time, and you know some rates but want another. The engine is the **chain rule** 链式法则: differentiate a relationship **with respect to time** t. Every variable becomes a function of t, so each derivative picks up a "$\,/\,dt$" factor. The **product rule** 乘积法则 and **quotient rule** 商法则 may also be needed.

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A rising balloon: related rates link how fast radius, volume and height change together

Solving Related Rates Problems

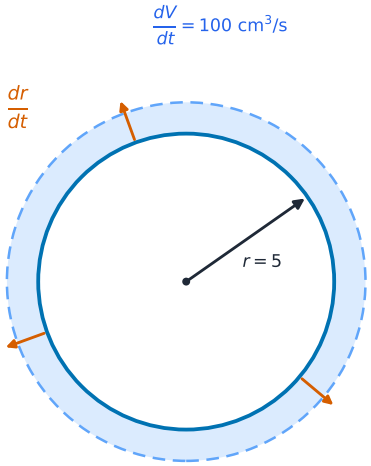
A reliable procedure – and a full-credit template on the exam:

1. **Name the variables** and write down the given rates and the unknown rate (e.g. "$\frac{dh}{dt} = -2$ cm/day, find $\frac{dV}{dt}$").
2. **Write an equation** relating the quantities (often a **geometric** 几何 or **volume** 体积 formula).
3. **Differentiate both sides with respect to** t (chain rule) – *before* substituting numbers.
4. **Substitute** the known values at the **instant** 时刻 of interest, and solve for the unknown rate.
5. State the answer with **units** and the correct **sign** (a decreasing quantity has a negative rate).

Substituting numbers too early is the classic error: differentiate the *general* relationship first, then plug in.

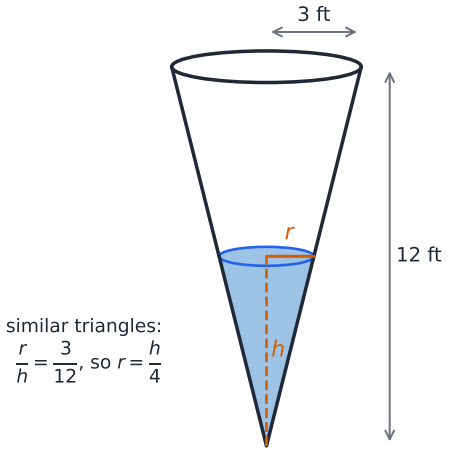
> **Worked example.** A spherical balloon's volume grows at $\frac{dV}{dt} = 100$ cm^3/s. From $V = \frac{4}{3}\pi r^3$, differentiate first: $\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$. At $r = 5$, $100 = 4\pi(25)\frac{dr}{dt}$, so $\frac{dr}{dt} = \frac{1}{\pi} \approx 0.32$ cm/s.

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An inflating balloon links dV/dt and dr/dt through the chain rule

> **Worked example: conical tank.** A tank is a cone with its vertex down, 12 ft deep with a top radius of 3 ft. Water drains out at 2 cubic feet per minute. How fast is the depth falling when the water is 4 ft deep?



The water forms a smaller cone similar to the tank, so $r = \frac{h}{4}$.

The volume of the water is $V = \frac{1}{3}\pi r^2 h$, which has **two** changing variables. The similar

triangles give $\frac{r}{h}=\frac{3}{12}$, so $r=\frac{h}{4}$. Substitute **before** differentiating:

$$V=\frac{1}{3}\pi\left(\frac{h}{4}\right)^2 h=\frac{\pi h^3}{48}\quad\Rightarrow\quad\frac{dV}{dt}=\frac{\pi h^2}{16}\frac{dh}{dt}.$$

Now substitute the instant: $\frac{dV}{dt}=-2$ (draining) and $h=4$:

$$-2=\frac{\pi(16)}{16}\frac{dh}{dt}\ \Rightarrow\ \frac{dh}{dt}=-\frac{2}{\pi}\approx-0.637\text{ ft/min}.$$

The depth is falling at about 0.637 feet per minute. Substituting $h=4$ before differentiating would turn h into a constant and give $\frac{dV}{dt}=0$, which is impossible.

> **Worked example: sliding ladder.** A 10 m ladder leans against a wall. Its foot slides away from the wall at 0.5 m/s. How fast is the top sliding down the wall when the foot is 6 m from the wall?

1. Variables: $x=$ distance from wall to foot, $y=$ height of the top. Given $\frac{dx}{dt}=0.5$ m/s; find $\frac{dy}{dt}$ when $x=6$.
2. Equation (**Pythagoras' theorem** 勾股定理): $x^2+y^2=100$.
3. Differentiate with respect to t: $2x\frac{dx}{dt}+2y\frac{dy}{dt}=0$.
4. At this instant $y=\sqrt{100-36}=8$, so $2(6)(0.5)+2(8)\frac{dy}{dt}=0$, which gives $\frac{dy}{dt}=-\frac{6}{16}=-0.375$ m/s.
5. The top is sliding **down** at 0.375 m/s (the negative sign means y is decreasing).

Note that $y=8$ came from the equation at that instant; it is not given in the question.

Common mistakes. Treating a constant as a variable or a variable as a constant: the ladder's length 10 never changes (its derivative is 0), but x and y both do. Dropping the sign: report $\frac{dy}{dt}=-0.375$ m/s, or say the top slides **down** at 0.375 m/s, never "down at -0.375 m/s".

Approximating Values Using Local Linearity and Linearization

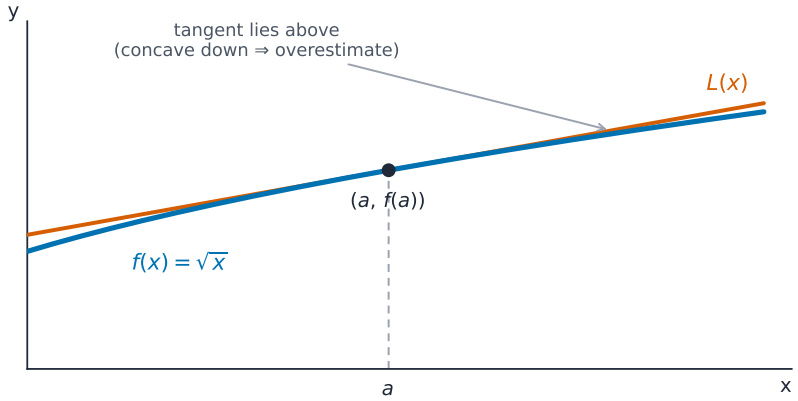
Near a **point of tangency** 切点, a **smooth curve** 光滑曲线 looks like its **tangent line** 切线 – this is **local linearity** 局部线性. So the tangent line gives a **linear approximation** 线性近似 (**linearization** 线性化) of the function near that point:

$$f(x)\approx L(x)=f(a)+f'(a)(x-a).$$

Use it to estimate f at an x close to a.

Over- or underestimate? The answer follows from **concavity** 凹凸性. If the graph is **concave up** 上凹 near a (it curves above its tangent), the tangent-line value is an **underestimate** 低估. If it is **concave down** 下凹, the tangent line lies above the curve, giving an **overestimate** 高估. Exam parts test this reasoning, so justify with the sign of f''.

> **Worked example.** $f(2)=5$, $f'(2)=-1$ and $f''(x)>0$ for all x. Estimate $f(2.1)$: $L(2.1)=5+(-1)(0.1)=4.9$. Because f is concave up, the estimate 4.9 is an underestimate of $f(2.1)$.



$f(x)\approx L(x)=f(a)+f'(a)(x-a)$ near $x=a$

The tangent line is a local linear approximation; concavity fixes over- or under-estimate

> **Worked example.** Estimate $\sqrt{4.1}$. Take $f(x)=\sqrt{x}$ and $a=4$: $f(4)=2$, and $f'(x)=\frac{1}{2\sqrt{x}}$ gives $f'(4)=\frac{1}{4}$. Then

$$L(4.1)=2+\tfrac{1}{4}(4.1-4)=2.025.$$

Here $f''(x)=-\frac{1}{4x^{3/2}}<0$, so the graph is concave down and 2.025 is an **overestimate** (the true value is $2.0248\ldots$).

Common mistake. Using the slope at the new point, $f'(4.1)$, instead of the known point, $f'(4)$. The whole idea is that $f(a)$ and $f'(a)$ are easy to find exactly.

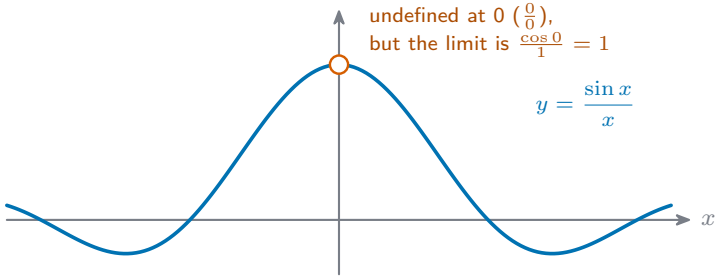
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Using L'Hospital's Rule for Indeterminate Forms

When **direct substitution** 直接代入 in a quotient of limits gives the **indeterminate form** 未定式 $\frac{0}{0}$ or $\frac{\infty}{\infty}$, you may use **L'Hospital's Rule** 洛必达法则:

$$\lim_{x\to c}\frac{f(x)}{g(x)}=\lim_{x\to c}\frac{f'(x)}{g'(x)},$$

provided the **numerator** 分子 and **denominator** 分母 are **differentiable** 可导 near the limiting input (excluding the input itself), the denominator derivative is nonzero there, and the derivative-ratio limit exists, finitely or infinitely. Use the corresponding **one-sided** 单侧 conditions when needed. **Differentiate the top and bottom separately** (this is *not* the quotient rule), then try the limit again. First confirm the form really is $\frac{0}{0}$ or $\frac{\infty}{\infty}$ – applying the rule to any other form is a mistake.



An indeterminate form can still have a limit.

Worked example. $\lim_{x\to 0}\frac{\sin x}{x}$ gives $\frac{0}{0}$, so differentiate top and bottom: $\lim_{x\to 0}\frac{\cos x}{1}=1$. And $\lim_{x\to 0}\frac{e^{2x}-1}{x}$ is also $\frac{0}{0}$; it becomes $\lim_{x\to 0}\frac{2e^{2x}}{1}=2$.

Worked example. Evaluate $\lim_{x\to\infty}\frac{x^2}{e^x}$. Both top and bottom grow without bound: the form is $\frac{\infty}{\infty}$. Apply the rule, check the new form, and apply it again:

$$\lim_{x\to\infty}\frac{x^2}{e^x}=\lim_{x\to\infty}\frac{2x}{e^x}\ \left(\text{still }\tfrac{\infty}{\infty}\right)=\lim_{x\to\infty}\frac{2}{e^x}=0.$$

This proves the Unit 1 claim that an **exponential** 指数函数 beats any **polynomial** 多项式. Check the form before **every** use: once it stops being $\frac{0}{0}$ or $\frac{\infty}{\infty}$, stop and substitute.

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Worked example (from given values). f has a continuous derivative near $x=2$, with $f(2)=3$ and $f'(2)=8$. Find $\lim_{x\to 2}\frac{f(x)-3}{x^2-4}$.

First show the form: f is continuous, so $\lim_{x\to 2}\big(f(x)-3\big)=3-3=0$, and $\lim_{x\to 2}(x^2-4)=0$. The form is $\frac{0}{0}$, so

$$\lim_{x\to 2}\frac{f(x)-3}{x^2-4}=\lim_{x\to 2}\frac{f'(x)}{2x}=\frac{f'(2)}{4}=\frac{8}{4}=2.$$

(The last step needs f' continuous at 2, which exam questions provide; then the derivative-ratio limit exists.) Writing "$=\frac{0}{0}$" as if it were a number, or skipping the check of **both** limits, loses the point for L'Hospital's Rule.

Exam tips

- In motion problems: velocity is the derivative of position, acceleration the derivative of velocity; **speed increases when velocity and acceleration share a sign**.
- For **related rates**, differentiate the relating equation with respect to **time**, then substitute the given values last.
- Use the tangent line for a **linear approximation** near a known point; it is accurate only close by.
- Read the sign of a rate: $\frac{dy}{dt}>0$ means y is increasing at that instant, $\frac{dy}{dt}<0$ means it is decreasing. A falling height or a shrinking volume has a negative rate.
- Always state units and interpret the answer in context.
- A tangent-line estimate is an underestimate where the curve is concave up, an overestimate where it is concave down.

Practice.

1. $P(t)$ is the number of people in a queue, t in minutes. Interpret $P'(10)=-4$.
2. A particle has $s(t)=t^3-9t^2+24t$. When is it at rest, and is it speeding up or slowing down at $t=3$?
3. A circle's radius grows at 0.5 cm/s. How fast is its area growing when $r=6$ cm?
4. A 1.8 m person walks away from a 6 m lamppost at 1.4 m/s. How fast is the length of their shadow growing?
5. Use linearization to estimate $\sqrt[3]{8.3}$, and say whether it is too big or too small.
6. Evaluate $\lim_{x\to 0}\frac{1-\cos x}{x^2}$.

Answers.

1. At $t=10$ minutes, the number of people in the queue is decreasing at 4 people per minute.

2. $v = 3t^2 - 18t + 24 = 3(t-2)(t-4)$: at rest at $t = 2$ and $t = 4$. At $t = 3$, $v = -3 < 0$ and $a = 6t - 18 = 0$, so it is neither speeding up nor slowing down at that instant.
3. $\dfrac{dA}{dt} = 2\pi r \dfrac{dr}{dt} = 2\pi(6)(0.5) = 6\pi \approx 18.8$ cm^2/s.
4. Similar triangles: $\dfrac{s}{1.8} = \dfrac{x+s}{6}$, so $4.2s = 1.8x$ and $s = \frac{3}{7}x$. Then $\dfrac{ds}{dt} = \frac{3}{7}(1.4) = 0.6$ m/s.
5. $f(x) = x^{1/3}$, $a = 8$: $f'(8) = \frac{1}{12}$, so $\sqrt[3]{8.3} \approx 2 + \frac{0.3}{12} = 2.025$. $f'' < 0$ for $x > 0$, so this is too big.
6. $\frac{0}{0}$: $\lim\limits_{x\to 0} \dfrac{\sin x}{2x}$, still $\frac{0}{0}$: $\lim\limits_{x\to 0} \dfrac{\cos x}{2} = \dfrac{1}{2}$.

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Analytical Applications of Differentiation

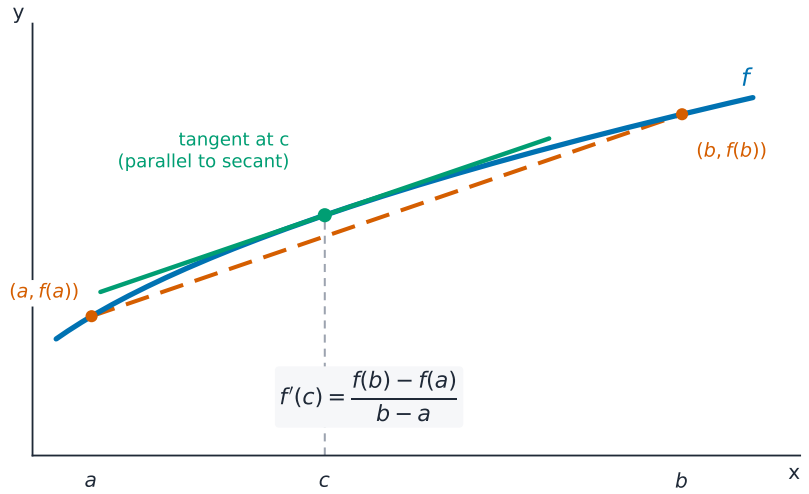
AP Calculus BC

Using the Mean Value Theorem

The **Mean Value Theorem** 中值定理 (MVT) links the **average rate of change** 平均变化率 to an instantaneous one:

> If f is **continuous** 连续 on $[a, b]$ and **differentiable** 可导 on (a, b), then there is at least one point c in (a, b) where
>
> $$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

In words: somewhere inside the **interval** 区间, the instantaneous rate equals the average rate. Geometrically, some **tangent line** 切线 is **parallel** 平行 to the line joining the **endpoints** 端点.



The Mean Value Theorem: some tangent is parallel to the secant over the interval

Exam skill. Like the IVT, the MVT is an existence theorem, and questions ask you to justify. Full credit needs: (1) state f is continuous on $[a, b]$ and differentiable on (a, b); (2) compute the average rate $\frac{f(b)-f(a)}{b-a}$; (3) conclude "by the MVT there is a c in (a, b) with $f'(c)$ equal to that value." Both hypotheses must be named.

> **Worked example.** For $f(x) = x^2$ on $[1, 3]$ the average rate is $\dfrac{9-1}{2} = 4$; setting $f'(c) = 2c = 4$ gives $c = 2$, which lies in $(1, 3)$ – the guaranteed point.

Why both hypotheses matter. Take $f(x) = |x|$ on $[-1, 1]$. It is continuous, and the average rate is $\dfrac{f(1) - f(-1)}{1 - (-1)} = \dfrac{1 - 1}{2} = 0$. But $f'(x)$ is -1 or 1 everywhere it exists, never 0. The MVT does not apply, because f is not differentiable at $x = 0$, which is inside $(-1, 1)$.

> **Worked example (MVT from a table).** A cyclist's velocity $v(t)$, in metres per second, is differentiable. Selected values:

t (s)	0	2	5	8
$v(t)$ (m/s)	3	7	7	1

(a) *Must there be a time in $0 < t < 8$ when the acceleration is 0?* v is differentiable, so it is continuous on $[2, 5]$ and differentiable on $(2, 5)$, and $\dfrac{v(5) - v(2)}{5 - 2} = \dfrac{7 - 7}{3} = 0$. By the Mean Value Theorem there is a c in $(2, 5)$ with $v'(c) = 0$. Yes.

(b) *Must there be a time when the acceleration is -2 m/s^2?* On $[5, 8]$: $\dfrac{1 - 7}{8 - 5} = -2$, so by the MVT, yes, for some c in $(5, 8)$.

Questions say "explain why there must be…": name the theorem, state both hypotheses (here they follow from "differentiable"), and show the average-rate calculation.

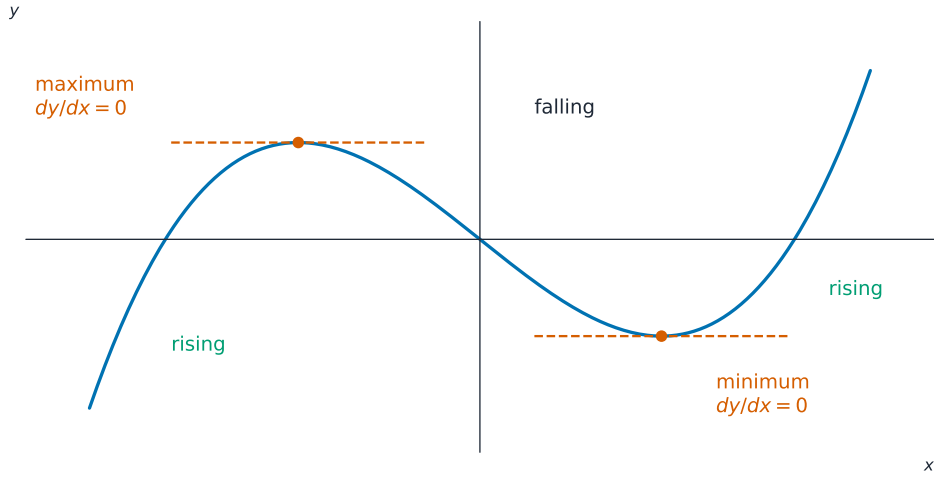
Extreme Values, Global vs Local Extrema, and Critical Points



A mountain peak: extrema and critical points mark highest and lowest values

The **Extreme Value Theorem** 极值定理 (EVT) guarantees extremes exist: a function **continuous** on a **closed interval** 闭区间 $[a, b]$ attains **both** an **absolute maximum** 最大值 and an **absolute minimum** 最小值 on it.

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At a differentiable interior maximum or minimum the derivative is zero

A **critical point** 临界点 is an **interior point** 内点 of the function's **domain** 定义域 where $f'(x) = 0$ **or** $f'(x)$ does not exist. All **local (relative) extrema** 局部极值 occur at critical points – but **not** every critical point is an **extremum** 极值. So critical points are the **candidates** 候选点; you must test each.

> **Worked example (a constant from a critical point).** $f(x) = \dfrac{1}{x^2 - kx}$, where k is a constant. For which k does f have a critical point at $x = 2$, and what kind is it?

$f'(x) = -\dfrac{2x - k}{(x^2 - kx)^2}$. At $x = 2$ the numerator must be 0: $4 - k = 0$, so $k = 4$. Then $f'(x) = -\dfrac{2x - 4}{(x^2 - 4x)^2}$. The denominator is positive, so the sign of f' is the sign of $-(2x - 4)$: positive just left of 2, negative just right. So f' changes from $+$ to $-$ and f has a **relative maximum** at $x = 2$, with value $f(2) = -\frac{1}{4}$. Note that $x = 0$ and $x = 4$ are not critical points: they are not in the domain.

Where a Function Increases or Decreases

The **first derivative** 一阶导数 tells you where f rises or falls:

- $f'(x) > 0$ on an interval $\Rightarrow$ f is **increasing** 递增 there;
- $f'(x) < 0 \Rightarrow$ f is **decreasing** 递减.

On the exam, "find the intervals where f is increasing" means: find the critical points, then test the sign of f' between them, and **justify with the sign of f'** (a stated reason, not just an interval).

Worked example. $f'(x) = (x-1)^2(x+2)$. Where is f increasing? The critical points are $x = -2$ and $x = 1$. The factor $(x-1)^2$ is never negative, so the sign of f' is the sign of $x+2$: negative for $x < -2$, positive for $x > -2$ (except $f'(1) = 0$). So f is decreasing on $(-\infty, -2]$ and increasing on $[-2, \infty)$ – **because $f' > 0$ except at the isolated zeros $x = -2$ and $x = 1$.** At $x = 1$, f' does not change sign, so $x = 1$ is a critical point but not an extremum.

The First Derivative Test for Local Extrema

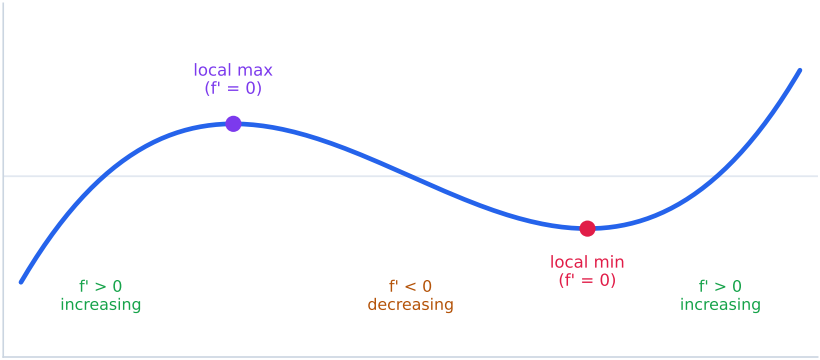
To classify a critical point $x = c$ as a local max, local min, or neither, check how f' changes sign there:

- f' changes $+$ **to** $-$ at $c \Rightarrow$ **local maximum** 极大值;
- f' changes $-$ **to** $+$ at $c \Rightarrow$ **local minimum** 极小值;
- f' does **not** change sign $\Rightarrow$ neither.

Always state the **sign change** 符号变化 as your **justification** 理由.

Worked example. For $f(x) = x^3 - 3x^2$, $f'(x) = 3x(x-2)$ is zero at $x = 0, 2$. Signs give $+, -, +$, so $x = 0$ is a **local maximum** ($f = 0$) and $x = 2$ a **local minimum** ($f = -4$).

Common mistakes. Justifying with "$f'(0) = 0$" alone: a zero of f' only makes $x = 0$ a candidate; the reason is the **sign change** of f'. Testing the sign of f instead of f'. Giving the x-value when the question asks for the maximum **value** ($f(0) = 0$), or the reverse.



$f' > 0 \Rightarrow$ increasing · $f' < 0 \Rightarrow$ decreasing · $f' = 0$ candidates for extrema

Where $f' > 0$ the graph rises and where $f' < 0$ it falls; a local max sits where f' turns $+$ to $-$, a local min where it turns $-$ to $+$.

The Candidates Test for Absolute Extrema

On a **closed interval**, absolute (global) extrema occur only at **critical points or endpoints**. The **candidates test** 候选点判别法:

1. List all critical points in $[a, b]$ **and** the two endpoints.
2. Evaluate f at each candidate.
3. The largest output is the absolute maximum; the smallest is the absolute minimum.

Show the table of values – the comparison is the argument.

Worked example. Find the absolute maximum and minimum of $f(x) = x^3 - 3x^2$ on $[1, 4]$.

$f'(x) = 3x(x-2)$ is zero at $x = 0$ and $x = 2$. Only $x = 2$ lies in $[1, 4]$, so the candidates are $x = 1$, $x = 2$ and $x = 4$:

x	1 **(endpoint)**	2 **(critical point)**	4 **(endpoint)**
$f(x)$	$1 - 3 = -2$	$8 - 12 = -4$	$64 - 48 = 16$

The absolute maximum is 16, at $x = 4$; the absolute minimum is -4, at $x = 2$. The maximum is at an endpoint where $f'(4) = 24 \neq 0$: this is why the endpoints must be on the list.

Concavity and Points of Inflection

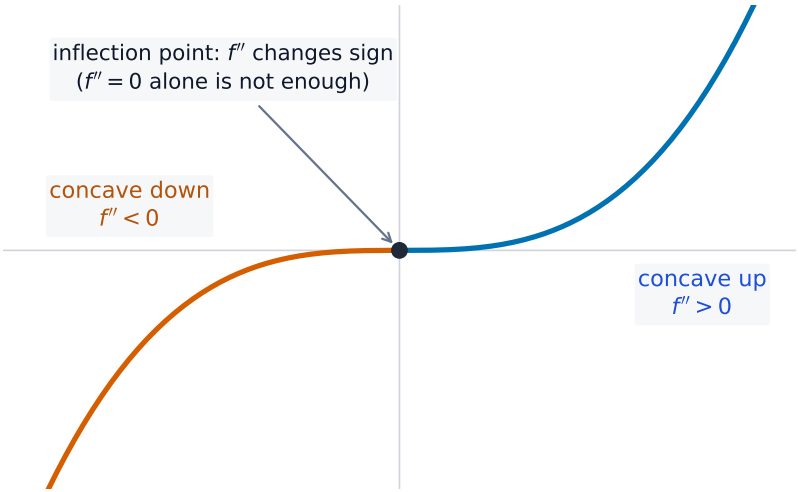
The **second derivative** 二阶导数 describes bending:

- $f'' > 0 \Rightarrow f$ is **concave up** 上凹 (curving like a cup; f' is increasing);
- $f'' < 0 \Rightarrow f$ is **concave down** 下凹 (f' is decreasing).

A **point of inflection** 拐点 is a point on a continuous graph where concavity **changes**. Where f'' exists on both sides, a sign change detects this (not merely $f'' = 0$); f'' may be undefined at the point itself. Report its x-coordinate and justify with the sign change of f''.

Worked example. For $f(x) = x^4 - 4x^3$, $f''(x) = 12x^2 - 24x = 12x(x-2)$. Its signs are $+, -, +$ around $x = 0$ and $x = 2$, so both are points of inflection. Compare $g(x) = x^4$: $g''(0) = 0$ but $g''(x) = 12x^2 \geq 0$ on both sides, so there is **no** inflection point at 0. A zero of f'' is only a candidate.

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Concavity comes from the sign of the second derivative; it flips at an inflection point

The Second Derivative Test for Extrema

An alternative way to classify a critical point c where $f'(c) = 0$:

- $f''(c) > 0 \Rightarrow$ concave up $\Rightarrow$ **local minimum**;
- $f''(c) < 0 \Rightarrow$ concave down $\Rightarrow$ **local maximum**;
- $f''(c) = 0 \Rightarrow$ the test is **inconclusive** 无法判定 – fall back on the **first derivative test** 一阶导数判别法.

> **Worked example (from a differential equation).** $y = f(x)$ is the solution of $\dfrac{dy}{dx} = xy - 6$ with $f(2) = 3$. Does f have a relative minimum, a relative maximum, or neither at $x = 2$?

At $(2, 3)$, $\dfrac{dy}{dx} = 2(3) - 6 = 0$, so $x = 2$ is a critical point. Differentiate the equation (product rule): $\dfrac{d^2y}{dx^2} = y + x\dfrac{dy}{dx}$. At $(2, 3)$ this is $3 + 2(0) = 3 > 0$. Since $f'(2) = 0$ and $f''(2) > 0$, f has a **relative minimum** at $x = 2$ by the second derivative test. You do not need to solve the differential equation.

Special case: if a continuous function has only **one** critical point on an interval and it is a local extremum, that point is also the **absolute** extremum there.

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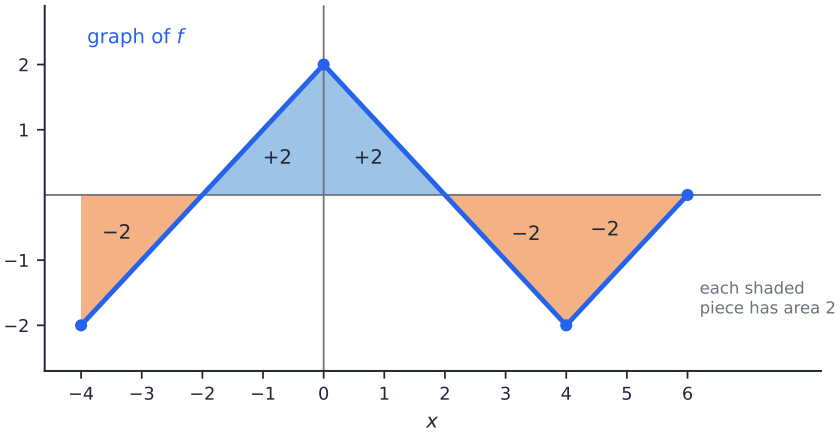
Sketching a Function and Its Derivative

Key features of f, f', and f'' mirror each other. To sketch or read graphs:

- $f' > 0$ on an interval $\Rightarrow$ f is increasing there; a change of f' from $+$ to $-$ at an interior point where f is continuous gives a local maximum.
- $f'' > 0$ on an interval $\Rightarrow$ f' is increasing and f is concave up there. A change in concavity at a point on a continuous graph gives an inflection point; f'' need not exist or be zero at that point.

Connecting f, f', and f''

This is the skill of reading one graph to describe another. A very common exam setup gives the graph of f' and asks about f: where is f increasing (where $f' > 0$), where are f's extrema (where f' crosses zero, with a sign change), where is f concave up (where f' is increasing). Answer questions about f using the *height and slope* of the f' graph.



The graph of f; $g(x) = \int_0^x f(t)\,dt$, so $g' = f$ and $g'' = f'$.

> **Worked example (the classic graph question).** Let $g(x) = \int_0^x f(t)\,dt$ for $-4 \le x \le 6$, with f shown above.

(a) *Where is g increasing and concave down?* g increases where $g' = f > 0$: on $(-2, 2)$. g is concave down where $g'' = f' < 0$, that is, where the graph of f falls: on $(0, 4)$. Both: $(0, 2)$.

(b) *Points of inflection of g.* $g'' = f'$ changes sign where f changes from rising to falling or back: at $x = 0$ and $x = 4$.

(c) *Absolute extrema of g on $[-4,6]$.* $g' = f = 0$ at $x = -2$ and $x = 2$. Use the candidates test with signed areas:

x	-4		-2	2	6
$g(x)$	$-(-2+2)=0$		-2	2	$2-2-2=-2$

The absolute maximum is 2, at $x = 2$; the absolute minimum is -2, reached at both $x = -2$ and $x = 6$. For $x < 0$, remember $g(x) = -\int_x^0 f(t)\,dt$: so $g(-2) = -(+2) = -2$.

Common mistake. Reading a graph of f' as if it were f: "f is increasing where the graph rises" is wrong. f increases where the graph of f' lies **above** the axis, and f has an inflection point where the graph of f' turns.

Introduction to Optimization Problems

Optimization 最优化 uses the derivative to find the largest or smallest value of a quantity on an interval. It is the candidates/derivative-test machinery applied to a real goal.

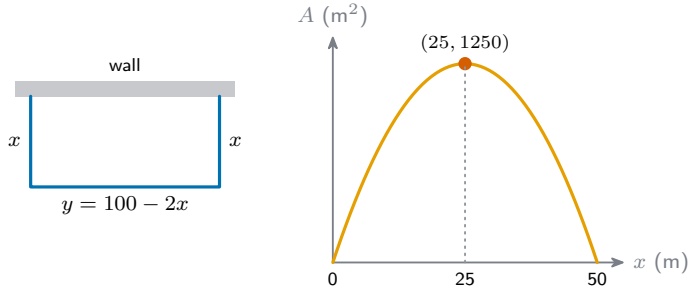


Fenced enclosures: optimization finds the dimensions that maximise area for a fixed fence length

Solving Optimization Problems

A dependable procedure:

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Use the constraint to get one variable, then optimise.

1. Write the quantity to optimize as a function of **one** variable (use a **constraint** 约束条件 equation to eliminate extras).
2. State the interval of allowed inputs.
3. Find critical points ($f' = 0$ or undefined) and test them (first- or second-derivative test, or candidates test if the interval is closed).
4. Answer the question asked, **with units and interpretation** in context – the maximum *area*, the minimum *cost*, etc.

> **Worked example.** With 100 m of fence for a rectangular pen against a wall (only three sides fenced), let the ends be x and the far side $y = 100 - 2x$. The area $A(x) = x(100 - 2x) = 100x - 2x^2$ has $A'(x) = 100 - 4x = 0$ at $x = 25$; since $A'' = -4 < 0$ this is the maximum, giving $y = 50$ and $A = 1250$ m^2.

Common mistakes. Trying to optimize a function of two variables without using the constraint to remove one; skipping the justification that $x = 25$ gives a **maximum** (here $A'' < 0$); and stopping at $x = 25$ when the question asks for the largest **area**.

> **Worked example (shortest distance).** Find the point on $y = \sqrt{x}$ closest to $(3, 0)$.

Minimize the **square** of the distance – it has the same minimum and no square root: $D(x) = (x-3)^2 + (\sqrt{x})^2 = x^2 - 5x + 9$ for $x \geq 0$. $D'(x) = 2x - 5 = 0$ at $x = \frac{5}{2}$, and $D'' = 2 > 0$, so this is the minimum (the only critical point, so it is the absolute minimum). The point is $\left(\frac{5}{2}, \sqrt{\frac{5}{2}}\right)$ and the distance is $\sqrt{D\left(\frac{5}{2}\right)} = \sqrt{\frac{11}{4}} = \frac{\sqrt{11}}{2}$.

Exploring Behaviors of Implicit Relations

All of this extends to **implicitly defined** 隐式定义 relations. A critical point of an implicit relation is where $\frac{dy}{dx} = 0$ (**horizontal tangent** 水平切线) or is **undefined** 无

定义 (**vertical tangent** 竖直切线). Because $\dfrac{dy}{dx}$ is usually a relation in x **and** y, and the second derivative involves x, y, and $\dfrac{dy}{dx}$, substitute your first-derivative expression back in when finding $\dfrac{d^2y}{dx^2}$, then reason about concavity from its sign.

> **Worked example.** For $x^2 + y^2 - 6x = 7$: $2x + 2y\dfrac{dy}{dx} - 6 = 0$, so $\dfrac{dy}{dx} = \dfrac{3 - x}{y}$.

- Horizontal tangents: $x = 3$, then $9 + y^2 - 18 = 7$, so $y = \pm 4$: the points $(3, 4)$ and $(3, -4)$.
- Vertical tangents: $y = 0$, then $x^2 - 6x - 7 = 0$, so $x = 7$ or $x = -1$: the points $(7, 0)$ and $(-1, 0)$.
- Concavity: $\dfrac{d^2y}{dx^2} = \dfrac{-y - (3 - x)\dfrac{dy}{dx}}{y^2} = -\dfrac{y^2 + (3 - x)^2}{y^3} = -\dfrac{16}{y^3}$, using the curve: $(x - 3)^2 + y^2 = 16$. So the curve is concave down where $y > 0$ and concave up where $y < 0$ – the top and bottom of a circle.

Exam tips

- $f' > 0$ means increasing, $f' < 0$ decreasing; candidates for **extrema** are where $f' = 0$ or is undefined.
- Classify a critical point with the **first-derivative sign change** or the **second-derivative test** 二阶导数判别法 ($f'' > 0$ minimum, $f'' < 0$ maximum).
- $f'' > 0$ is concave up, $f'' < 0$ concave down; a **point of inflection** is where concavity changes (f'' changes sign).
- For an **absolute** extremum on a closed interval, also check the endpoints.
- Justify every conclusion by citing the sign of f' or f'' – the exam demands the reasoning, not just the answer.

Practice.

1. Does the MVT apply to $f(x) = x^{2/3}$ on $[-1, 8]$? Explain.
2. Find the absolute extrema of $f(x) = x - 2\sqrt{x}$ on $[0, 9]$.
3. $f'(x) = x(x - 3)^2$. Find the x-coordinates of the relative extrema of f and classify them.
4. Find the points of inflection of $f(x) = xe^{-x}$.
5. $\dfrac{dy}{dx} = x^2 - y$ and the solution passes through $(-2, 4)$. Does it have a relative minimum, relative maximum, or neither there?
6. A box with a square base and no lid must hold 32 cm^3. Find the dimensions that use the least material.

Answers.

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1. No: $f'(x) = \dfrac{2}{3x^{1/3}}$ does not exist at $x = 0$, which lies in $(-1, 8)$.
2. $f'(x) = 1 - \dfrac{1}{\sqrt{x}} = 0$ at $x = 1$. Candidates: $f(0) = 0$, $f(1) = -1$, $f(9) = 3$. Minimum -1 at $x = 1$; maximum 3 at $x = 9$.
3. Critical points $x = 0$ and $x = 3$. f' changes $-$ to $+$ at 0 (relative minimum); $f' \geq 0$ on both sides of 3 (neither).
4. $f''(x) = (x - 2)e^{-x}$ changes sign at $x = 2$: inflection point $(2,\ 2e^{-2})$.
5. $\dfrac{dy}{dx} = 4 - 4 = 0$. $\dfrac{d^2y}{dx^2} = 2x - \dfrac{dy}{dx} = -4 - 0 = -4 < 0$: relative maximum.
6. $x^2h = 32$; area $A = x^2 + 4xh = x^2 + \dfrac{128}{x}$. $A' = 2x - \dfrac{128}{x^2} = 0$ at $x = 4$, $A'' > 0$: base 4 cm by 4 cm, height 2 cm.

Integration and Accumulation of Change

AP Calculus BC

Exploring Accumulations of Change



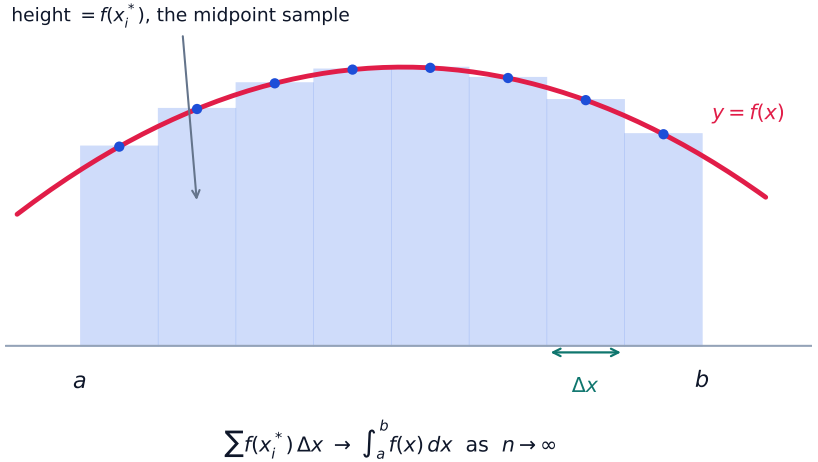
A water tank: integration accumulates change – total volume from a rate of flow

Where a **derivative** 导数 measures a **rate**, an **integral** 积分 measures an **accumulation** 累积 – a total built up from a rate. If a rate of change acts over an interval, the **area** 面积 between its graph and the axis gives the net accumulated change. This "area = total change" idea is the foundation of **integral calculus** 积分学.

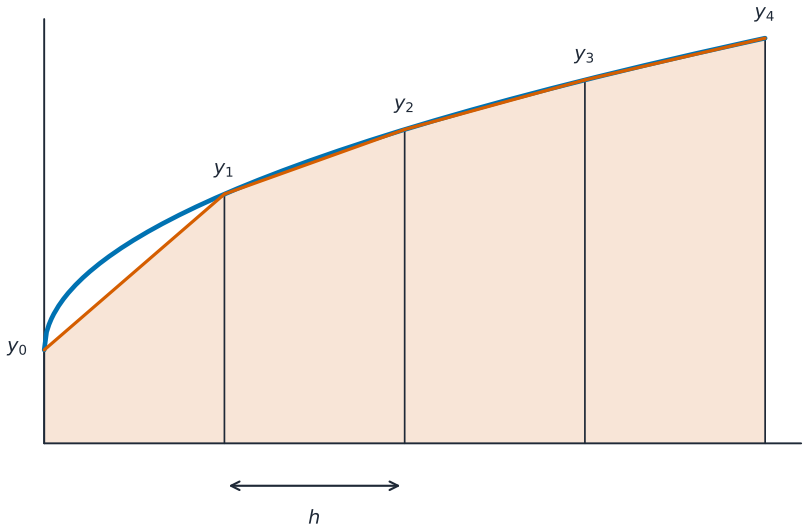
Approximating Areas with Riemann Sums

A **Riemann sum** 黎曼和 estimates the area under a curve by adding the areas of thin **rectangles** 矩形. Split $[a, b]$ into **subintervals** 子区间 and use the function's height at the **left** endpoint, **right** endpoint, or **midpoint** 中点 of each. A **trapezoidal sum** 梯形法 uses **trapezoids** 梯形 instead, averaging the two endpoint heights – usually more accurate. With more, thinner rectangles the estimate improves.

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A Riemann sum approximates the area under a curve with rectangles



Strips of width h approximate the area under a curve

Exam skill: be able to compute left, right, midpoint, and trapezoidal estimates from a table or graph, and state whether each **over- or under-estimates** based on whether the function is increasing/decreasing or concave up/down.

The rules. For an **increasing** function, a left sum underestimates (each rectangle

sits below the curve) and a right sum overestimates; for a **decreasing** function it is the other way round. A **trapezoidal** sum overestimates when the function is **concave up** (each chord lies above the curve) and underestimates when it is **concave down**. Give the reason with the answer: "overestimate, because f is increasing" earns the point; "overestimate" alone does not.

> **Worked example.** A table gives $f(0)=3$, $f(2)=5$, $f(4)=8$, $f(6)=9$, and f is increasing. A **right** Riemann sum with three strips of width 2 gives
>
> $$2\big(f(2)+f(4)+f(6)\big)=2(5+8+9)=44.$$
>
> Since f is increasing, this right sum is an **overestimate** 高估; the left sum $2(3+5+8)=32$ is an **underestimate** 低估, so the true value lies between 32 and 44.

> **Worked example: trapezoidal sum, unequal widths.** Water flows into a tank at rate $R(t)$ litres per minute.

t (minutes)	0	2	5	10
$R(t)$ (litres/min)	4	6	7	3

Estimate $\int_0^{10} R(t)\,dt$ with a trapezoidal sum. The widths are 2, 3 and 5, so each trapezoid needs its own width:

$$\int_0^{10} R(t)\,dt \approx 2\cdot\frac{4+6}{2}+3\cdot\frac{6+7}{2}+5\cdot\frac{7+3}{2}=10+19.5+25=54.5 \text{ litres}.$$

A left Riemann sum on the same table is $2(4)+3(6)+5(7)=61$ litres. Using one width for every strip when the table steps are unequal is a common error.

Riemann Sums, Summation Notation, and Definite Integral Notation

Writing a Riemann sum with **summation notation** 求和记号 $\sum_{k=1}^{n} f(x_k)\,\Delta x$ and letting the number of rectangles grow without bound gives the exact area – the **definite integral** 定积分:

$$\int_a^b f(x)\,dx=\lim_{n\to\infty}\sum_{k=1}^{n} f(x_k)\,\Delta x.$$

The integral is the **limit of Riemann sums**; a and b are the **limits of integration** 积分限.

> **Worked example (limit of a sum to an integral).** Write $\lim_{n\to\infty}\sum_{k=1}^{n}\frac{3}{n}\left(1+\frac{3k}{n}\right)^2$ as a definite integral and evaluate it.

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Match the pieces: $\Delta x=\frac{3}{n}$, so the interval has width 3; $x_k=1+\frac{3k}{n}$ runs from 1 to 4; and $f(x)=x^2$. So the limit is

$$\int_1^4 x^2\,dx=\frac{4^3-1^3}{3}=21.$$

The same limit is also $\int_0^3 (1+x)^2\,dx$; both are correct as long as f and the limits match.

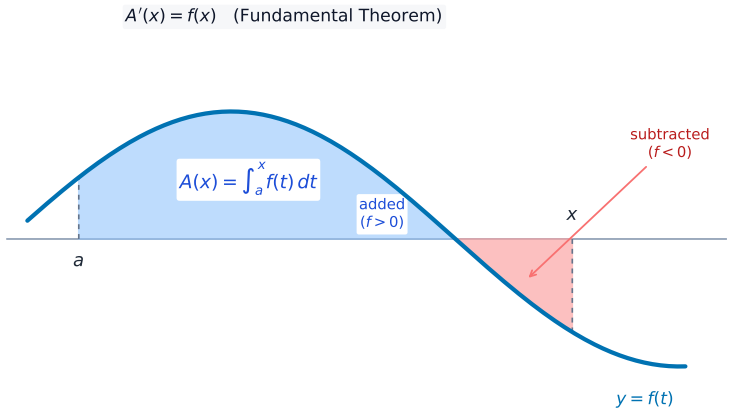
The Fundamental Theorem of Calculus and Accumulation Functions

An **accumulation function** 累积函数 $g(x)=\int_a^x f(t)\,dt$ gives the accumulated area from a up to x. The **Fundamental Theorem of Calculus (FTC)** 微积分基本定理 says, when f is continuous, its derivative is the **integrand** 被积函数:

$$\frac{d}{dx}\int_a^x f(t)\,dt=f(x).$$

Differentiation and integration are **inverse** operations. With a variable **upper limit** 上限 and the **chain rule** 链式法则, $\frac{d}{dx}\int_a^{u(x)} f(t)\,dt=f(u(x))\,u'(x)$.

Common mistakes. Forgetting the chain-rule factor: $\frac{d}{dx}\int_1^{x^2}\cos t\,dt=\cos(x^2)\cdot 2x$, not $\cos(x^2)$. Differentiating f instead of evaluating it: the derivative of $\int_a^x f(t)\,dt$ is $f(x)$, not $f'(x)$.



The accumulation function adds signed area; the FTC says its derivative is f

Interpreting the Behavior of Accumulation Functions

Because $g'(x) = f(x)$, the graph of f tells you everything about g: g **increases** where $f > 0$, **decreases** where $f < 0$, has **extrema** where f crosses zero, and is **concave up** where f is increasing. Reading these connections off a graph of f is a classic free-response task.

Common mistake. Adding an area below the axis instead of subtracting it. Suppose f lies above the axis on $[0, 3]$ with area 6 and below it on $[3, 5]$ with area 2. Then $g(5) = \int_0^5 f(t)\,dt = 6 - 2 = 4$, and g has its maximum at $x = 3$, where f changes from positive to negative. The total area, 8, answers a different question.

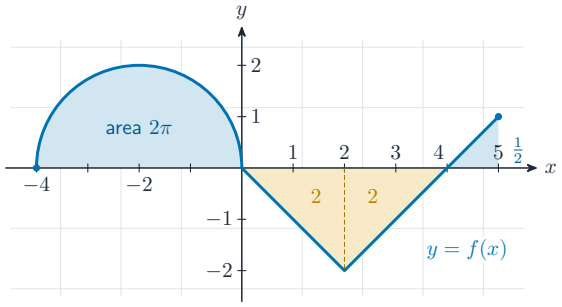
Worked example (values and derivatives of g). $g(x) = 5 + \int_1^x f(t)\,dt$, where $\int_1^3 f(t)\,dt = 4$ and $f(3) = -2$.

- $g(3) = 5 + 4 = 9$, and $g'(3) = f(3) = -2$.
- For $h(x) = \dfrac{g(x)}{5x}$, the quotient rule gives $h'(3) = \dfrac{g'(3)\cdot 15 - g(3)\cdot 5}{15^2} = \dfrac{-30 - 45}{225} = -\dfrac{1}{3}$.

The FTC turns "the derivative of g" into "the value of f", so a graph or table of f gives g' directly.

■ Worked example: the graph-of-f question

This is one of the most common free-response questions. The graph of f on $[-4, 5]$ is an upper semicircle of radius 2 and three line segments. Let $g(x) = \int_0^x f(t)\,dt$.



The graph of f: a semicircle and three line segments, with the area of each region marked

(a) Find $g(-4)$, $g(2)$ and $g(5)$. Use the areas, with a sign. The semicircle has area $\frac{1}{2}\pi(2)^2 = 2\pi$, and each triangle below the axis has area $\frac{1}{2}(2)(2) = 2$.

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- $g(-4) = \int_0^{-4} f(t)\,dt = -\int_{-4}^0 f(t)\,dt = -2\pi$ (backwards over a positive area).
- $g(2) = -2$ (one triangle below the axis).
- $g(5) = -2 - 2 + \frac{1}{2} = -3.5$.

(b) On which intervals is g increasing? Find the x-values of the relative extrema of g. Since $g'(x) = f(x)$, g is increasing on $[-4, 0]$ and $[4, 5]$, where $f \geq 0$. g has a **relative maximum** at $x = 0$, because $g' = f$ changes from positive to negative there, and a **relative minimum** at $x = 4$, because $g' = f$ changes from negative to positive.

(c) Find the x-values of the points of inflection of g. $g''(x) = f'(x)$, so look at where f changes from increasing to decreasing, or back. f increases on $(-4, -2)$, decreases on $(-2, 2)$ and increases on $(2, 5)$, so g has points of inflection at $x = -2$ and $x = 2$. ($x = 0$ is not one: f is decreasing on both sides of it.)

(d) Find the absolute maximum and minimum values of g on $[-4, 5]$. Use the candidates test: check the endpoints and the critical points.

x	$g(x)$	**Why it is a candidate**
-4	$-2\pi \approx -6.28$	endpoint
0	0	critical point: $g'(0) = f(0) = 0$
4	-4	critical point: $g'(4) = f(4) = 0$
5	-3.5	endpoint

The absolute maximum is 0, at $x = 0$; the absolute minimum is -2π, at $x = -4$.

(e) Write an equation of the line tangent to g at $x = 2$. $g(2) = -2$ and $g'(2) = f(2) = -2$, so $y = -2 - 2(x - 2)$.

Exam tip. Every justification names $g' = f$ (or $g'' = f'$) and says what **changes sign**: "$g' = f$ changes from positive to negative at $x = 0$" earns the point; "the graph goes down" does not. Read values of g from areas, never from the height of the graph of f: the height gives g'.

Applying Properties of Definite Integrals

Definite integrals obey useful rules: reversing the limits negates the value ($\int_b^a = -\int_a^b$), an integral over a zero-width interval is 0, they **add over adjacent intervals** 在相邻区间上相加 ($\int_a^c = \int_a^b + \int_b^c$), and constants factor out. Use these to combine or split given integral values.

Worked example. Given $\int_0^5 f(x)\,dx = 7$ and $\int_0^2 f(x)\,dx = 3$, find $\int_5^2 \big(2f(x) + 1\big)\,dx$.

Splitting gives $\int_2^5 f = 7 - 3 = 4$. Then $\int_2^5 \big(2f(x) + 1\big)\,dx = 2(4) + 3 = 11$, because

the constant 1 integrates to the width 3. Reversing the limits changes the sign: the answer is -11.

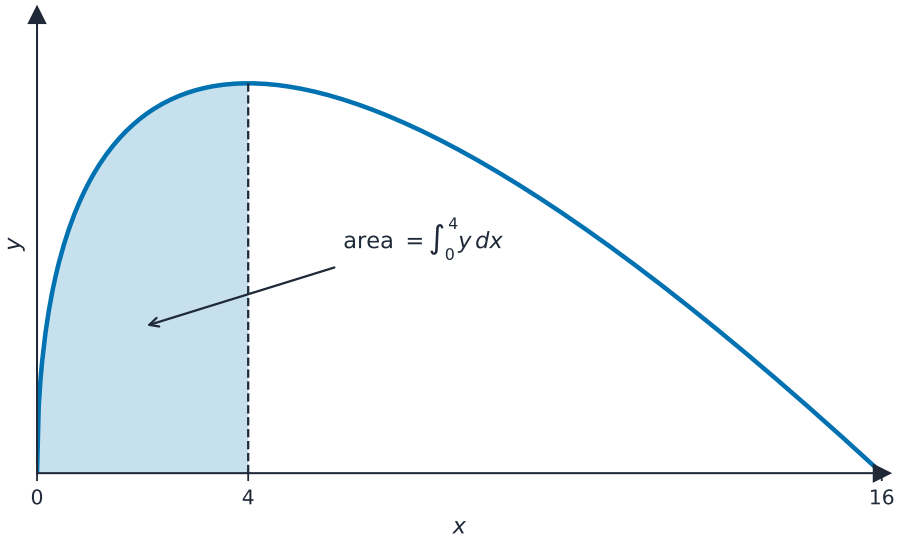
The Fundamental Theorem of Calculus and Definite Integrals

The **evaluation** form of the FTC computes a definite integral from an **antiderivative** 原函数 F (where $F' = f$):

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

So integrating a rate of change over $[a, b]$ gives the **net change** 净变化 in the quantity – the single most-used result in the course.

Worked example. $\int_1^3 (2x+1)\,dx$: an antiderivative is $F(x) = x^2 + x$, so the value is $F(3) - F(1) = 12 - 2 = 10$.



A definite integral is the signed area between the curve and the x-axis

Finding Antiderivatives and Indefinite Integrals

An **indefinite integral** 不定积分 $\int f(x)\,dx = F(x) + C$ is the family of all antiderivatives (hence the **constant of integration** 积分常数 C). Reverse each derivative rule: the **power rule** 幂法则 becomes $\int x^n\,dx = \dfrac{x^{n+1}}{n+1} + C$ (for $n \neq -1$), with $\int \frac{1}{x}\,dx = \ln|x| + C$, and the antiderivatives of e^x, $\sin x$, $\cos x$, and $\sec^2 x$ come straight from their derivatives.

Common mistakes. Leaving out $+C$; losing the minus sign in $\int \sin x\,dx = -\cos x + C$; and using the power rule on x^{-1}, which would divide by 0.

Integrating Using Substitution

u-substitution 换元积分 reverses the chain rule: choose $u = g(x)$ so that $g'(x)$ also appears, turning $\int f(g(x))g'(x)\,dx$ into $\int f(u)\,du$. Remember to convert dx to du and, for a definite integral, either change the limits to u-values or convert back to x at the end.

Worked example. Evaluate $\int_0^1 x\,(x^2+1)^3\,dx$. Let $u = x^2 + 1$, so $du = 2x\,dx$ and $x\,dx = \frac{1}{2}\,du$. Change the limits: $x = 0$ gives $u = 1$, and $x = 1$ gives $u = 2$:

$$\int_0^1 x\,(x^2+1)^3\,dx = \frac{1}{2}\int_1^2 u^3\,du = \frac{1}{2}\left[\frac{u^4}{4}\right]_1^2 = \frac{1}{8}\,(16-1) = \frac{15}{8}.$$

Integrating Using Long Division and Completing the Square

When a **rational** 有理 integrand is "top-heavy" (numerator degree $\geq$ denominator degree), **long division** 长除法 rewrites it as a **polynomial** 多项式 plus a **proper fraction** 真分式 you can integrate. **Completing the square** 配方法 in a **denominator** 分母 turns it into a form like $u^2 + a^2$, leading to an **arctangent** 反正切 antiderivative $\frac{1}{a}\arctan\frac{u}{a} + C$.

Worked example: long division. Since $x^2 + 1 = (x+1)(x-1) + 2$,

$$\int \frac{x^2+1}{x+1}\,dx = \int \left(x - 1 + \frac{2}{x+1}\right) dx = \frac{x^2}{2} - x + 2\ln|x+1| + C.$$

Worked example: completing the square. Since $x^2 + 2x + 5 = (x+1)^2 + 4$, take $u = x + 1$ and $a = 2$:

$$\int \frac{1}{x^2+2x+5}\,dx = \int \frac{1}{(x+1)^2+4}\,dx = \frac{1}{2}\arctan\left(\frac{x+1}{2}\right) + C.$$

Integrating Using Integration by Parts

Integration by parts 分部积分 reverses the product rule:

$$\int u\,dv = uv - \int v\,du.$$

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Choose u to simplify when differentiated and dv to be easy to integrate (the **LIATE** guide: logs, inverse-trig, algebraic, trig, exponential). It handles products like $\int xe^x\,dx$ and $\int x\ln x\,dx$, sometimes applied twice.

Worked example. For $\int xe^x\,dx$, choose $u = x$ $(du = dx)$ and $dv = e^x\,dx$ $(v = e^x)$:

$$\int xe^x\,dx = xe^x - \int e^x\,dx = xe^x - e^x + C = e^x(x-1) + C.$$

Worked example: definite. Evaluate $\displaystyle\int_1^e \ln x\,dx$. There is no obvious product, so take $u = \ln x$ $(du = \frac{1}{x}\,dx)$ and $dv = dx$ $(v = x)$:

$$\int_1^e \ln x\,dx = \Big[x\ln x\Big]_1^e - \int_1^e x\cdot\frac{1}{x}\,dx = (e\cdot 1 - 1\cdot 0) - (e-1) = 1.$$

Apply the limits to the uv part too; dropping them there is a frequent slip.

Worked example: by parts twice. $\displaystyle\int x^2e^x\,dx$. Take $u = x^2$, $dv = e^x\,dx$: $x^2e^x - \displaystyle\int 2xe^x\,dx$. The new integral needs parts again (or use the result above): $\displaystyle\int 2xe^x\,dx = 2e^x(x-1)$. So

$$\int x^2e^x\,dx = x^2e^x - 2xe^x + 2e^x + C = e^x\left(x^2 - 2x + 2\right) + C.$$

Check by differentiating: $e^x\left(x^2-2x+2\right) + e^x(2x-2) = x^2e^x$.

Integrating Using Linear Partial Fractions

Partial fractions 部分分式 split a rational function with a factorable denominator into a sum of simpler fractions:

$$\frac{1}{(x-a)(x-b)} = \frac{A}{x-a} + \frac{B}{x-b},$$

each of which integrates to a **logarithm** 对数. This technique is essential for the **logistic differential equation** 逻辑斯谛微分方程 in the next unit.

Worked example. Find $\displaystyle\int \frac{x+7}{(x-1)(x+3)}\,dx$. Write $\dfrac{x+7}{(x-1)(x+3)} = \dfrac{A}{x-1} + \dfrac{B}{x+3}$ and multiply through by the denominator:

$$x + 7 = A(x+3) + B(x-1).$$

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Put $x = 1$: $8 = 4A$, so $A = 2$. Put $x = -3$: $4 = -4B$, so $B = -1$. Then

$$\int \frac{x+7}{(x-1)(x+3)}\,dx = \int\left(\frac{2}{x-1} - \frac{1}{x+3}\right)dx = 2\ln|x-1| - \ln|x+3| + C.$$

Check by recombining: $2(x+3) - (x-1) = x + 7$.

Common mistakes. Splitting a top-heavy fraction without dividing first (when the top's degree is at least the bottom's, do long division before partial fractions), and writing $\int \frac{1}{x+3}\,dx$ with the wrong sign or as $\ln(x+3)$ without the absolute value.

Evaluating Improper Integrals

An **improper integral** 反常积分 has an **infinite limit** of integration or an **infinite discontinuity** 无穷间断点 in the integrand. Evaluate it as a **limit**: $\int_a^\infty f\,dx = \lim_{b\to\infty}\int_a^b f\,dx$. If the limit is a finite number the integral **converges** 收敛; otherwise it **diverges** 发散. Split at every interior **singularity** 奇点, and split an integral over the whole real line at a finite point. Each resulting one-sided improper integral must converge separately; cancellation of infinite parts is not convergence.

Worked example. $\displaystyle\int_1^\infty \frac{1}{x^2}\,dx = \lim_{b\to\infty}\left[-\frac{1}{x}\right]_1^b = \lim_{b\to\infty}\left(1 - \frac{1}{b}\right) = 1$, so it **converges** to 1. By contrast $\int_1^\infty \frac{1}{x}\,dx$ gives $\lim_{b\to\infty}\ln b = \infty$ and **diverges** – the same integrand-shape can go either way.

Worked example: infinite discontinuity. $\dfrac{1}{\sqrt{x}}$ is unbounded at $x = 0$, so

$$\int_0^1 \frac{1}{\sqrt{x}}\,dx = \lim_{a\to 0^+}\Big[2\sqrt{x}\Big]_a^1 = \lim_{a\to 0^+}\left(2 - 2\sqrt{a}\right) = 2,$$

which **converges**. An integral like this looks ordinary, so check the integrand for a **vertical asymptote** 竖直渐近线 on $[a,b]$ before using $F(b) - F(a)$ directly.

Common mistake. Missing a vertical asymptote **inside** the interval. Blindly writing

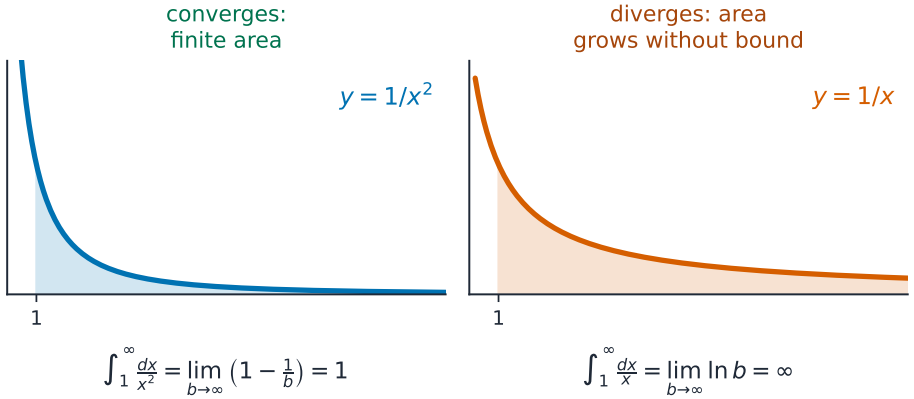
$$\int_{-1}^1 \frac{1}{x^2}\,dx = \Big[-\frac{1}{x}\Big]_{-1}^1 = -2$$

gives a negative answer for a positive integrand. The integrand is unbounded at $x = 0$: split there, and $\int_0^1 \frac{1}{x^2}\,dx$ diverges, so the whole integral **diverges**.

Worked example (partial fractions inside an improper integral). Use $\dfrac{3}{2x^2-7x+5} = \dfrac{2}{2x-5} - \dfrac{1}{x-1}$ to evaluate $\displaystyle\int_5^\infty \frac{3}{2x^2-7x+5}\,dx$.

$$\int_5^b \left(\frac{2}{2x-5} - \frac{1}{x-1}\right) dx = \Big[\ln|2x-5| - \ln|x-1|\Big]_5^b = \ln\frac{2b-5}{b-1} - \ln\frac{5}{4}.$$

As $b \to \infty$, $\dfrac{2b-5}{b-1} \to 2$, so the integral converges to $\ln 2 - \ln\frac{5}{4} = \ln\frac{8}{5}$. Combine the logarithms **before** taking the limit: separately, each term goes to infinity, and "$\infty - \infty$" tells you nothing.



The area under $1/x^2$ out to infinity is finite; under $1/x$ it is not

Selecting Techniques for Antidifferentiation

The BC exam expects you to **recognize which method fits**: basic rules, substitution (a chain-rule pattern), by parts (a product), partial fractions (a factorable rational), or long division/completing the square. Being able to look at an integral and pick the right tool quickly is itself a tested skill.

Worked example (choose the method).

integral	clue	method
$\displaystyle\int x\cos\left(x^2\right)dx$	the inside's derivative $2x$ is present	substitution: $\frac{1}{2}\sin\left(x^2\right)+C$
$\displaystyle\int x\cos x\,dx$	a product with no inside function	parts: $x\sin x+\cos x+C$
$\displaystyle\int \frac{1}{x^2-x}\,dx$	factorable denominator	partial fractions: $\ln\left\lvert\frac{x-1}{x}\right\rvert+C$
$\displaystyle\int \frac{1}{x^2+2x+2}\,dx$	denominator does not factor	complete the square: $\arctan(x+1)+C$

Exam tips

- For $n \neq -1$, use $\int x^n\,dx = \frac{x^{n+1}}{n+1} + C$ on an interval where the power rule applies. For $n = -1$, use $\ln|x| + C$. Do not forget $+C$.
- The **Fundamental Theorem** links the two: $\int_a^b f'(x)\,dx = f(b) - f(a)$, and $\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$.
- Approximate a definite integral with **Riemann sums** or the trapezoidal rule from a table of values.
- A definite integral is a **signed area** 有符号面积 (below the axis counts negative); split at sign changes for total area.
- Use **u-substitution** and remember to change the limits (or back-substitute) accordingly.

Practice.

1. Write $\lim\limits_{n\to\infty}\sum\limits_{k=1}^{n}\frac{2}{n}\sqrt{\frac{2k}{n}}$ as a definite integral and evaluate it.
2. Find $\dfrac{d}{dx}\displaystyle\int_x^3 \sqrt{1+t^4}\,dt$.
3. Find $\displaystyle\int x\ln x\,dx$.
4. Find $\displaystyle\int \frac{5}{x^2-6x}\,dx$.
5. Evaluate $\displaystyle\int_0^\infty xe^{-x}\,dx$ or show that it diverges.

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6. Does $\displaystyle\int_0^2 \frac{1}{(x-1)^2}\,dx$ converge?

Answers.

1. $\displaystyle\int_0^2 \sqrt{x}\,dx = \frac{2}{3}\cdot 2^{3/2} = \frac{4\sqrt{2}}{3}$.
2. $-\sqrt{1+x^4}$ (the variable is in the **lower** limit).
3. $u = \ln x$, $dv = x\,dx$: $\dfrac{x^2}{2}\ln x - \dfrac{x^2}{4} + C$.
4. $\dfrac{5}{x(x-6)} = \dfrac{5/6}{x-6} - \dfrac{5/6}{x}$, so $\dfrac{5}{6}\ln\left|\dfrac{x-6}{x}\right| + C$.
5. $\displaystyle\int_0^b xe^{-x}\,dx = 1-(b+1)e^{-b} \to 1$: converges to 1.
6. No: the integrand is unbounded at $x=1$, and $\displaystyle\int_1^2 \frac{dx}{(x-1)^2}$ diverges.

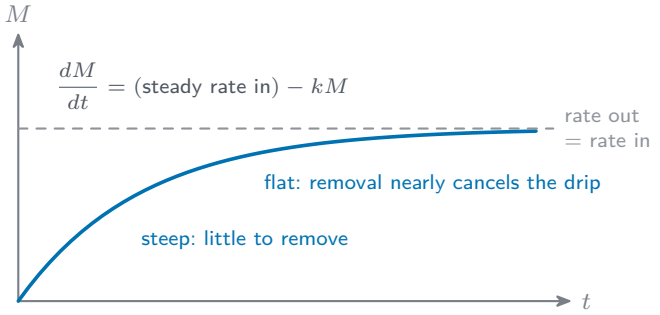
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Differential Equations

AP Calculus BC

Modeling Situations with Differential Equations

A nurse sets up a drip. Medicine flows into the blood at a steady rate, and the body removes it at a rate that grows with the amount already there. Nobody can write a formula for the amount $M(t)$ straight away, but everyone can say how fast it **changes**. That sentence is a differential equation; this unit teaches you to write one, read its picture, approximate it and solve it.



Read the picture from the rate.

A **differential equation** 微分方程 relates a function to its **derivatives** 导数. Many real situations are described by a rate: "the population grows at a rate proportional to its size" becomes $\dfrac{dP}{dt} = kP$, where k is a constant. Setting up the equation from a verbal description – identifying what changes and what it is **proportional** 成正比 to – is the first skill.

> **Worked example: from words to an equation.** Read "is proportional to" as "equals a constant k times", and "is inversely proportional to" as "equals k divided by".

sentence	differential equation
The temperature T of a drink changes at a rate proportional to the difference between T and the room temperature 20°C.	$\dfrac{dT}{dt} = k(T - 20)$
The depth h of water in a tank decreases at a rate proportional to $\sqrt{h}$.	$\dfrac{dh}{dt} = -k\sqrt{h}$, with $k > 0$
A rumour spreads at a rate proportional to the product of the number N who have heard it and the number $1000 - N$ who have not.	$\dfrac{dN}{dt} = kN(1000 - N)$

The word "decreases" gives the minus sign. The last equation is logistic (see the end of this unit).

Verifying Solutions for Differential Equations

A **solution** 解 is a function that satisfies the equation. To **verify** 验证 a proposed solution, differentiate it and substitute into the equation, checking that both sides agree. A **general solution** 通解 contains a constant C; a **particular solution** 特解 fixes C from a condition.

> **Worked example: an unknown linear solution.** For what values of m and b is the line $y = mx + b$ a solution of $\dfrac{dy}{dx} = 2x - y$?

Substitute: the left side is m; the right side is $2x - (mx + b) = (2 - m)x - b$. For $m = (2-m)x - b$ to hold for **every** x, the x-terms must match ($0 = 2 - m$, so $m = 2$) and the constants must match ($m = -b$, so $b = -2$). The solution is $y = 2x - 2$. Check: $\dfrac{dy}{dx} = 2$ and $2x - (2x - 2) = 2$.

Common mistake. Checking a single point. A solution must make the equation true at every x in its domain.

Sketching Slope Fields

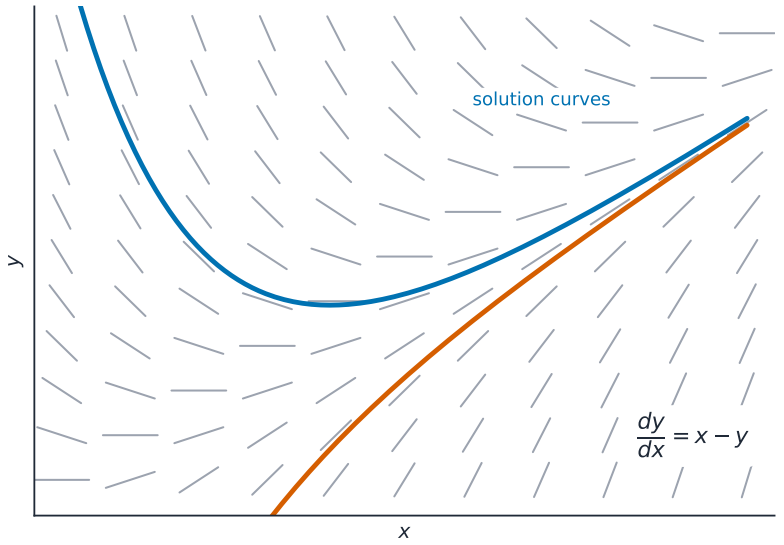
A **slope field** 斜率场 draws a short line segment at many points, each with the slope $\dfrac{dy}{dx}$ the equation gives there. It pictures the family of solution curves without solving. To sketch one, evaluate the right-hand side at each grid point and draw a segment of that slope.

> **Worked example.** For $\dfrac{dy}{dx} = x - y$, substitute each point into $x - y$:

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point (x, y)	$(0,0)$	$(1,0)$	$(0,1)$	$(1,1)$	$(2,1)$	$(-1,1)$
slope $x - y$	0	1	-1	0	1	-2

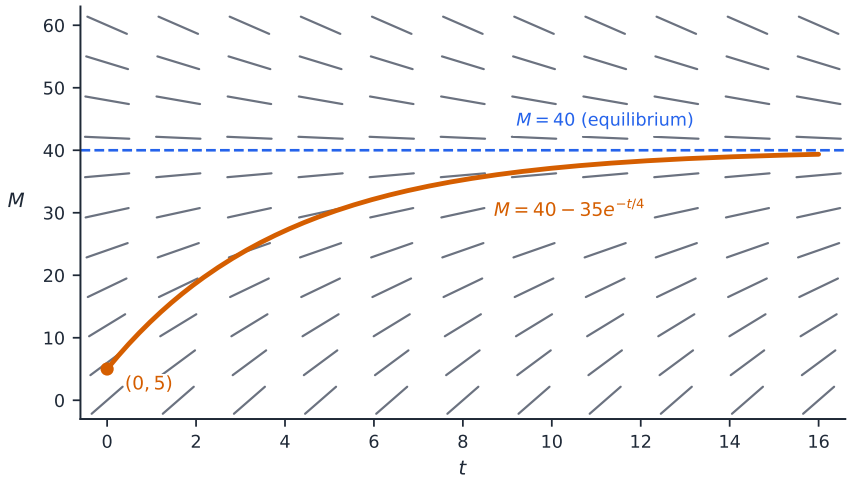
Every point on the line $y = x$ has slope 0 (flat segments), and every point on $y = x - 1$ has slope 1.



A slope field shows the gradient everywhere; solution curves follow it

Reasoning Using Slope Fields

A **solution curve** 解曲线 **follows the segments** like a boat following a current. From a slope field you can sketch the particular solution through a given point, describe long-run behavior, and locate where solutions level off (slopes near zero) – reasoning about solutions purely from the picture.



Slope field of $\frac{dM}{dt} = \frac{1}{4}(40 - M)$*: flat along* $M = 40$*, rising below it and falling above it. The solution through* $(0, 5)$ *climbs quickly, then levels off towards* 40.

Reading the field above. The right-hand side depends only on M, so every horizontal row of segments is identical. On the line $M = 40$ the slope is 0 at every t, so that line is itself a solution: an **equilibrium solution** 平衡解. Below it solutions rise; above it they fall; none crosses the line.

Exam skill: "Explain why this could not be the slope field." Find one point where the picture disagrees with the equation. For $\frac{dH}{dt} = -\frac{1}{15}(H - 20)$, every segment with $H > 20$ must have **negative** slope; a field showing positive slopes above $H = 20$, or slopes that change along a horizontal row, cannot be it. Name the point and both slopes.

Concavity from the equation. Differentiate the equation itself: $\frac{d^2M}{dt^2} = -\frac{1}{4}\frac{dM}{dt} = -\frac{1}{16}(40 - M)$. This is negative for $M < 40$, so the solution through $(0, 5)$ is concave down.

Approximating Solutions Using Euler's Method

Euler's method 欧拉方法 approximates a solution numerically by stepping along the slope field. Starting from a known point, take a small step Δx and update:

$$y_{\text{new}} = y_{\text{old}} + \frac{dy}{dx} \cdot \Delta x.$$

Repeat for each step. Smaller steps give a better **approximation** 近似值. (This is a BC-only technique.)

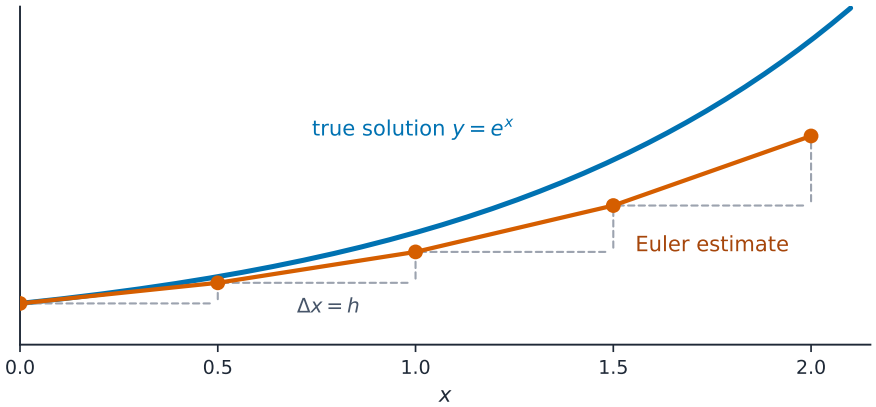
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Exam skill: be able to carry out two or three Euler steps by hand from a table, and know that Euler's method **underestimates** 低估 or **overestimates** 高估 depending on the solution's concavity.

> **Worked example.** Approximate $y(1)$ for $\frac{dy}{dx} = x + y$, $y(0) = 1$, with step $\Delta x = 0.5$. Step 1: slope at $(0, 1)$ is $0 + 1 = 1$, so $y(0.5) \approx 1 + 1(0.5) = 1.5$. Step 2: slope at $(0.5, 1.5)$ is $0.5 + 1.5 = 2$, so $y(1) \approx 1.5 + 2(0.5) = 2.5$.

Common mistakes. Forgetting to multiply the slope by Δx, and working out each slope from the **original** point instead of the point you have just reached: step 2 uses (0.5, 1.5).

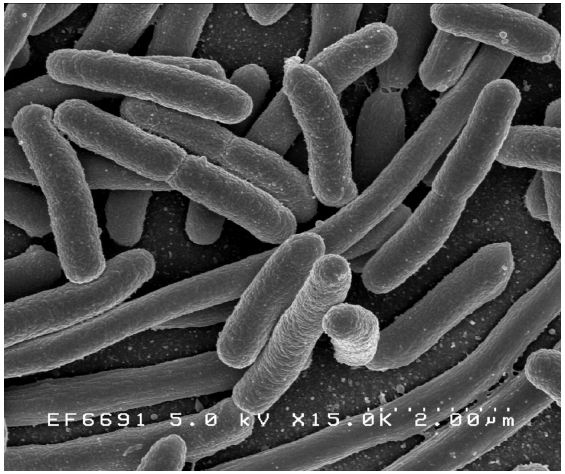
Over or under? Differentiate the equation: $\frac{d^2y}{dx^2} = 1 + \frac{dy}{dx} = 1 + x + y$, which is positive near these points. The solution is concave up, so each tangent step lies **below** the curve and 2.5 is an **underestimate**. (The exact solution is $y = 2e^x - x - 1$, so $y(1) = 2e - 2 \approx 3.44$.)



$\frac{dy}{dx} = y,\ y(0) = 1\text{:}\quad y_{n+1} = y_n + h\,y_n$

Euler's method steps along the tangent; the estimate drifts from the true solution

Finding General Solutions Using Separation of Variables



E. coli under a microscope: differential equations model exponential growth

In **separation of variables** 分离变量法, a **separable** 可分离 differential equation is written with all the y's on one side and all the x's on the other, then integrated (where the divided factor is nonzero):

$$\frac{dy}{dx} = g(x)h(y) \;\Rightarrow\; \int \frac{dy}{h(y)} = \int g(x)\,dx.$$

This produces the general solution (with a $+C$), the main analytic method for solving differential equations in this course. Also check **constant solutions** 常数解 with $h(y) = 0$ separately; division by $h(y)$ can lose them.

Worked example. Solve $\dfrac{dy}{dx} = xy$ with $y(0) = 2$. Separating, $\int \frac{dy}{y} = \int x\,dx$ gives $\ln|y| = \frac{x^2}{2} + C$, so $y = Ae^{x^2/2}$. The condition $y(0) = 2$ gives $A = 2$, so $y = 2e^{x^2/2}$.

Common mistakes. Adding the constant at the end: $y = e^{x^2/2} + C$ is wrong, because the constant enters when you integrate and becomes the multiplier $A = e^C$. Trying to separate an equation such as $\dfrac{dy}{dx} = x + y$: a sum cannot be written as $g(x)\,h(y)$.

Worked example (integration by parts inside). Find the particular solution of $\dfrac{dy}{dx} = y\,(x \ln x)$ with $f(1) = 4$.

$$\int \frac{dy}{y} = \int x \ln x\,dx \;\Rightarrow\; \ln|y| = \frac{x^2}{2}\ln x - \frac{x^2}{4} + C.$$

(By parts with $u = \ln x$, $dv = x\,dx$.) At $(1, 4)$: $\ln 4 = 0 - \frac{1}{4} + C$, so $C = \ln 4 + \frac{1}{4}$. Since $y > 0$,

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$$y = 4\,e^{\frac{x^2}{2}\ln x - \frac{x^2}{4} + \frac{1}{4}}.$$

Worked example (a critical point from the equation). $\dfrac{dH}{dt} = \frac{1}{2}(H - 1)\cos\left(\frac{t}{2}\right)$ with $H(0) = 4$. Separate: $\int \frac{dH}{H-1} = \int \frac{1}{2}\cos\left(\frac{t}{2}\right)dt$, so $\ln|H - 1| = \sin\left(\frac{t}{2}\right) + C$. At $t = 0$: $\ln 3 = C$, and

$$H = 1 + 3e^{\sin(t/2)}.$$

For $0 < t < 5$, $H > 1$, so $\dfrac{dH}{dt} = 0$ only when $\cos\left(\frac{t}{2}\right) = 0$, at $t = \pi$. The factor $\cos\left(\frac{t}{2}\right)$ changes from positive to negative there, so $\dfrac{dH}{dt}$ does too: H has a **relative maximum** at $t = \pi$. You can answer this from the differential equation alone, without the solution.

Finding Particular Solutions Using Initial Conditions

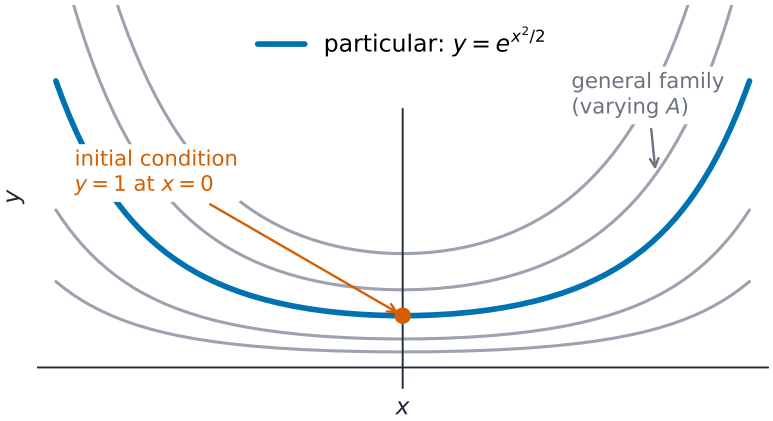
An **initial condition** 初始条件 (a known point, e.g. $y(0) = 5$) pins down the constant C. Solve for the general solution, substitute the condition to find C, then write the particular solution. Watch the **domain** 定义域 – a particular solution is valid only on the interval containing the initial point.

Worked example. Solve $\dfrac{dy}{dx} = \dfrac{x}{y}$ with $y(0) = -3$. Separating gives $\dfrac{y^2}{2} = \dfrac{x^2}{2} + C$. Substitute the initial point now: $\frac{9}{2} = C$, so $y^2 = x^2 + 9$. The initial value is negative, so take the negative root:

$$y = -\sqrt{x^2 + 9}.$$

The positive root fails $y(0) = -3$; choosing the sign from the initial condition is a scored step.

Worked example (domain). $\dfrac{dy}{dx} = xy^2$ with $y(1) = 1$: $-\dfrac{1}{y} = \dfrac{x^2}{2} + C$ gives $C = -\frac{3}{2}$, so $y = \dfrac{2}{3 - x^2}$. The formula breaks at $x = \pm\sqrt{3}$; the solution lives on the interval that contains $x = 1$, so its domain is $-\sqrt{3} < x < \sqrt{3}$.



The constant gives a family of curves; an initial condition picks out one

Exponential Models with Differential Equations

The equation $\dfrac{dy}{dt} = ky$ says the rate of change is **proportional** to the amount – giving **exponential growth or decay** 指数增长或衰减. Separating variables yields

$$y = y_0 e^{kt},$$

with $k > 0$ for growth and $k < 0$ for decay. This models unrestricted population growth, **radioactive decay** 放射性衰变, and **continuously compounded interest** 连续复利.



A cooling cup of coffee: Newton's law of cooling is a classic differential equation model

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> **Worked example: Newton's law of cooling.** Coffee at 90°C cools in a room at 20°C by $\dfrac{dT}{dt} = -k\,(T - 20)$, with t in minutes. After 5 minutes it is 70°C. When does it reach 40°C?

The difference $T - 20$ decays exponentially from $90 - 20 = 70$, so

$$T = 20 + 70e^{-kt}.$$

From $T(5) = 70$: $50 = 70e^{-5k}$, so $k = \dfrac{\ln(7/5)}{5} \approx 0.0673$. For $T = 40$: $20 = 70e^{-kt}$, so $t = \dfrac{\ln 3.5}{k} \approx 18.6$ minutes. Writing $T = 90e^{-kt}$ (cooling towards 0°C) is the common error.

> **Worked example: limited growth.** The amount of medicine M, in milligrams, satisfies $\dfrac{dM}{dt} = \frac{1}{4}(40 - M)$, t in hours, $M(0) = 5$.

(a) Separate: $-\ln|40-M| = \dfrac{t}{4} + C$, and $C = -\ln 35$. Since $M < 40$, $M = 40 - 35e^{-t/4}$.

(b) $\lim\limits_{t\to\infty} M(t) = 40$: over a long time the amount of medicine in the blood approaches 40 milligrams. Give the units and the context.

(c) The tangent line at $t = 0$ has slope $M'(0) = 8.75$, so $M(2) \approx 5 + 8.75(2) = 22.5$ mg. Since $\dfrac{d^2M}{dt^2} < 0$ (from the slope-field section), the curve is concave down and lies below its tangents: 22.5 is an **overestimate**. (Exact: $M(2) \approx 18.8$ mg.)

Logistic Models with Differential Equations

Real growth is limited by resources, so the **logistic model** 逻辑斯蒂模型 adds a **carrying capacity** 环境容纳量 L:

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{L}\right).$$

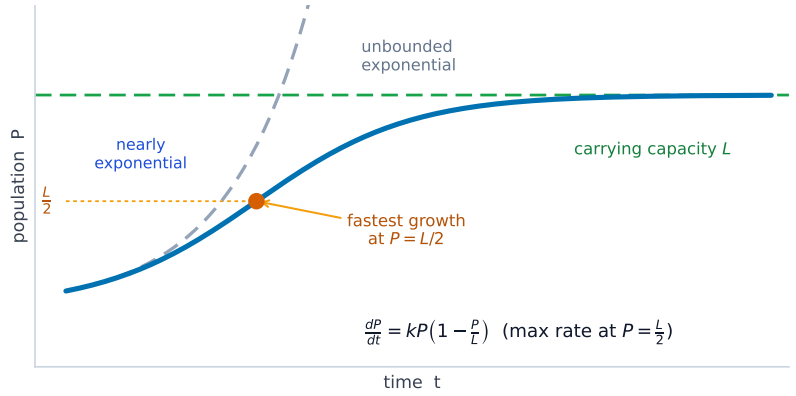
For $k > 0$, $L > 0$ and $0 < P(0) < L$, growth is nearly exponential when P is small, then the growth rate falls after $P = L/2$ as the population approaches L. The states $P = 0$ and $P = L$ are **equilibrium solutions**; a population initially at zero stays zero. Key BC facts: the population **levels off at** L ($\lim_{t\to\infty} P = L$), and it grows **fastest when** $P = \frac{L}{2}$ (the inflection point of the S-shaped curve). You are expected to read L and the fastest-growth value directly from the equation. If $P(0) > L$, the population instead **decreases** towards L, so $\lim_{t\to\infty} P = L$ still holds.

Worked example. For $\dfrac{dP}{dt} = 0.05\,P\left(1 - \dfrac{P}{2000}\right)$, the carrying capacity is $L = 2000$ (the population levels off there), and growth is fastest when $P = \dfrac{L}{2} = 1000$ – both read straight off the equation, no solving needed.

Common mistakes. Saying growth is fastest when $P = L$: at $P = L$ the growth rate is 0. Giving a **time** when the question asks for the population at which growth is fastest ($P = 1000$).

Why $\frac{L}{2}$? Write $\dfrac{dP}{dt} = kP - \dfrac{k}{L}P^2$. Differentiate with respect to t: $\dfrac{d^2P}{dt^2} = \left(k - \dfrac{2k}{L}P\right)\dfrac{dP}{dt}$. For $0 < P < L$, $\dfrac{dP}{dt} > 0$, so $\dfrac{d^2P}{dt^2}$ changes sign where $k - \dfrac{2k}{L}P = 0$, at $P = \dfrac{L}{2}$. That is where the growth rate is largest.

Worked example (reading the logistic form). $\dfrac{dP}{dt} = \dfrac{P(600 - P)}{300}$ with $P(0) = 800$. Rewrite: $\dfrac{dP}{dt} = 2P\left(1 - \dfrac{P}{600}\right)$, so $k = 2$ and $L = 600$. Because $P(0) > 600$, $\dfrac{dP}{dt} < 0$: the population decreases towards 600, and $\lim\limits_{t\to\infty} P(t) = 600$. It never passes through $\frac{L}{2} = 300$, so it has no point of fastest growth.



logistic: $dP/dt = kP(1 - P/L)$ · $P \to L$ as $t \to \infty$

The logistic model grows fastest at P=L/2 and levels off at the carrying capacity L

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Exam tips

- Solve a **separable** equation by getting all y on one side and all x on the other, then integrating both sides (add $+C$ once).
- Use the initial condition to find C (a particular solution).
- Sketch or read a **slope field**: the little segments show $\frac{dy}{dx}$ at each point, and a solution curve follows them.
- Recognise **exponential** models $\frac{dy}{dt} = ky \Rightarrow y = Ce^{kt}$ (growth/decay).
- A differential equation gives the slope – you must integrate to recover the function.
- **Logistic** $\frac{dP}{dt} = kP\left(1 - \frac{P}{L}\right)$: the limit is L (for $P(0) > 0$) and growth is fastest at $P = \frac{L}{2}$.

Practice.

1. Verify that $y = 3e^{2x} - 1$ is a solution of $\dfrac{dy}{dx} = 2y + 2$.
2. For $\dfrac{dy}{dx} = y - x$, use Euler's method with two steps of size 0.5 from $y(0) = 2$ to estimate $y(1)$.
3. Find the particular solution of $\dfrac{dy}{dx} = \dfrac{x^2}{y}$ with $y(0) = -2$.
4. $\dfrac{dA}{dt} = \dfrac{12 - A}{3}$, $A(0) = 0$, with A in grams and t in minutes. Find $A(t)$ and interpret $\lim\limits_{t\to\infty} A(t)$.
5. A population satisfies $\dfrac{dP}{dt} = 0.4P - 0.0002P^2$ with $P(0) = 300$. Find $\lim\limits_{t\to\infty} P(t)$ and the population when it grows fastest.
6. $\dfrac{dy}{dt} = 0.5(y - 10)$ with $y(0) = 4$. Find $\dfrac{d^2y}{dt^2}$ in terms of y, use the tangent line at $t = 0$ to estimate $y(1)$, and decide whether the estimate is too big or too small.

Answers.

1. $\dfrac{dy}{dx} = 6e^{2x}$ and $2y + 2 = 6e^{2x}$; the sides agree for all x.
2. Slope at $(0, 2)$ is 2, so $y(0.5) \approx 2 + 0.5(2) = 3$. Slope at $(0.5, 3)$ is 2.5, so $y(1) \approx 3 + 0.5(2.5) = 4.25$.
3. $\dfrac{y^2}{2} = \dfrac{x^3}{3} + 2$, and $y(0) < 0$, so $y = -\sqrt{\dfrac{2x^3}{3} + 4}$ for $x > -\sqrt[3]{6}$.
4. $A = 12 - 12e^{-t/3}$; the limit is 12: over a long time the amount approaches 12 grams.
5. $0.4P\left(1 - \dfrac{P}{2000}\right)$: $L = 2000$, so the limit is 2000; fastest growth at $P = 1000$.
6. $\dfrac{d^2y}{dt^2} = 0.25(y - 10)$. $y'(0) = -3$, so $y(1) \approx 1$. For $y < 10$ the solution is concave down, so the estimate is too big (exact $10 - 6e^{0.5} \approx 0.11$).

Applications of Integration

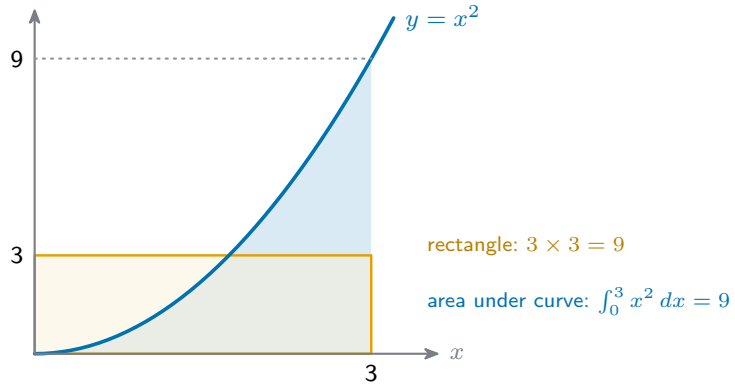
AP Calculus BC

Finding the Average Value of a Function on an Interval

The **average value** 平均值 of f over $[a, b]$ is the **integral** 积分 divided by the width:

$$f_{\text{avg}} = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

It is the constant height a **rectangle** 矩形 would need to have the same area as the region under f. Do not confuse this with the average **rate of change** 变化率 (which uses the **derivative** 导数).



The average value is the height of the equal-area rectangle.

Worked example. The average value of $f(x) = x^2$ on $[0, 3]$ is $\frac{1}{3}\int_0^3 x^2\,dx = \frac{1}{3}\left[\frac{x^3}{3}\right]_0^3 = \frac{1}{3}(9) = 3$.

Worked example (average from a table). The height H of a tree, in metres, is measured at times t in years.

t	2	3	5	7	10
$H(t)$	1.5	2	6	11	15

Use a trapezoidal sum with the four subintervals to estimate the average height for $2 \le t \le 10$. The widths are **unequal** $(1, 2, 2, 3)$, so compute each trapezoid separately:

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$$\int_2^{10} H\,dt \approx 1\cdot\frac{1.5+2}{2} + 2\cdot\frac{2+6}{2} + 2\cdot\frac{6+11}{2} + 3\cdot\frac{11+15}{2} = 1.75 + 8 + 17 + 39 = 65.75.$$

The average height is about $\frac{65.75}{10-2} \approx 8.22$ metres. Forgetting to divide by $b - a = 8$ is the most common error.

When is the function equal to its average? If a question asks for the time t at which $A(t)$ equals its average value on $[0, 30]$, first compute $A_{\text{avg}} = \frac{1}{30}\int_0^{30} A(t)\,dt$ on the calculator, then solve $A(t) = A_{\text{avg}}$ with the calculator. The Mean Value Theorem for integrals promises such a t exists when A is continuous.

Connecting Position, Velocity, and Acceleration Using Integrals

For straight-line motion, **integration** 积分法 reverses **differentiation** 求导:

$$v(t) = \int a(t)\,dt, \qquad s(t) = \int v(t)\,dt.$$

Two key distinctions: **displacement** 位移 is $\int_a^b v\,dt$ (**net change** 净变化 in position), while **total distance** 总路程 is $\int_a^b |v|\,dt$ (splitting where v changes sign). **Speed** 速率 is $|v|$.



A speedometer shows speed at each instant; integrating velocity over time gives displacement

Worked example. A particle has $v(t) = t^2 - 4t + 3$ for $0 \le t \le 3$ (metres per second).

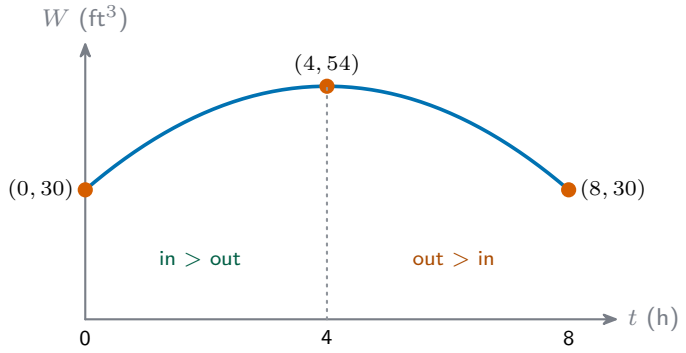
- Displacement: $\int_0^3 (t^2 - 4t + 3)\,dt = \left[\frac{t^3}{3} - 2t^2 + 3t\right]_0^3 = 9 - 18 + 9 = 0$ m. The particle ends where it started.
- Distance: $v = (t-1)(t-3)$ is negative on $(1, 3)$. $\int_0^1 v\,dt = \frac{4}{3}$ and $\int_1^3 v\,dt = -\frac{4}{3}$, so the total distance is $\frac{4}{3} + \frac{4}{3} = \frac{8}{3}$ m.

Position from a known point. If a train is 300 m east of the station at $t = 2$, its position is $x(t) = 300 + \int_2^t v(s)\,ds$. The lower limit is the time you **know** the position, not always 0.

Interpreting with units. "$\int_0^{40} |v(t)|\,dt$" with v in metres per second means "the total distance, in metres, that the object travels from $t = 0$ to $t = 40$ seconds". Name the quantity, the units and the time interval.

Using Accumulation Functions and Definite Integrals in Applied Contexts

When a rate is given (flow rate, sales per day), the **definite integral** 定积分 gives the **accumulated total** 累积总量, and $\int_a^b R(t)\,dt$ carries the **units** of R times time. A common setup: **initial amount** 初始量 $+ \int(\text{rate in} - \text{rate out})\,dt$ gives the amount at a later time. Always interpret the answer **in context, with units**.



Initial amount plus the integral of rate in minus rate out.

Worked example (rate in, rate out). Water flows into a pipe at 12 cubic feet per hour and drains out at $D(t) = 3t$ cubic feet per hour, for $0 \le t \le 8$. At $t = 0$ the pipe holds 30 cubic feet.

(a) *Amount.* $W(t) = 30 + \int_0^t (12 - 3s)\,ds = 30 + 12t - 1.5t^2$.

(b) *Increasing or decreasing at $t = 3$?* Compare the rates, not the amounts: $W'(3) = 12 - 3(3) = 3 > 0$, so the amount is increasing, because water enters faster than it leaves.

(c) *Minimum amount.* $W'(t) = 0$ at $t = 4$. Use the **candidates test** – check the critical point and both endpoints:

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t	0	4	8
$W(t)$	30	54	30

The minimum is 30 cubic feet, at $t = 0$ and $t = 8$; $t = 4$ is the maximum. A sign chart of W' alone can find the maximum, but only the table settles which endpoint is lowest.

Finding the Area Between Curves Expressed as Functions of x

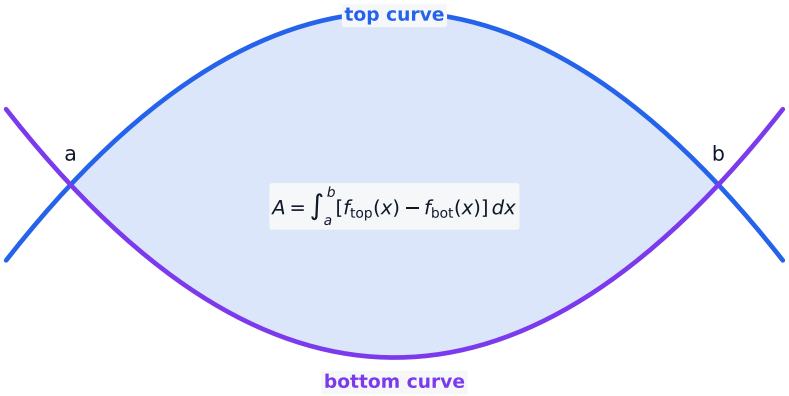


A potter shaping a vessel: volumes of revolution rotate a plane region about an axis

The area between $y = f(x)$ (top) and $y = g(x)$ (bottom) from a to b is

$$\int_a^b \big(f(x) - g(x)\big)\,dx.$$

Find the **intersection points** 交点 for the **limits of integration** 积分限, and always subtract **top minus bottom**.



The area between two curves is the integral of top minus bottom

Worked example. Between $y = x$ and $y = x^2$ (crossing at $x = 0, 1$, with $y = x$ on top), the area is $\displaystyle\int_0^1 (x - x^2)\,dx = \left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 = \frac{1}{6}$.

Finding the Area Between Curves Expressed as Functions of y

When curves are easier to describe as $x = f(y)$, integrate with respect to y instead, using **right minus left**:

$$\int_c^d \big(f_{\text{right}}(y) - g_{\text{left}}(y)\big)\,dy.$$

Choosing to integrate in y can avoid splitting the region into several pieces.

Worked example. Find the same area between $y = x$ and $y = x^2$ with horizontal slices. Solve each curve for x: the line is $x = y$ and the **parabola** 抛物线 is $x = \sqrt{y}$. For $0 < y < 1$, $\sqrt{y} > y$ (try $y = 0.25$: $0.5 > 0.25$), so the parabola is on the right:

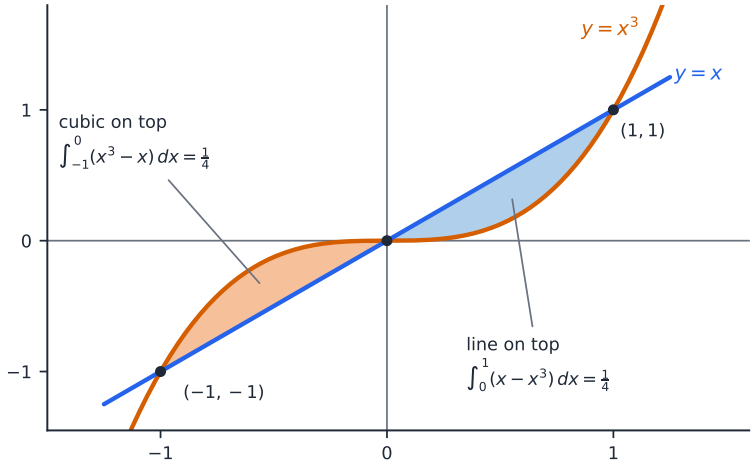
$$A = \int_0^1 \left(\sqrt{y} - y\right) dy = \left[\frac{2}{3}y^{3/2} - \frac{y^2}{2}\right]_0^1 = \frac{2}{3} - \frac{1}{2} = \frac{1}{6}.$$

Both methods give $\frac{1}{6}$, as they must.

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Finding the Area Between Curves That Intersect More Than Twice

If two curves cross several times, the top and bottom **switch**. Split the region at each intersection and integrate each piece with the correct top-minus-bottom (or use $\int |f - g|$), then add the pieces.



$y = x^3$ and $y = x$ cross three times, so the top curve switches at $x = 0$.

Worked example. Find the area between $y = x^3$ and $y = x$. They meet where $x^3 = x$, at $x = -1, 0, 1$. Test a point in each piece: at $x = -\frac{1}{2}$, $x^3 = -\frac{1}{8} > -\frac{1}{2}$, so the cubic is on top; at $x = \frac{1}{2}$ the line is on top.

$$A = \int_{-1}^{0} (x^3 - x)\,dx + \int_0^1 (x - x^3)\,dx = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}.$$

Common mistake. Writing $\displaystyle\int_{-1}^{1} (x - x^3)\,dx = 0$: the two pieces cancel. On the calculator, $\displaystyle\int_{-1}^{1} \left|x - x^3\right| dx$ gives $\frac{1}{2}$ in one step.

Volumes with Cross Sections: Squares and Rectangles

If a solid's **cross sections** 横截面 **perpendicular** 垂直 to the x-axis are **squares** 正方形 or rectangles, integrate their area to find the **volume** 体积. With side length equal to the distance between two curves, a square cross section gives

$$V = \int_a^b \big(f(x) - g(x)\big)^2\,dx.$$

The method is always "**integrate the cross-sectional area**."

> **Worked example.** The base of a solid is the region under $y = \sqrt{x}$ from $x = 0$ to $x = 4$, and cross sections perpendicular to the x-axis are squares. Each square's side is $s(x) = \sqrt{x}$, so $A(x) = x$ and
>
> $$V = \int_0^4 x\,dx = \left[\frac{x^2}{2}\right]_0^4 = 8.$$

With semicircles on the same base instead, $A(x) = \frac{\pi}{8}\left(\sqrt{x}\right)^2$, so $V = \frac{\pi}{8} \cdot 8 = \pi$: only the area formula changes.

> **Worked example.** The base of a solid is the region between $y = x$ and $y = x^2$. Cross sections perpendicular to the x-axis are squares. Find the volume.

The side of each square is the slice length $s(x) = x - x^2$, so $A(x) = (x - x^2)^2 = x^2 - 2x^3 + x^4$ and

$$V = \int_0^1 \left(x^2 - 2x^3 + x^4\right) dx = \frac{1}{3} - \frac{1}{2} + \frac{1}{5} = \frac{10 - 15 + 6}{30} = \frac{1}{30}.$$

Expand the square before integrating; $\int (x - x^2)^2\,dx$ is **not** $\left(\int (x - x^2)\,dx\right)^2$.

Volumes with Cross Sections: Triangles and Semicircles

Same idea, different area formula: for **equilateral triangle** 等边三角形 cross sections use $A = \frac{\sqrt{3}}{4}s^2$, and for **semicircular** 半圆形的 ones $A = \frac{\pi}{8}s^2$ (with s the distance between the curves). Substitute the area formula and integrate.

> **Worked example.** The base is the region between $y = x$ and $y = x^2$ for $0 \le x \le 1$. Cross sections perpendicular to the x-axis have side (or diameter) $s = x - x^2$, and the squares section found $\displaystyle\int_0^1 (x - x^2)^2\,dx = \frac{1}{30}$.

- Equilateral triangles: $V = \dfrac{\sqrt{3}}{4}\displaystyle\int_0^1 (x - x^2)^2\,dx = \frac{\sqrt{3}}{4} \cdot \frac{1}{30} = \frac{\sqrt{3}}{120}$.
- Semicircles: $V = \dfrac{\pi}{8} \cdot \dfrac{1}{30} = \dfrac{\pi}{240}$.

Only the constant in front changes: square 1, equilateral triangle $\frac{\sqrt{3}}{4}$, semicircle $\frac{\pi}{8}$. The semicircle area is $\frac{1}{2}\pi r^2$ with radius $r = \frac{s}{2}$: $\frac{1}{2}\pi\frac{s^2}{4} = \frac{\pi}{8}s^2$. Using s as the radius is the usual slip.

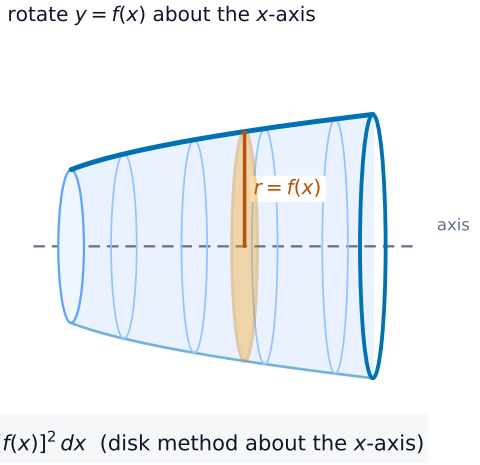
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Volume with Disc Method: Revolving Around the x- or y-Axis

Revolving a region around an axis makes a **solid of revolution** 旋转体 whose cross sections are **discs**. The **disc method** 圆盘法 integrates $\pi(\text{radius})^2$:

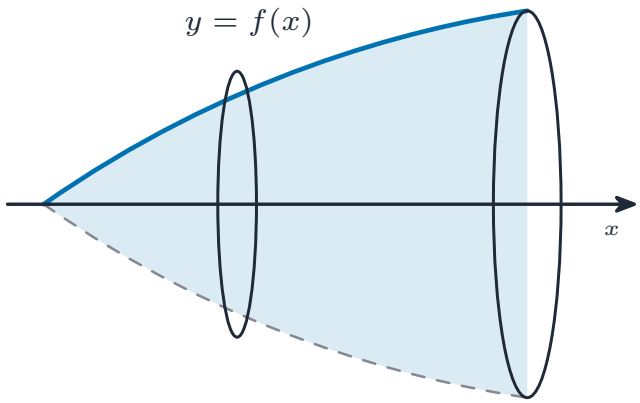
$$V = \pi \int_a^b \left(R(x)\right)^2 dx,$$

where the **radius** 半径 R is the distance from the curve to the axis. Use dy when revolving around the y-axis.



The disc method: rotating y=f(x) about the axis sweeps out disks of radius f(x)

> **Worked example.** Revolving the region under $y = \sqrt{x}$ from 0 to 4 about the x-axis gives discs of radius $\sqrt{x}$: $V = \pi\displaystyle\int_0^4 (\sqrt{x})^2\,dx = \pi\int_0^4 x\,dx = 8\pi$.



rotate the region around the x-axis to sweep a solid

Rotating a region about an axis sweeps out a solid of revolution

Volume with Disc Method: Revolving Around Other Axes

When the **axis of revolution** 旋转轴 is a horizontal or vertical line like $y = k$ (not an axis), the **radius adjusts**: $R = |f(x) - k|$. Set up the radius as the distance from the curve to that line, then integrate πR^2 as before.

> **Worked example.** The region between $y = \sqrt{x}$ and the line $y = 2$, for $0 \le x \le 4$, is revolved about $y = 2$. Each disc has radius $R = 2 - \sqrt{x}$ (line minus curve, since the line is above):

$$V = \pi \int_0^4 \left(2 - \sqrt{x}\right)^2 dx = \pi \int_0^4 \left(4 - 4\sqrt{x} + x\right) dx = \pi \left(16 - \frac{64}{3} + 8\right) = \frac{8\pi}{3}.$$

For a **write, but do not evaluate** question, the integral with correct limits and radius is the whole answer – do not waste time evaluating it.

Volume with Washer Method: Revolving Around the x- or y-Axis

If the region does not touch the axis, revolving leaves a hole, so cross sections are **washers** (**rings** 圆环). The **washer method** 垫圈法 subtracts the inner disc:

$$V = \pi \int_a^b \left(R_{\text{outer}}^2 - R_{\text{inner}}^2\right) dx.$$

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Identify the outer and inner radii as distances from each curve to the axis.

Volume with Washer Method: Revolving Around Other Axes

As with discs, revolving around a line $y = k$ or $x = k$ shifts both radii – each becomes the distance from its curve to that line. Sketch the region and the axis, label R_{outer} and R_{inner}, then integrate the difference of squares.

> **Worked example.** Revolve the region between $y = \sqrt{x}$ and $y = x^2$ $(0 \le x \le 1)$ about (a) the x-axis and (b) the line $y = -1$.

(a) On $(0, 1)$, $\sqrt{x} > x^2$, so $y = \sqrt{x}$ is farther from the x-axis: $R = \sqrt{x}$ and $r = x^2$.

$$V = \pi \int_0^1 \left(x - x^4\right) dx = \pi \left(\frac{1}{2} - \frac{1}{5}\right) = \frac{3\pi}{10}.$$

(b) Each radius is now a distance to $y = -1$, so add 1 to each height: $R = \sqrt{x} + 1$ and $r = x^2 + 1$. Expand both squares:

$$(\sqrt{x} + 1)^2 - (x^2 + 1)^2 = \left(x + 2\sqrt{x} + 1\right) - \left(x^4 + 2x^2 + 1\right) = x + 2\sqrt{x} - x^4 - 2x^2.$$

$$V = \pi \int_0^1 \left(x + 2\sqrt{x} - x^4 - 2x^2\right) dx = \pi \left(\frac{1}{2} + \frac{4}{3} - \frac{1}{5} - \frac{2}{3}\right) = \frac{29\pi}{30}.$$

Common mistakes. Writing $\pi \int (R - r)^2\, dx$ instead of $\pi \int (R^2 - r^2)\, dx$; and, about $y = -1$, using $R = \sqrt{x} - 1$ (subtracting) instead of $R = \sqrt{x} - (-1) = \sqrt{x} + 1$.

The Arc Length of a Smooth Curve and Distance Traveled

The **arc length** 弧长 of $y = f(x)$ from a to b is

$$L = \int_a^b \sqrt{1 + \left(f'(x)\right)^2}\, dx.$$

This BC-only formula comes from summing tiny **hypotenuses** 斜边 $\sqrt{dx^2 + dy^2}$. The same idea gives the distance a **particle** 质点 travels along a curved path.

> **Worked example (perimeter).** R is the region between $y = x$ and $y = x^2$ for $0 \le x \le 1$. Its boundary is a straight segment from $(0, 0)$ to $(1, 1)$, of length $\sqrt{2}$, and the parabola, of length $\int_0^1 \sqrt{1 + (2x)^2}\, dx$. So

$$\text{perimeter} = \sqrt{2} + \int_0^1 \sqrt{1 + 4x^2}\, dx \approx \sqrt{2} + 1.479 \approx 2.893.$$

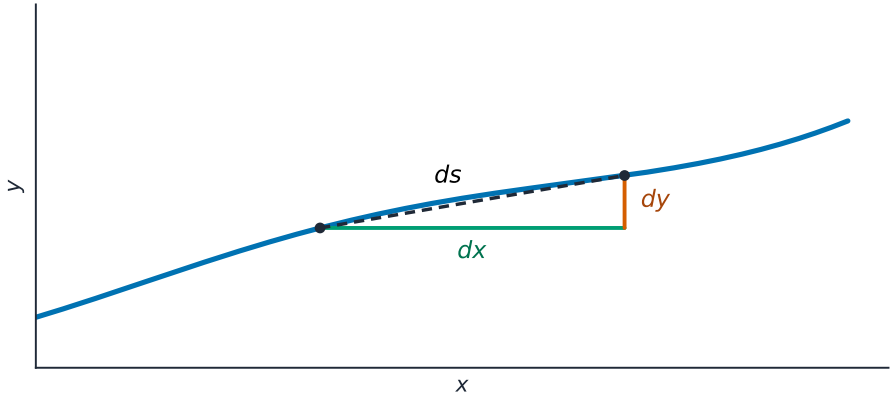
Arc-length integrals rarely have a neat antiderivative; set them up and use the calculator.

Worked example. Find the arc length of $y = \tfrac{2}{3}x^{3/2}$ from $x = 0$ to $x = 3$. Here $f'(x) = x^{1/2}$, so $1 + (f')^2 = 1 + x$ and

$$L = \int_0^3 \sqrt{1+x}\,dx = \left[\tfrac{2}{3}(1+x)^{3/2}\right]_0^3 = \tfrac{2}{3}(8-1) = \tfrac{14}{3}.$$



The bridge's main cable hangs as a smooth curve; an integral gives its exact length



$$ds = \sqrt{dx^2 + dy^2}$$

Each small piece of a curve is the hypotenuse of a dx-by-dy triangle

Exam tips

- Area between curves is $\int(\text{top} - \text{bottom})\,dx$ – find the intersection points for the limits and keep top minus bottom.
- For a **volume of revolution**, add up disc/washer cross-sections of area πr^2 (or $\pi(R^2 - r^2)$).
- The **average value** of f on $[a,b]$ is $\frac{1}{b-a}\int_a^b f\,dx$.
- Accumulated change is $\int$ of a rate: total = initial value + $\int_a^b(\text{rate})\,dt$.
- Integration means "adding up infinitely many tiny pieces" – set up the integrand as one thin slice.
- **Total distance** uses $|v|$; **displacement** uses v. For a minimum or maximum amount, use the candidates test with the endpoints.

Practice.

1. Find the average value of $f(x) = \sin x$ on $[0, \pi]$.
2. $v(t) = 2t - 4$ m/s for $0 \le t \le 5$. Find the displacement and the total distance.
3. Find the area enclosed by $y = x^2$ and $y = 4$.
4. Find the volume when the region under $y = x^2$, $0 \le x \le 2$, is revolved about the x-axis.
5. The region between $y = x$ and $y = x^2$ is revolved about the line $y = -1$. Write, but do not evaluate, an integral for the volume.
6. Write an integral for the length of $y = \ln x$ from $x = 1$ to $x = e$.

Answers.

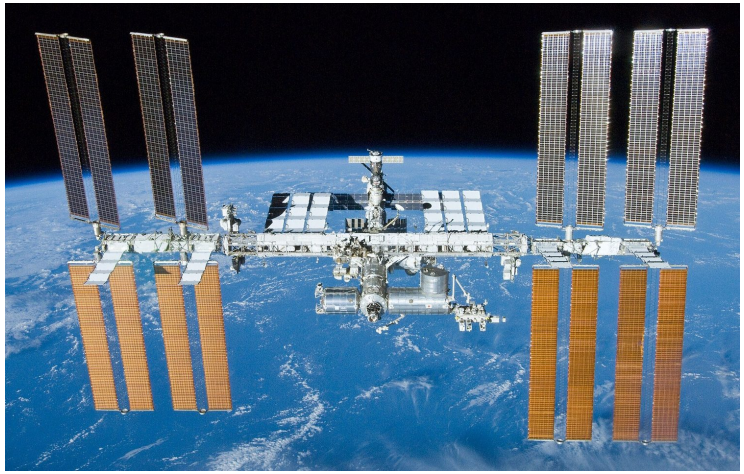
1. $\dfrac{1}{\pi}\displaystyle\int_0^{\pi} \sin x\,dx = \frac{2}{\pi}$.
2. Displacement $\displaystyle\int_0^5 (2t-4)\,dt = 25-20 = 5$ m. $v < 0$ on $(0,2)$: distance $= 4+9 = 13$ m.
3. They meet at $x = \pm 2$: $\displaystyle\int_{-2}^{2}(4 - x^2)\,dx = 16 - \frac{16}{3} = \frac{32}{3}$.
4. $\pi\displaystyle\int_0^2 x^4\,dx = \frac{32\pi}{5}$.
5. $\pi\displaystyle\int_0^1 \left[(x+1)^2 - (x^2+1)^2\right]dx$.
6. $\displaystyle\int_1^e \sqrt{1 + \frac{1}{x^2}}\,dx$.

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Parametric Equations, Polar Coordinates, and Vector-Valued Functions

AP Calculus BC

Defining and Differentiating Parametric Equations



The same parameter identifies both coordinates of a point on its path.

Parametric equations 参数方程 give coordinates $x = f(t)$ and $y = g(t)$ in terms of a **parameter** 参数 t. A robot's position needs both coordinates at the same time. For $x = t^2$, $y = t^3$, increasing t through $-1, 0, 1, 2$ gives $(1,-1), (0,0), (1,1), (4,8)$. The same x-coordinate can have different y-coordinates, so the whole curve need not be one function $y = f(x)$.

Keep the parameter interval and direction. For $x = \cos t$, $y = \sin t$, $0 \le t \le \pi$ traces only the upper semicircle, from $(1,0)$ to $(-1,0)$. The interval $[0, 2\pi]$ traces a full circle counterclockwise. Eliminating t gives $x^2 + y^2 = 1$ but loses this interval and travel order.

Where $dx/dt \neq 0$, the chain rule gives the **slope** 斜率

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}.$$

Differentiate both coordinates, then evaluate at the same parameter. This is not the derivative of y/x or the product of the coordinate rates.

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> **Worked example: slope and tangent.** For $x = t^2$, $y = t^3$ at $t = 2$, the point is $(4, 8)$ and the derivatives are 4 and 12. The slope is 3, so the tangent is $y - 8 = 3(x - 4)$. For the related curve $x = t^2$, $y = t^3 - t$, the slope is $(3t^2 - 1)/(2t)$ for $t \neq 0$; at $t = 2$ it is $11/4$.

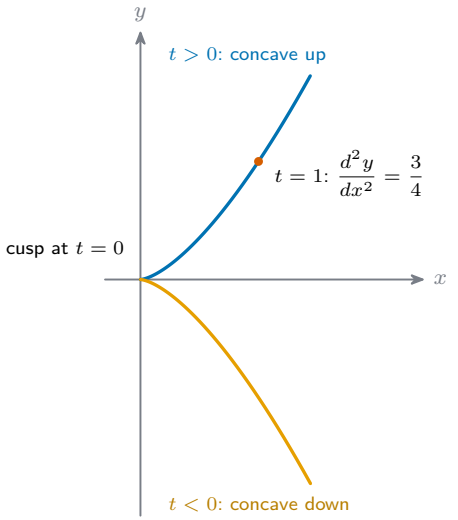
A regular horizontal tangent has $dy/dt = 0$ and $dx/dt \neq 0$; a regular vertical tangent has $dx/dt = 0$ and $dy/dt \neq 0$. If both derivatives vanish, these tests do not decide the tangent. For $x = t^3$, $y = t^3$ at zero, both vanish but the path is the line $y = x$. Zero coordinate rates do not prove that no geometric tangent exists.

> **Try it.** For $x = t^2 + 2$, $y = t^3 - 12t$, $-3 \le t \le 3$, find the regular horizontal and vertical tangent lines.

Answer. The derivatives are $2t$ and $3t^2 - 12$. Horizontal tangents occur at $t = \pm 2$, with points $(6, -16)$ and $(6, 16)$ and lines $y = -16$, $y = 16$. At $t = 0$, the y-coordinate rate is $-12 \neq 0$, so the vertical tangent is $x = 2$ at $(2, 0)$. All three parameters lie in the stated interval.

Second Derivatives of Parametric Equations

The **second derivative** 二阶导数 measures change of slope with x. Write the first slope as $m(t) = dy/dx$. Where $dx/dt \neq 0$,



Differentiate the slope in t, then divide by dx/dt again.

$$\frac{d^2y}{dx^2} = \frac{dm/dt}{dx/dt}.$$

Differentiate the first slope in t, then divide by dx/dt again. It is not generally the ratio of the second coordinate derivatives. Simplify first when helpful; a quotient rule is needed only if the remaining expression requires it. The equivalent formula is

$$\frac{d^2y}{dx^2} = \frac{x'y'' - y'x''}{(x')^3}, \qquad x' \neq 0.$$

Here primes on x and y denote derivatives in t.

> **Worked example: two different derivatives.** For $x = t^2$, $y = t^3$ and $t \neq 0$, the first slope is $3t/2$, whose t-derivative is $3/2$. Dividing by $2t$ gives $3/(4t)$. At $t = 1$ the second x-derivative is $3/4$, rather than the intermediate value $3/2$ or the incorrect ratio $6t/2 = 3$.

Its sign gives **concavity** 凹凸性 on a smooth local graph: positive means concave up, negative means concave down. The upper branch of this curve is concave up for $t > 0$; the lower branch is concave down for $t < 0$. At zero the formula is unavailable, and the branches meet at a cusp. A sign change across that singular parameter does not by itself establish an inflection of one smooth graph.

> **Worked example: decreasing x.** For $x = -t$, $y = t^2$, the slope is $-2t$ and its t-derivative is -2. Dividing by $dx/dt = -1$ gives $d^2y/dx^2 = 2$. The graph is $y = x^2$, concave up despite the negative intermediate derivative.

> **Try it.** For $x = t^3 + 2t$, $y = t^2$, find the second x-derivative and its sign at $t = 1$.

Answer. The first slope is $2t/(3t^2+2)$; its t-derivative is $2(2-3t^2)/(3t^2+2)^2$. Divide again to obtain $2(2-3t^2)/(3t^2+2)^3$. At $t = 1$ this is $-2/125$, so the curve is locally concave down at $(3, 1)$. The denominator $3t^2 + 2$ is always positive.

A zero second derivative alone is not an inflection test: $y = x^4$ has second derivative zero at the origin but is concave up on both sides. Check a change of concavity and the regularity of the graph.

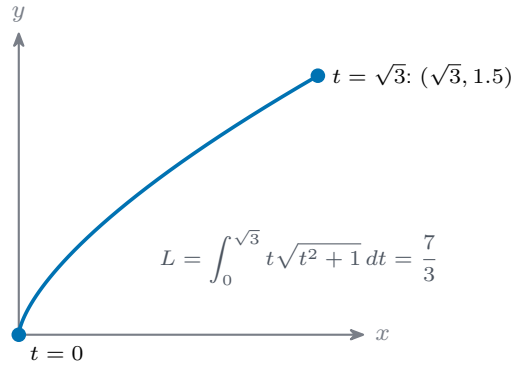
> **Worked example.** For the curve $x = t^2$, $y = t^3 - t$, find $\dfrac{d^2y}{dx^2}$ at $t = 2$. First write $\dfrac{dy}{dx} = \dfrac{3t^2-1}{2t} = \dfrac{3}{2}t - \dfrac{1}{2}t^{-1}$, and differentiate with respect to t:
>
> $$\frac{d}{dt}\left(\frac{dy}{dx}\right) = \frac{3}{2} + \frac{1}{2t^2}.$$
>
> Then divide by $\dfrac{dx}{dt} = 2t$:
>
> $$\frac{d^2y}{dx^2} = \frac{\frac{3}{2} + \frac{1}{2t^2}}{2t}, \qquad \text{at } t = 2: \quad \frac{1.5 + 0.125}{4} = \frac{13}{32}.$$

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It is positive, so the curve is concave up at $t = 2$. Forgetting the final division by $\dfrac{dx}{dt}$ is the usual error.

Finding Arc Lengths of Curves Given by Parametric Equations

For a smooth or piecewise smooth path on $a \le t \le b$, its **arc length** 弧长 is



Arc length integrates the speed.

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt.$$

The integrand is **speed** 速率 when t is time. Square and add both coordinate rates under a nonnegative root. Use parameter bounds; a zero component or isolated zero speed is allowed because no rate is in a denominator.

> **Worked example: a curved route.** For $x = t^2$, $y = t^3$, the speed is
>
> $$\sqrt{4t^2 + 9t^4} = |t|\sqrt{4 + 9t^2}.$$
>
> On $[0, 1]$, use $|t| = t$. Substitution $u = 4 + 9t^2$ gives
>
> $$L = \int_0^1 t\sqrt{4 + 9t^2}\, dt = \frac{13^{3/2} - 8}{27} \approx 1.440.$$
>
> On $[-1, 0]$, use $|t| = -t$ to obtain the same positive length. On $[-1, 1]$, double the half-interval length to get approximately 2.879. Always retain $\sqrt{t^2} = |t|$ when simplifying.

> **Worked example: retracing.** For $x = t^2$, $y = 0$ on $[-1, 1]$, moving from x=1 to 0 and back gives $\int_{-1}^{1} 2|t|\, dt = 2$. The segment counted once has length 1; endpoint distance is 0. The integral counts repeated traversals.

Try it. Find the length for $x = t^2$, $y = t^3$ on $[-2, 0]$.

Answer. Integrate $(-t)\sqrt{4+9t^2}$ on this interval. The length is $(40^{3/2}-8)/27$. Using t instead of -t would incorrectly produce a negative integrand.

Worked example. Find the length of $x = \frac{1}{3}t^3$, $y = \frac{1}{2}t^2$ for $0 \le t \le \sqrt{3}$. Here $\dfrac{dx}{dt} = t^2$ and $\dfrac{dy}{dt} = t$, so for $t \ge 0$

$$\sqrt{t^4+t^2} = t\sqrt{t^2+1}, \qquad L = \int_0^{\sqrt{3}} t\sqrt{t^2+1}\,dt = \left[\frac{1}{3}\left(t^2+1\right)^{3/2}\right]_0^{\sqrt{3}} = \frac{1}{3}(8-1) = \frac{7}{3}.$$

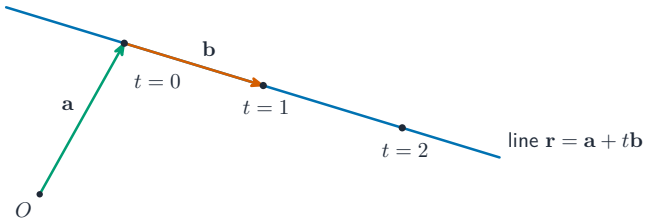
Most exam arc lengths need a calculator; this one was chosen so the integral works by hand ($u = t^2+1$).

Defining and Differentiating Vector-Valued Functions

A **vector-valued function** 向量值函数 $\mathbf{r}(t) = \langle x(t), y(t)\rangle$ gives a vector for each parameter value. A **position vector** 位置向量 is a vector from the chosen origin to a point, giving its coordinates. Differentiate component by component:

$$\mathbf{v}(t) = \mathbf{r}'(t) = \langle x'(t), y'(t)\rangle, \qquad \mathbf{a}(t) = \mathbf{r}''(t) = \langle x''(t), y''(t)\rangle.$$

The **velocity** 速度 and **acceleration** 加速度 are vectors. Speed is the scalar **magnitude** 模长 $|\mathbf{v}| = \sqrt{x'^2+y'^2}$. Differentiate the functions before substituting a parameter value; differentiating a fixed position vector loses the nearby change information.



A vector-valued line: start at a, then slide by t lots of the direction b

Worked example: the complete chain. For $\mathbf{r}(t) = \langle t^2, \sin t\rangle$, velocity is $\langle 2t, \cos t\rangle$ and acceleration is $\langle 2, -\sin t\rangle$. At zero, position is $\langle 0, 0\rangle$, velocity is $\langle 0, 1\rangle$, acceleration is $\langle 2, 0\rangle$ and speed is 1. Being at the origin does not mean being stationary.

Nonzero velocity gives the tangent direction. A zero velocity vector has no direction, but does not rule out a geometric tangent: $\mathbf{r}(t) = \langle t^3, t^3\rangle$ has zero velocity at zero while its path is the line $y = x$. Position and velocity vectors need not point in the same direction.

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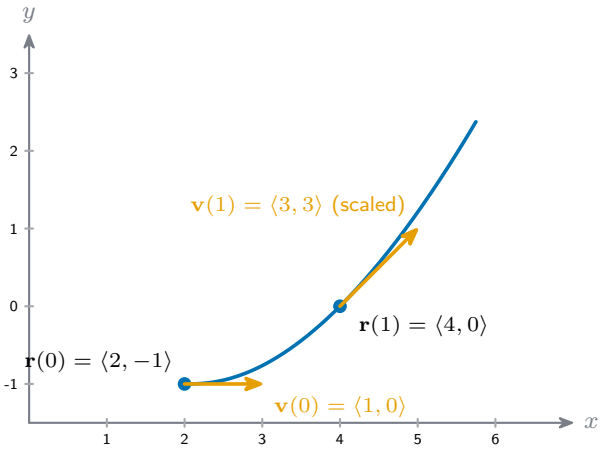
Worked example: nontrivial speed. For $\mathbf{r}(t) = \langle t^2, t^3-t\rangle$, velocity is $\langle 2t, 3t^2-1\rangle$ and acceleration is $\langle 2, 6t\rangle$. At $t = 1$, the point is $(1, 0)$, velocity is $\langle 2, 2\rangle$, acceleration is $\langle 2, 6\rangle$ and speed is $2\sqrt{2}$. At $t = -1$, the point is again $(1, 0)$ but velocity $\langle -2, 2\rangle$ points left and up.

Try it. For $\mathbf{r}(t) = \langle 3t^2+1, t^3-2t\rangle$, find position, velocity, acceleration and speed at $t = 1$.

Answer. The values are $\langle 4, -1\rangle$, $\langle 6, 1\rangle$, $\langle 6, 6\rangle$ and $\sqrt{37}$. Motion is right and up, determined by velocity rather than the negative y-coordinate.

Integrating Vector-Valued Functions

A **constant vector** 常向量 is a vector whose components do not vary with the parameter. Integrate each component and retain an independent constant in each component: $\int\langle f, g\rangle\,dt = \langle F + C_1, G + C_2\rangle$ when $F' = f$, $G' = g$. An **initial condition** 初始条件 fixes each component's constant independently. Each new integration introduces another constant vector.



Each initial condition fixes its own constants.

Worked example: recover velocity and position. Given $\mathbf{a}(t) = \langle 2, 6t\rangle$ and $\mathbf{v}(0) = \langle 1, 0\rangle$, integration gives $\mathbf{v}(t) = \langle 2t+1, 3t^2\rangle$. If also $\mathbf{r}(0) = \langle 2, -1\rangle$, integrate again to obtain $\mathbf{r}(t) = \langle t^2+t+2, t^3-1\rangle$. The velocity condition and position condition fix different constants.

Equivalently, include the initial values with definite integrals:

$$\mathbf{v}(t) = \mathbf{v}(t_0) + \int_{t_0}^{t} \mathbf{a}(u)\,du, \qquad \mathbf{r}(t) = \mathbf{r}(t_0) + \int_{t_0}^{t} \mathbf{v}(u)\,du.$$

The vector integrals give changes; add the initial vector to recover the state. The dummy variable u is distinct from the upper limit t. These signed integrals can also recover an earlier state when $t < t_0$.

Worked example: data at a nonzero time. Keep acceleration $\langle 2, 6t\rangle$ but use $\mathbf{v}(1) = \langle 4, -2\rangle$, $\mathbf{r}(1) = \langle 3, 7\rangle$. The resulting functions are $\mathbf{v}(t) = \langle 2t+2, 3t^2-5\rangle$ and $\mathbf{r}(t) = \langle t^2+2t, t^3-5t+11\rangle$. Substitute t=1 into both and differentiate to verify. The constants are not simply the initial coordinates copied into the antiderivative.

Try it. Given $\mathbf{a}(t) = \langle 4, 6t\rangle$, $\mathbf{v}(1) = \langle 1, -2\rangle$, $\mathbf{r}(1) = \langle 3, 5\rangle$, recover velocity and position, then find position at $t = 2$.

Answer. Velocity is $\langle 4t-3, 3t^2-5\rangle$, position is $\langle 2t^2-3t+4, t^3-5t+9\rangle$, and position at 2 is $\langle 6, 7\rangle$. Check both derivative equations and both initial vectors.

Worked example. A **particle** 质点 has velocity $\vec{v}(t) = \langle 2t,\ 3t^2-1\rangle$ and is at $(1, -2)$ when $t = 0$. Find its position at $t = 2$. Add the integral of each component to its starting value:

$$x(2) = 1 + \int_0^2 2t\,dt = 1 + 4 = 5, \qquad y(2) = -2 + \int_0^2 (3t^2-1)\,dt = -2 + (8-2) = 4.$$

The particle is at $(5, 4)$. Leaving out the starting point is the classic lost mark.

Solving Motion Problems Using Parametric and Vector-Valued Functions

Displacement 位移 is the vector change in position, $\int_a^b \mathbf{v}(t)\,dt = \mathbf{r}(b) - \mathbf{r}(a)$. **Distance traveled** 路程 is the total length of the route, found by integrating its nonnegative speed. For $a \le b$, this scalar is $\int_a^b |\mathbf{v}(t)|\,dt$. Final position is initial position plus displacement. Distance is at least the displacement magnitude, with equality possible on a straight route traced in one direction.

Worked example: compare the quantities. For $\mathbf{v}(t) = \langle 2t, 3\rangle$ and $\mathbf{r}(0) = \langle 1, -2\rangle$, position is $\langle 1+t^2, -2+3t\rangle$ and acceleration is $\langle 2, 0\rangle$. At t=2, position is $\langle 5, 4\rangle$, velocity is $\langle 4, 3\rangle$ and speed is 5. On $[0, 2]$, displacement is $\langle 4, 6\rangle$.

The traveled distance is $D = \int_0^2 \sqrt{4t^2+9}\,dt = 5 + \frac{9}{4}\ln 3 \approx 7.472$.

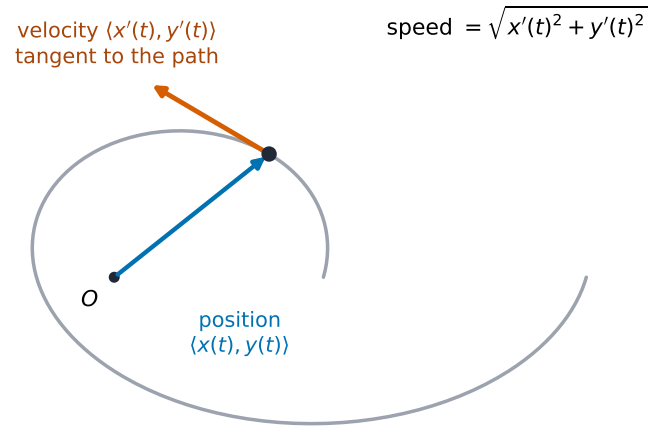
The endpoint distance is $\sqrt{52} \approx 7.211$, shorter than this curved route. Keep position, velocity, acceleration, speed, displacement and total distance distinct.

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Where speed is positive, its derivative is $(\mathbf{v}\cdot\mathbf{a})/|\mathbf{v}|$. A positive dot product means speeding up; a negative one means slowing down. At t=2 in the example, the rate is $8/5 > 0$. At zero speed, investigate separately instead of dividing by zero. Constant speed can coexist with nonzero acceleration when velocity changes direction, as on a circle.

Worked position example. For $\mathbf{r}(t) = \langle t^2, t^3-t\rangle$, velocity at 1 is $\langle 2, 2\rangle$, speed is $2\sqrt{2}$, and position at 2 is $\langle 4, 6\rangle$.

Common mistake. Adding the components for the speed $(2+2=4)$. Speed is the **length** of the velocity vector, $\sqrt{(dx/dt)^2 + (dy/dt)^2}$.



The pictured velocity is nonzero and tangent to the path; speed is its magnitude.

Try it. For $\mathbf{r}(t) = \langle (t-2)^2, 0\rangle$ on $[0, 3]$, find displacement, traveled distance and the reversal time.

Answer. The endpoints are $(4, 0)$ and $(1, 0)$, giving displacement $\langle -3, 0\rangle$. Speed is $2|t-2|$; splitting its integral at 2 gives distance 4+1=5. The point slows down before t=2, stops and reverses there, then speeds up afterward.

Defining Polar Coordinates and Differentiating in Polar Form



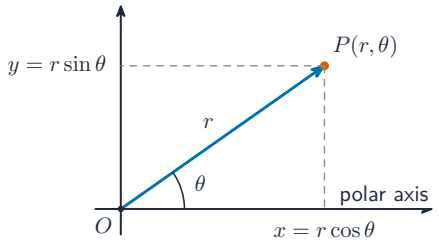
A spiral is simple in polar form: r grows steadily as θ turns

Polar coordinates 极坐标 locate a point by a signed radius r and an angle θ, measured from the positive x-axis. A positive radius moves along that ray; a negative radius moves in the opposite direction. The distance from the origin is $|r|$. The **pole** 极点 is the origin of the polar coordinate system.

Cartesian 直角坐标 refers to a coordinate system with perpendicular axes. Convert with $x = r\cos\theta$, $y = r\sin\theta$. These formulas also work for negative r.

> **Worked example: converting points.** $(2, \pi/3)$ gives $(1, \sqrt{3})$, while $(-2, \pi/6)$ gives $(-\sqrt{3}, -1)$.

Polar representations are not unique. Adding a full turn leaves a point unchanged, and $(-r, \theta+\pi)$ gives the same point as (r, θ). To choose a nonnegative radius from x and y, use $r = \sqrt{x^2+y^2}$ and select an angle in the correct quadrant. The point $(-1, \sqrt{3})$ has angle $2\pi/3$ in $[0, 2\pi)$; $\arctan(y/x) = -\pi/3$ would choose the wrong quadrant. Handle axes without dividing by x; at the pole every angle represents the same point.



Polar coordinates give a point by its signed radius r and angle theta

For $r = f(\theta)$, use radians and treat the angle as a parameter. With primes denoting

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angle derivatives,

$$x' = r'\cos\theta - r\sin\theta, \qquad y' = r'\sin\theta + r\cos\theta.$$

The **tangent slope** 切线斜率 is the rate of change of y with x along a curve, where that derivative is finite. Where $x' \neq 0$, it is $dy/dx = y'/x'$, not $dr/d\theta$.

> **Worked example: point and tangent.** For $r = 2\cos\theta$ at $\pi/4$, $r = \sqrt{2}$ and $r' = -\sqrt{2}$. The point is $(1, 1)$, and the coordinate rates are $x' = -2$, $y' = 0$. Hence the slope is 0 and the tangent is $y = 1$, although the radial rate is $-\sqrt{2}$. At $\pi/2$, the point is the pole with $x' = 0$, $y' = -2$, so the tangent is $x = 0$.

Regular horizontal and vertical tangents require the complementary coordinate rate to be nonzero. If both vanish, investigate further: for $r = 1 + \cos\theta$ at π, $x' = y' = 0$, so the regular ratio test is inconclusive. A zero radius alone does not cause this; for $r = \theta$ at zero, $x' = 1$, $y' = 0$ and the tangent is $y = 0$.

For the second x-derivative, differentiate the slope in the angle and divide by x' again:

$$\frac{d^2y}{dx^2} = \frac{d(y'/x')/d\theta}{x'} = \frac{x'y'' - y'x''}{(x')^3}, \qquad x' \neq 0.$$

Here $x'' = (r'' - r)\cos\theta - 2r'\sin\theta$ and $y'' = (r'' - r)\sin\theta + 2r'\cos\theta$. For $r = 2$ at $\pi/4$, the slope is $-\cot\theta$, whose angle derivative is 2 at that point. Dividing by $x' = -\sqrt{2}$ gives $d^2y/dx^2 = -\sqrt{2}$, consistent with the upper circle being concave down.

> **Try it.** For $r = 3\sin\theta$ at $\pi/6$, find the Cartesian point, coordinate rates and tangent line.

Answer. Here $r = 3/2$, $r' = 3\sqrt{3}/2$ and $(x, y) = (3\sqrt{3}/4, 3/4)$. The rates are $x' = 3/2$, $y' = 3\sqrt{3}/2$, so the slope is $\sqrt{3}$ and the tangent is $y - 3/4 = \sqrt{3}(x - 3\sqrt{3}/4)$.

Finding the Area of a Polar Region

A **sector** 扇形 is a circular region bounded by two radii and the arc joining their endpoints. Its area is $R^2\Delta\theta/2$ for a radian angle $\Delta\theta$. Thin sectors lead to the sweep integral

$$A = \frac{1}{2}\int_{\alpha}^{\beta} \big(f(\theta)\big)^2\, d\theta, \qquad \alpha \le \beta.$$

The integrand is nonnegative even for negative r. It gives geometric area when the radial sweep covers the intended region once; repeated coverage is counted repeatedly. Use ordered radian bounds, the complete squared radius and square units.

A **petal** 花瓣 is a loop of a **rose curve** 玫瑰线 from the pole back to the pole. Find its start and end angles, rather than automatically integrating through a full turn.

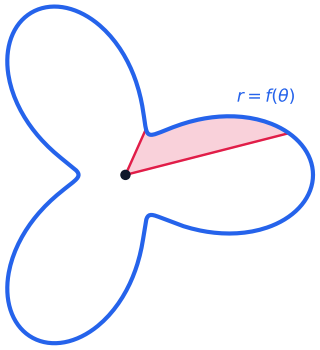
Worked example. Find the area of one petal of the **rose curve** $r = 2\sin(2\theta)$ (traced for θ from 0 to $\frac{\pi}{2}$). Using $\sin^2 u = \frac{1}{2}(1 - \cos 2u)$,

$$A = \frac{1}{2}\int_0^{\pi/2} (2\sin 2\theta)^2\, d\theta = \int_0^{\pi/2} (1 - \cos 4\theta)\, d\theta = \left[\theta - \tfrac{\sin 4\theta}{4}\right]_0^{\pi/2} = \frac{\pi}{2}.$$

Worked example: a different rose. For $r = \cos 3\theta$, the right-hand petal is traced from $-\pi/6$ to $\pi/6$, where r is nonnegative. Its area is

$$\frac{1}{4}\int_{-\pi/6}^{\pi/6} (1 + \cos 6\theta)\, d\theta = \left[\frac{\theta}{4} + \frac{\sin 6\theta}{24}\right]_{-\pi/6}^{\pi/6} = \frac{\pi}{12}.$$

Symmetry 对称性 is a property in which a shape matches itself after a reflection or rotation. This rose has three equal petals, giving total area $\pi/4$. It is traced once on $[0, \pi]$; the integral on $[0, 2\pi]$ is $\pi/2$ because it counts the rose twice. Likewise, $r = 2\cos\theta$ traces its circle once on $[0, \pi]$, with area π, and twice on $[0, 2\pi]$.



The shaded region is a fan of thin sectors swept from α to β; each has area $\frac{1}{2}r^2\, d\theta$, so the total is $\frac{1}{2}\int_\alpha^\beta r^2\, d\theta$.

Try it. Find the full area and upper-half area for $r = 2 + 2\cos\theta$.

Answer. Use $[0, 2\pi]$ for the full cardioid. Expanding its square gives

$$\frac{1}{2}\int_0^{2\pi} (4 + 8\cos\theta + 4\cos^2\theta)\, d\theta = 6\pi.$$

On $[0, \pi]$, r is nonnegative and y is nonnegative, so this interval sweeps the upper half once. Reflection symmetry gives upper-half area 3π. All radii are twice those of $r = 1 + \cos\theta$, so the full area is four times $3\pi/2$, not twice it.

Finding the Area of the Region Bounded by Two Polar Curves

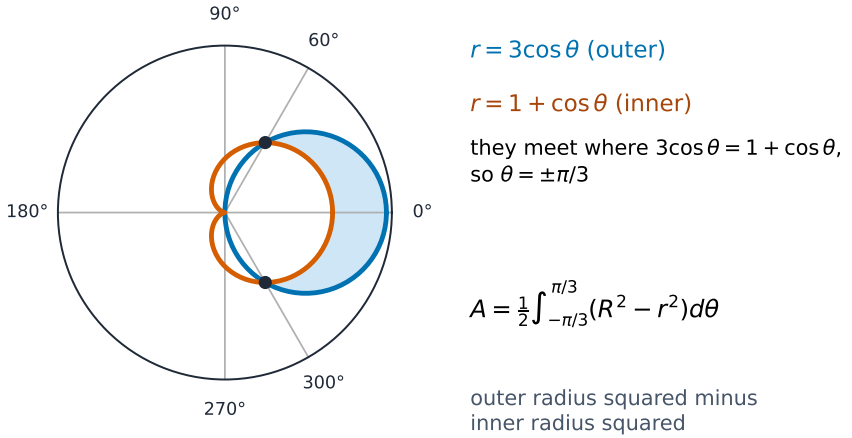
An **outer polar radius** 外极径 is the nonnegative distance to the farther boundary on a common ray. An **inner polar radius** 内极径 is the nonnegative distance to the nearer boundary on that ray. For a radial band with $r_{\text{out}} \ge r_{\text{in}} \ge 0$, subtract the sectors:

$$A = \frac{1}{2}\int_\alpha^\beta \left(r_{\text{out}}^2 - r_{\text{in}}^2\right) d\theta.$$

Square each radius before subtracting; this is not $(r_{\text{out}} - r_{\text{in}})^2/2$. Use common physical rays and nonnegative distances, rather than comparing signed parameters at unrelated angles.

An **intersection** 交点 is a point shared by two curves. Equating radii at the same angle finds candidates; also check the pole and other parameter representations.

Bounds can come from given rays or a full turn. The concentric circles $r = 2$ and $r = 1$ never cross, yet their ring has area $\frac{1}{2}\int_0^{2\pi}(4 - 1)\, d\theta = 3\pi$.



Area inside $r = 3\cos\theta$ and outside $r = 1 + \cos\theta$: outer radius squared minus inner radius squared

Worked example: the pictured region. For $r_A = 3\cos\theta$ and $r_B = 1 + \cos\theta$, common nonnegative crossing radii satisfy $\cos\theta = 1/2$, giving angles $\pm\pi/3$ and radius $3/2$. Both curves also reach the pole, but at different parameters: A at $\pi/2$ and B at π.

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Inside A but outside B requires $3\cos\theta \geq 1+\cos\theta \geq 0$, so keep $[-\pi/3, \pi/3]$. The area is

$$\frac{1}{2}\int_{-\pi/3}^{\pi/3}\left[9\cos^2\theta-(1+\cos\theta)^2\right]d\theta=\left[\frac{3\theta}{2}+\sin 2\theta-\sin\theta\right]_{-\pi/3}^{\pi/3}=\pi.$$

Where A is nearer than B, that requested radial band is empty. Taking an absolute difference there would add the opposite region, inside B but outside A. Do not use an absolute value to repair bounds chosen for a different request.

Worked example: switching outer boundary. For the band between $r_A = 1+\cos\theta$ and $r_B = 1$ on $[0, 2\pi]$, split at $\pi/2$ and $3\pi/2$. A is outer on the first and last intervals; B is outer in the middle. Let

$$f(\theta)=\frac{1}{2}\left[(1+\cos\theta)^2-1\right], \qquad F(\theta)=\sin\theta+\frac{\theta}{4}+\frac{\sin 2\theta}{8}.$$

The two A-outer pieces total $2+\pi/4$; the middle B-outer area is $2-\pi/4$. Their sum is 4 square units, while the unsplit signed integral of f is only $\pi/2$. For this band, integrate $|f|$, or reverse the subtraction on the middle interval.

For inside both radial regions, use the smaller nonnegative reach and integrate $\frac{1}{2}\min(r_A, r_B)^2$. For the concentric circles 2 and 1, the common disk has area π, rather than the ring area 3π.

Try it. Find the area inside $r = 3$, outside $r = 1$, between the rays $\pi/6$ and $5\pi/6$.

Answer. The given rays supply the limits. The angle span is $2\pi/3$, so the area is $\frac{1}{2}(9-1)(2\pi/3) = 8\pi/3$ square units. No intersection calculation is needed for these concentric circles.

Exam tips

- For **parametric** curves, $\frac{dy}{dx}=\frac{dy/dt}{dx/dt}$ where $dx/dt \neq 0$; speed is $\sqrt{(dx/dt)^2+(dy/dt)^2}$.
- A **vector-valued** function carries the same information – differentiate/integrate it component by component.
- In **polar**, distance is $|r|$; convert with $x = r\cos\theta$, $y = r\sin\theta$. The slope ratio needs $dx/d\theta \neq 0$, and geometric area needs a once-only radial sweep.
- Watch the direction of tracing using both coordinate rates and set correct θ-limits for polar area.
- Eliminate the parameter to recover the ordinary y-vs-x shape when it helps.

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Infinite Sequences and Series

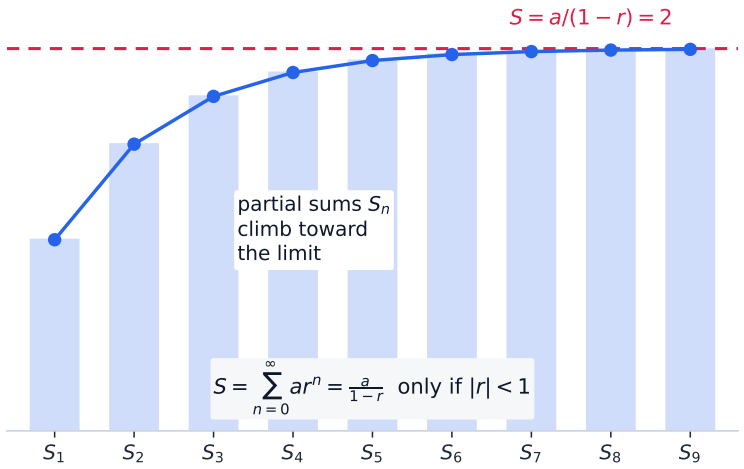
AP Calculus BC

Defining Convergent and Divergent Infinite Series



Nested matryoshka dolls: an infinite series keeps adding terms – some converge, some diverge

An **infinite series** 无穷级数 adds infinitely many terms, $\sum_{n=1}^{\infty} a_n$. Its value is defined as the **limit** 极限 of the **partial sums** 部分和 $S_N = a_1+a_2+\cdots+a_N$. If S_N approaches a finite number L, the series **converges** 收敛 to L; otherwise it **diverges** 发散. Every convergence question is really a question about the limit of the partial sums.



For the geometric series with $a = 1$, $r = \frac{1}{2}$, the partial sums $S_1, S_2, S_3, \dots$ climb toward the limit $\dfrac{a}{1-r} = 2$ – that limit is the series' value.

Working with Geometric Series



A bouncing ball under a strobe: each bounce reaches a fixed fraction of the last height, so the total distance is a geometric series

A **geometric series** 几何级数 adds terms obtained by multiplying by a fixed **ratio** 公比 r:

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$$a + ar + ar^2 + \cdots .$$

Here a is the actual first term. For nonzero terms, dividing a term by the previous one gives r. Its first N terms have sum

$$S_N = \sum_{n=0}^{N-1} ar^n = \frac{a(1 - r^N)}{1 - r} \quad (r \neq 1).$$

To derive this, subtract rS_N from S_N; the middle terms cancel. When $r = 1$, $S_N = Na$ instead. An individual term is not a partial sum.

For $a \neq 0$, the infinite series converges exactly when $|r| < 1$. Then $r^N \to 0$ and

$$S = \lim_{N\to\infty} S_N = \frac{a}{1 - r}.$$

The ratio may be negative or irrational; its magnitude is what matters. If $a = 0$, every term is zero for any fixed finite r, so the sum is 0. Zero terms do not determine a unique ratio by division. In the case $r = 1$, handle the zero series directly rather than evaluating $0/0$.

> **Worked example: finite versus infinite.** For $3 + 3/2 + 3/4 + \cdots$, $a = 3$ and $r = 1/2$. The infinite sum is $3/(1 - 1/2) = 6$. The first four-term sum is $3(1 - (1/2)^4)/(1 - 1/2) = 5.625$, leaving 0.375 still to be added.

> **Worked example: a negative ratio.** The series $4-2+1-1/2+\cdots$ has $r = -1/2$ and sum $4/(1 + 1/2) = 8/3$. Its partial sums $4, 2, 3, 2.5, \dots$ alternate around that limit. Shrinking oscillations can converge. By contrast, with a nonzero first term and $r = -1$, the partial sums alternate between a and 0 and do not have a limit. At $r = 1$, they are Na; for $|r| > 1$, the terms do not tend to zero.

> **Worked example: read the starting index.** For $\sum_{n=1}^{\infty} 3 \cdot 2^{-n}$, the first term is $3/2$, not 3. The ratio is $1/2$, so the sum is 3. For $\sum_{n=2}^{\infty} 12(-1/3)^n$, the first term is $4/3$ and the ratio is $-1/3$, giving sum 1.

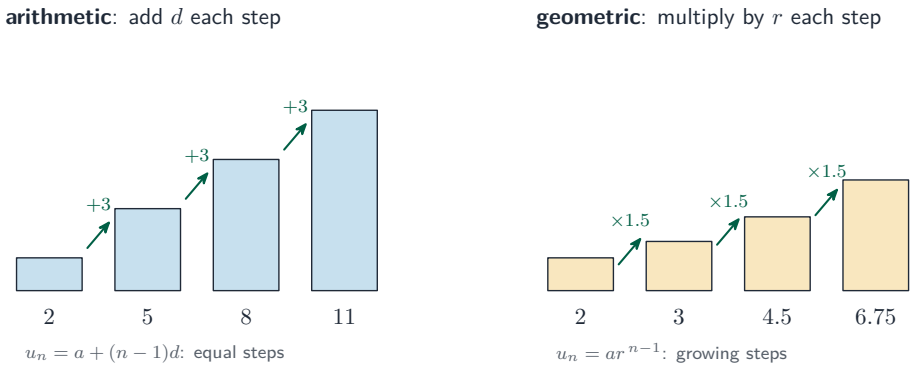
A finite sum exists even when the infinite series diverges. For $3+6+12+\cdots$, the first three terms add to 21, but the infinite series has no finite sum. The formal expression $3/(1 - 2) = -3$ is not an ordinary infinite sum of these positive terms.

> **Try it.** For $\sum_{k=3}^{\infty} 16(1/2)^k$, find the actual first term, infinite sum, first four-term sum and remaining amount.

Answer. The first term is 2 and the ratio is $1/2$. The infinite sum is 4. The first four terms are $2, 1, 1/2, 1/4$, with sum $15/4 = 3.75$. The remaining amount is $1/4$.

Common mistakes. Using $\dfrac{a}{1 - r}$ when $|r| \geq 1$ (the series diverges), and taking a

from the formula instead of the series: $\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n$ starts with $\frac{1}{2}$, so its sum is $\frac{1/2}{1-1/2} = 1$, not 2.



A geometric sequence multiplies by the same ratio at each step

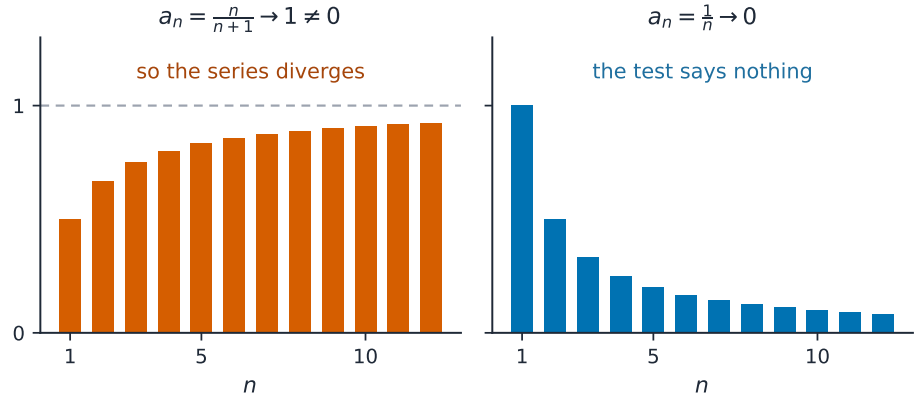
The nth Term Test for Divergence

A series converges when its partial sums $S_n = a_1 + \cdots + a_n$ tend to a finite limit S. Since $a_n = S_n - S_{n-1}$, convergence gives $a_n \to S - S = 0$. Therefore **if the terms do not tend to zero, the series diverges**. This includes a nonzero finite term limit, unbounded terms or a term limit that does not exist.

> **Worked example: a nonzero limit.** For $a_n = n/(2n+1)$, divide by n to obtain $a_n = 1/(2 + 1/n) \to 1/2$. Thus $\sum n/(2n+1)$ diverges by the nth Term Test. The number $1/2$ is the term limit, not the series sum.

> **Worked example: no term limit.** For $a_n = (-1)^{n-1}$, odd terms equal 1 and even terms equal -1. The terms have no single limit, so the series diverges. Its partial sums alternate between 1 and 0: bounded partial sums need not converge.

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A zero term limit makes this test **inconclusive** 无法判定. Both $\sum 1/n$ and $\sum 1/2^{n-1}$ have terms tending to zero, yet the first diverges and the second converges to 2. For the harmonic series, the block from $n = 2^{k-1}+1$ to 2^k contains 2^{k-1} terms each at least $1/2^k$, so each block adds at least $1/2$. The geometric-series theorem gives the second series' sum. These are separate arguments; the nth Term Test alone decides neither.

Alternating signs alone do not make the term limit fail: $(-1)^{n-1}/n \to 0$ because its absolute value tends to zero. The nth Term Test is inconclusive for this series too. Changing finitely many finite terms does not change convergence.

> **Try it.** Apply only the nth Term Test to $\sum 3n^2/(n^2+4)$, $\sum[2 + (-1)^n]$ and $\sum 2/(n+1)$, with $n \geq 1$.

Answer. The first has term limit $3 \neq 0$, so it diverges. The second has odd terms 1 and even terms 3, so its term limit does not exist and the series diverges. The third has term limit 0, so this test is inconclusive; another argument is needed for its classification.

Integral Test for Convergence

The **integral test** 积分判别法 compares a series tail with a matching **improper integral** 反常积分. Choose f with $a_n = f(n)$, and an integer N such that f is positive, continuous and non-increasing (**decreasing** 递减 in the non-strict sense) on $[N, \infty)$. Flat parts are allowed. Then

$$\sum_{n=N}^{\infty} a_n \quad \text{and} \quad \int_N^{\infty} f(x)\,dx$$

converge or diverge together. Their values are not the same quantity. Finitely many finite terms before the valid tail do not affect convergence. Do not automatically use

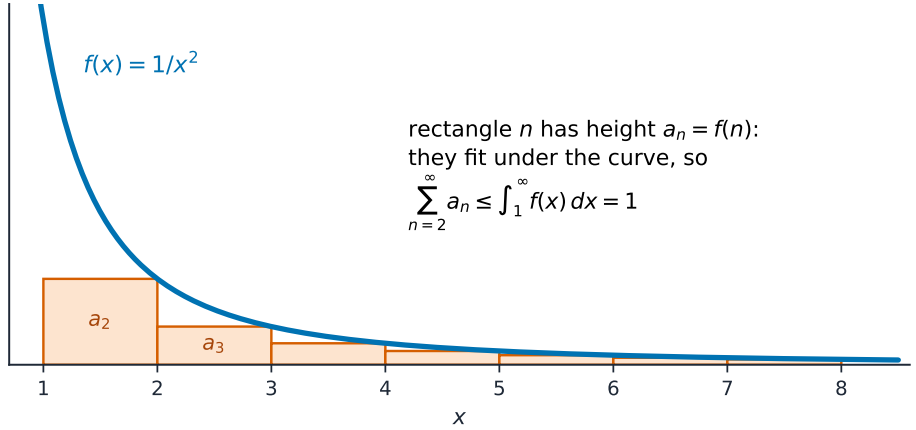
lower limit 1: the function may be undefined there.

Rectangle comparison. On $[N, N+m]$, unit-width left rectangles have total $U = \sum_{k=N}^{N+m-1} f(k)$ and lie above the area. Right rectangles have total $L = \sum_{k=N+1}^{N+m} f(k)$ and lie below. Thus $L \le I \le U$, where $I = \int_N^{N+m} f(x)\,dx$. This comparison bounds the positive partial sums when the improper integral is finite; if the integral grows without bound, so does the series.

Worked example: a finite integral. For $\sum_{n=1}^{\infty} 1/n^2$, $f(x) = 1/x^2$ is positive and continuous on $[1, \infty)$, with $f'(x) = -2/x^3 < 0$. Compute the improper integral as a limit:

$$\int_1^{\infty} \frac{1}{x^2}\,dx = \lim_{b\to\infty} \left[-\frac{1}{x}\right]_1^b = \lim_{b\to\infty} \left(1 - \frac{1}{b}\right) = 1.$$

The series converges. Its sum is not 1, since its first two terms already add to 1.25. For comparison, $\int_1^b dx/x = \ln b \to \infty$, so the harmonic series diverges despite its terms tending to zero.



The original diagram omits $f(1)$: add that first term separately when bounding the series from $n = 1$.

Worked example: a shifted function. For $\sum_{n=1}^{\infty} 1/(n+2)^2$, use $f(x) = 1/(x+2)^2$. It meets the conditions for $x \ge 1$, and its improper integral is $[-1/(x+2)]_1^{\infty} = 1/3$. The series converges; $1/3$ is the integral value, not the exact series sum.

Worked example: choose the lower limit. For $\sum_{n=2}^{\infty} 1/(n \ln n)$, the matching function is positive and continuous for $x \ge 2$. Its derivative is $-(\ln x + 1)/[x^2(\ln x)^2] < 0$. Since

$$\int_2^b \frac{dx}{x \ln x} = \ln(\ln b) - \ln(\ln 2) \to \infty,$$

the series diverges. Lower limit 1 would be invalid because the denominator is zero there. If a chosen matching function fails a test condition, that failure alone does not prove divergence.

Try it. Use the Integral Test for $\sum_{n=2}^{\infty} 1/[n(\ln n)^2]$. Explain the conditions and the integral.

Answer. The matching function is positive and continuous on $[2, \infty)$, with derivative $-(\ln x + 2)/[x^2(\ln x)^3] < 0$. Its antiderivative is $-1/\ln x$, so the improper integral from 2 is $1/\ln 2$, finite. The series converges. Changing any finite prefix of finite terms does not change this conclusion.

Harmonic Series and p-Series

A **p-series** p 级数 $\sum_{n=1}^{\infty} 1/n^p$, with fixed real p, **converges exactly when** $p > 1$ and **diverges for** $p \le 1$. The **harmonic series** 调和级数 is the boundary $p = 1$ and diverges despite its terms tending to zero.

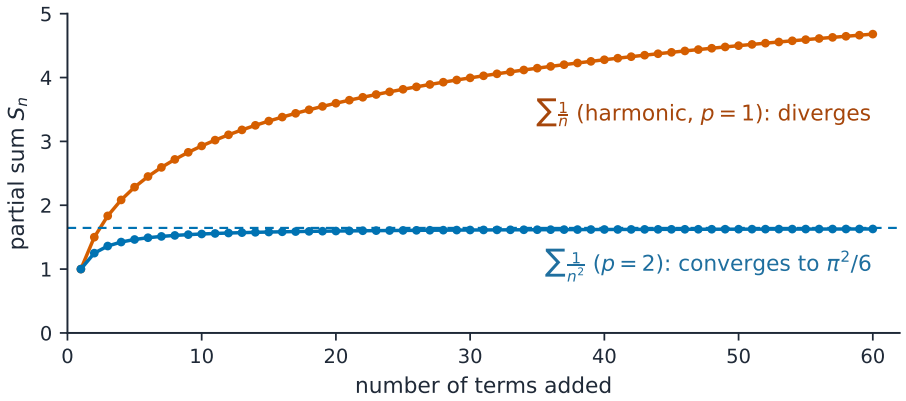
Why the rule holds. For $p > 0$, $f(x) = x^{-p}$ is positive, continuous and decreasing on $[1, \infty)$. If $p \ne 1$,

$$\int_1^b x^{-p}\,dx = \frac{b^{1-p} - 1}{1 - p}.$$

For $p > 1$, the improper integral is $1/(p-1)$, finite, so the series converges. For $0 < p < 1$, the integral grows without bound, so the series diverges. At $p = 1$, use $\int_1^b dx/x = \ln b \to \infty$ separately. An integral's value is not the exact series sum.

For $p = 0$, the terms equal 1; for $p < 0$, they grow without bound. These terms do not tend to zero, so the nth Term Test proves divergence. For negative p, the matching function is increasing, so the decreasing-function Integral Test has not been justified.

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$\sum \frac{1}{n^p}$ converges when $p > 1$ and diverges when $p \le 1$

Recognise the power. Since $n\sqrt{n} = n^{3/2}$, $\sum 1/(n\sqrt{n})$ converges. Since $\sqrt[3]{n^2} = n^{2/3}$, $\sum 1/\sqrt[3]{n^2}$ diverges. A fixed nonzero factor and a finite change in the starting index preserve convergence. For example,

$$\sum_{n=1}^{\infty} \frac{3}{(n+2)^{3/2}} = 3\sum_{k=3}^{\infty} \frac{1}{k^{3/2}}$$

converges. A zero factor gives the all-zero series directly. The expression $1/(n^2+1)$ is not exactly a p-series term; use comparison rather than deleting the $+1$.

The series $\sum 1/2^n$ is geometric, with the changing index in the exponent.

Alternating harmonic series. The signed series $\sum_{n=1}^{\infty}(-1)^{n-1}/n$ converges by the Alternating Series Test because its positive magnitudes decrease to zero. Its magnitude series $\sum 1/n$ diverges. The positive p-series rule alone does not classify signed series; the alternating test is explained below.

> **Try it.** Classify $\sum_{n=4}^{\infty} 5/(n\sqrt{n})$, and find all real q for which $\sum_{n=1}^{\infty} 1/n^{2q-1}$ converges.

Answer. The first has power $3/2 > 1$ and converges; factor 5 and the finite omitted prefix do not change this. The second requires $2q - 1 > 1$, hence $q > 1$. At $q = 1$ it is harmonic and diverges.

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Comparison Tests for Convergence

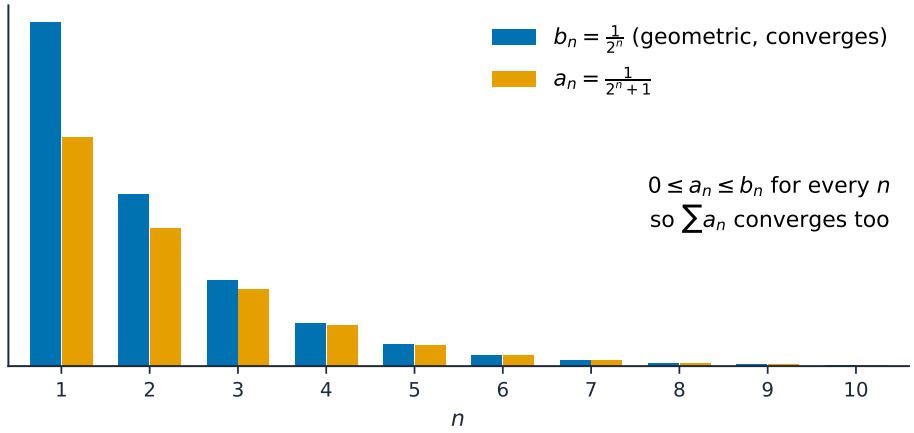
Comparison relates an unfamiliar series to a known p-series or geometric series. Check the terms and the direction of the inequality on a valid tail.

- **Direct comparison** 直接比较: if $0 \le a_n \le b_n$ eventually and $\sum b_n$ converges, the smaller $\sum a_n$ converges. If $a_n \ge b_n \ge 0$ eventually and $\sum b_n$ diverges, the larger $\sum a_n$ diverges.
- **Limit comparison** 极限比较: for eventually positive terms, if $\lim a_n/b_n = L$ with $0 < L < \infty$, both series have the same convergence behaviour. Classify the known series first; the limit does not by itself say that both converge.

Convergent upper bound. Since $0 < 1/(n^2+1) < 1/n^2$ for $n \ge 1$ and the p-series with power 2 converges, $\sum 1/(n^2+1)$ converges.

Divergent lower bound. For $n \ge 1$, $n+1 \le 2n$, so $1/(n+1) \ge 1/(2n) > 0$. Half the harmonic series diverges, and therefore so does $\sum 1/(n+1)$.

> **Worked example.** Does $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n^2-1}$ converge? For large n the terms behave like $\dfrac{1}{n^2}$. Direct comparison points the wrong way here ($\frac{1}{n^2-1} > \frac{1}{n^2}$, and being bigger than a convergent series proves nothing). Use the limit comparison test with $b_n = \dfrac{1}{n^2}$:
>
> $$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n^2}{n^2-1} = 1,$$
>
> a finite positive number. Since $\displaystyle\sum \frac{1}{n^2}$ converges ($p = 2 > 1$), $\displaystyle\sum \frac{1}{n^2-1}$ **converges** too.



Avoid the wrong direction. Being smaller than a divergent series does not establish divergence: $1/n^2 \le 1/n$, but the smaller series converges. Being larger than a convergent series does not establish convergence. A finite picture does not prove a bound for every later index.

Evaluate the full ratio. For $a_n = (3n^2+1)/(n^4+5)$ and $b_n = 1/n^2$,

$$\frac{a_n}{b_n} = \frac{3n^4+n^2}{n^4+5} = \frac{3+1/n^2}{1+5/n^4} \longrightarrow 3.$$

The terms are positive and the comparison p-series converges, so the given series converges. For $a_n = (2n+1)/(n^2+4)$ and $b_n = 1/n$, the ratio is $(2+1/n)/(1+4/n^2) \to 2$. Positivity and harmonic divergence give divergence of the latter series.

Check the tail. For $n \ge 3$, $n^2 - 4 \ge n^2/2$, so $0 < 1/(n^2-4) \le 2/n^2$. Thus the series starting at $n = 3$ converges. A term at $n = 2$ would be undefined: finite-prefix invariance requires finite, defined terms. Signed terms need a separate argument or study of their absolute values.

If a ratio limit is 0 or infinity, the finite-positive Limit Comparison theorem does not justify equal convergence behaviour. For example, $(1/n^2)/(1/n) \to 0$ even though the numerator series converges and the denominator series diverges. Choose another comparison or a valid direct bound; these limits are not automatic divergence verdicts. Comparison also does not equate exact sums.

> **Try it.** Classify $\sum_{n=2}^{\infty} 1/(n^2-1)$, $\sum \sqrt{n}/(n^2+1)$ and $\sum (4n^2+3)/(n^3+2)$.

Answer. The first is bounded above by $2/n^2$ for $n \ge 2$, so it converges. For the second, comparison with $1/n^{3/2}$ gives ratio $n^2/(n^2+1) \to 1$ and convergence. For the third, comparison with $1/n$ gives ratio $(4+3/n^2)/(1+2/n^3) \to 4$ and divergence. All the chosen tails are positive.

Alternating Series Test for Convergence

An **alternating series** 交错级数 has signs that switch. Write $a_n = (-1)^{n-1} b_n$ for a positive first sign, or $a_n = (-1)^n b_n$ for a negative first sign. The magnitudes b_n are positive.

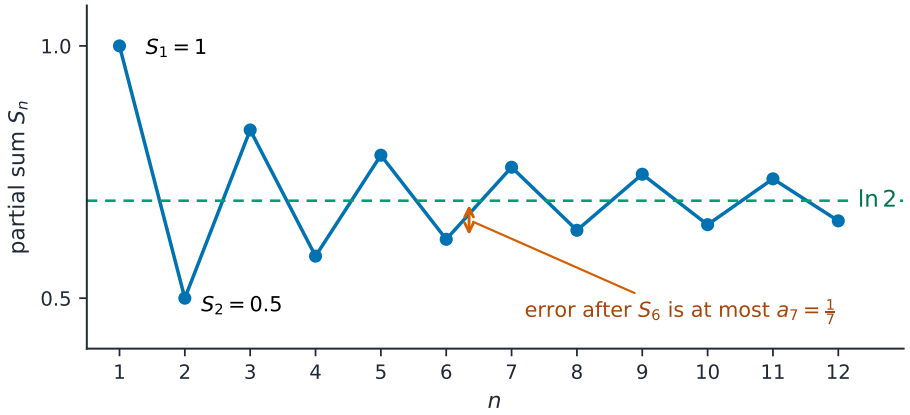
The Alternating Series Test gives sufficient conditions: if, from some index onward, $b_{n+1} \le b_n$ and $b_n \to 0$, the series converges. Equal consecutive magnitudes are allowed. A finite prefix of finite terms does not affect convergence.

Why it works. In the positive-first case with non-increasing magnitudes from the start, the even partial sums increase and the odd partial sums decrease; each group bounds the other. The gap $S_{2k-1} - S_{2k} = b_{2k}$ tends to zero, so both groups approach the same finite limit. Apply this reasoning to a valid tail for eventual conditions.

Alternating harmonic example. For $b_n = 1/n$, positivity, decrease and the zero limit all hold for $n \ge 1$, so $\sum (-1)^{n-1}/n$ converges.

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The positive harmonic series diverges. Starting with $(-1)^n$ multiplies the signed series by -1 and preserves convergence. A zero term limit is not the exact series sum.



$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$: terms shrink to 0 and alternate, so it converges

Eventually decreasing example. For $b_n = n/(n^2+9)$, $f(x) = x/(x^2+9)$ has derivative $(9-x^2)/(x^2+9)^2 \le 0$ for $x \ge 3$. Also $b_n = (1/n)/(1+9/n^2) \to 0$. The alternating series converges although the first magnitudes increase.

Separate two failures. If $b_n = 1 + 1/n$, the magnitudes decrease but tend to 1; the signed terms do not approach zero, and the nth Term Test proves divergence. If only eventual non-increase fails, the Alternating Series Test is not justified; that failure alone does not prove divergence.

For example, let $b_{2k-1} = 1/k$ and $b_{2k} = 1/k + 1/k^2$. There is a magnitude increase in every pair. Yet the positive-first adjacent signed pairs sum to $-1/k^2$. Their even partial sums converge, and the difference to the next odd sum tends to zero, so the original series converges by this separate argument. Adjacent pairing keeps the original order.

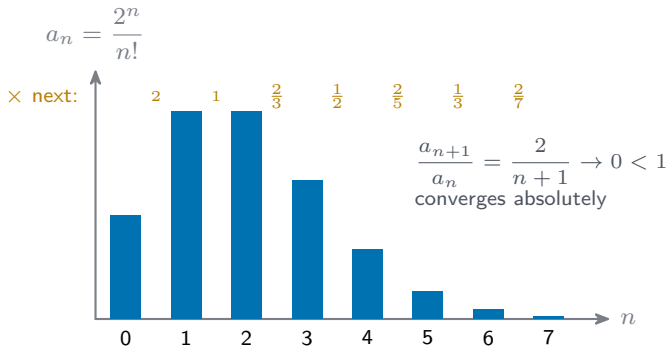
The test does not by itself classify convergence of $\sum b_n$ or calculate an exact sum. Absolute/conditional convergence and numerical error bounds are separate topics below.

> **Try it.** Test the alternating series with positive magnitudes $2/\sqrt{n+3}$, $n/(n^2+16)$ and $(3n+1)/(n+2)$, for $n \ge 1$.

Answer. The first decreases to zero and gives convergence. The second tends to zero and has derivative $(16-x^2)/(x^2+16)^2 \le 0$ from $x = 4$, so its alternating series converges. The third has magnitude limit 3, so its signed terms fail to tend to zero and the nth Term Test gives divergence.

Ratio Test for Convergence

The **ratio test** 比值判别法 uses the absolute next/current ratio $q_n = |a_{n+1}/a_n|$. The terms must be nonzero eventually so that the quotient is defined on a tail. When $L = \lim q_n$ exists:



Once the ratio stays below 1, the terms collapse.

- $L < 1$: the series **converges absolutely**, because the tail magnitudes are eventually bounded by a convergent geometric series.
- $L > 1$, including $+\infty$: the series diverges.
- $L = 1$: the test is inconclusive; another test is needed.

Finitely many finite early terms, including zeros, do not affect convergence. An all-zero tail converges directly; do not evaluate $0/0$. Recurring zero denominators or a nonexistent ratio limit require another argument, rather than an automatic divergence verdict.

It is the go-to test for series with **factorials** 阶乘 or n**th powers**, and it is exactly how you find the radius of convergence of a power series.

Factorial cancellation. For $a_n = 2^n/n!$, first write $a_{n+1} = 2^{n+1}/(n+1)!$. Then

$$q_n = \left|\frac{2^{n+1}}{(n+1)!} \cdot \frac{n!}{2^n}\right| = \frac{2}{n+1} \longrightarrow 0.$$

The series converges absolutely. Use $(n+1)! = (n+1)n!$, not $n! + 1$. Early ratios need not all be below 1: for $4^n/n!$, the first ratio is 2, but $4/(n+1) \to 0$ still proves absolute convergence.

Signs and quotient direction. For $(-2)^n/n!$, the signed ratio is $-2/(n+1)$; its absolute value tends to zero. For $(-3)^n$, the absolute ratio is $3 > 1$, so the series diverges despite its negative signed ratio. For $n!/2^n$, use the correct quotient $(n+1)/2 \to +\infty$, proving divergence. Its reciprocal is not the Ratio Test quotient.

A changing exponent. For $a_n = n!/n^n$,

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)!}{(n+1)^{n+1}} \cdot \frac{n^n}{n!} = \left(\frac{n}{n+1}\right)^n = \left(1 + \frac{1}{n}\right)^{-n} \longrightarrow e^{-1} < 1.$$

Hence the series converges absolutely. Both the base and the exponent change in the next denominator.

The boundary is inconclusive. The harmonic terms have $q_n = n/(n+1) \to 1$ and their series diverges. The terms $1/n^2$ have $q_n = (n/(n+1))^2 \to 1$ and their series converges. The p-series theorem supplies these classifications. In particular, a finite ratio below 1 at every index is not enough; the harmonic series is a counterexample. The ratio limit is not the series sum.

For fixed real $c \neq 0$, $\sum_{n=1}^{\infty} c^n/n!$ has absolute-ratio limit 0 and converges absolutely. For $c = 0$, it is an all-zero series and converges directly, without a ratio calculation.

> **Try it.** Apply the Ratio Test to $\sum 3^n/(n+2)!$, $\sum n^3/5^n$ and $\sum (n+1)!/3^n$, starting at $n = 1$.

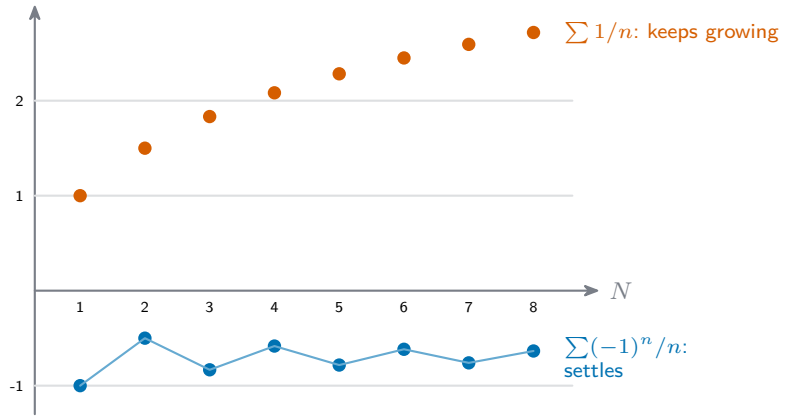
Answer. Their absolute ratios are $3/(n+3) \to 0$, $((n+1)/n)^3/5 \to 1/5$ and $(n+2)/3 \to +\infty$. The first two series converge absolutely; the third diverges. Each next term changes every occurrence of the index.

Common mistake. Reading a ratio-test limit of exactly 1 as convergence. Both $\sum \frac{1}{n}$ (diverges) and $\sum \frac{1}{n^2}$ (converges) give a limit of 1: the test is **inconclusive** there, so use another test.

Determining Absolute or Conditional Convergence

A series **converges absolutely** 绝对收敛 if $\sum |a_n|$ converges. It **converges conditionally** 条件收敛 if $\sum a_n$ converges but $\sum |a_n|$ diverges. A divergent signed series is neither of these convergent categories.

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Conditional: the signed series settles, the magnitudes do not.

Two separate questions. A useful procedure is to test the magnitude series first. If it converges, absolute convergence guarantees convergence of the original series. If it diverges, test the signed series separately: signed convergence gives conditional convergence; signed divergence gives divergence. Divergence of $\sum |a_n|$ alone does not distinguish these last two outcomes.

For real terms, $a_n^+ = \max(a_n, 0)$ and $a_n^- = \max(-a_n, 0)$ satisfy $0 \le a_n^{\pm} \le |a_n|$. If the magnitude series converges, both series of parts converge by comparison. Since $a_n = a_n^+ - a_n^-$, their finite limits give convergence of the signed series.

Conditional example. For $\sum(-1)^n/n$, the magnitude series is divergent harmonic. The signed series passes the Alternating Series Test because its positive magnitudes decrease to zero. These two facts together establish conditional convergence.

Absolute example. For $\sum(-1)^n/n^2$, the magnitude p-series converges. Therefore the signed series is absolutely convergent; its alternating signs do not make it conditional, and no extra convergence proof is needed.

Divergent example. For $(-1)^n n/(n+1)$, the magnitudes tend to 1, so the signed terms do not tend to zero. The nth Term Test gives divergence, not conditional convergence. This and the conditional harmonic example both have divergent magnitude series.

A shifted example is $\sum_{n=1}^{\infty}(-1)^{n-1}2/\sqrt{n+3}$. Its magnitude series is twice a divergent p-series tail with power $1/2$; its signed series passes the Alternating Series Test. Thus it is conditionally convergent. A Ratio Test result of $L = 1$ does not establish conditional convergence: both the harmonic and squared-denominator alternating examples have that ratio limit.

Rearrangement and finite changes. Any regrouping or rearrangement of an absolutely convergent series has the same sum. A conditionally convergent series has no such rearrangement guarantee; a new order may change the sum or fail to converge. Use the stated order when classifying. Changing finitely many finite terms, or multiplying by a fixed nonzero constant, preserves the classification. A zero multiplier gives an absolutely convergent all-zero series directly.

> **Try it.** Classify $\sum(-1)^{n-1}/n^{3/2}$, $\sum(-1)^n/\sqrt{n+1}$ and $\sum(-1)^n(2n+1)/(n+4)$, for $n \ge 1$.

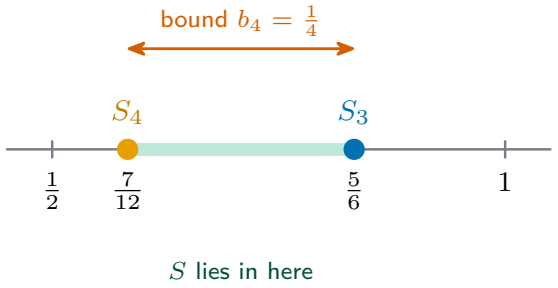
■ Worked answers

The first is absolutely convergent because its magnitude p-series has power $3/2 > 1$. The second is conditional: the magnitude p-series tail has power $1/2$ and diverges, while the signed series passes the Alternating Series Test. The third diverges by the nth Term Test since its magnitudes tend to 2. Give both signed and magnitude arguments when claiming conditional convergence.

Alternating Series Error Bound

For a series starting at $n = 1$, let S_N contain its first N terms. If the omitted tail alternates with positive, non-increasing magnitudes tending to zero, then

$$|S - S_N| \le b_{N+1}.$$



The first omitted term bounds the error.

The bound uses the **first omitted term** 第一个略去的项 (its magnitude), not the last included one. The exact sum lies between S_N and S_{N+1}. A positive next term makes S_N a lower estimate; a negative next term makes it an upper estimate. Convergence alone, without the omitted-tail conditions, does not justify this theorem.

Exact harmonic example. For $1 - 1/2 + 1/3 - \cdots$, $S_3 = 5/6$ and the next term is $-1/4$. Thus the bound is $1/4$ and $7/12 \le S \le 5/6$. The bound is not the actual error, and the exact infinite sum is not needed to certify this enclosure.

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Match the target inequality. A sufficient next-term condition for error at most ε is $b_{N+1} \le \varepsilon$. For a strict certificate using the non-strict bound alone, use $b_{N+1} < \varepsilon$. With harmonic magnitudes and target 0.01, these conditions give N≥99 and N≥100 respectively. Stop before the omitted term, whose index is 100 or 101. These are certificate counts, not claims about the smallest possible actual-error count.

Keep the coefficient. For magnitudes $3/n^2$, requiring $3/(N+1)^2 \le 0.001$ gives $(N+1)^2 \ge 3000$. The smallest integer satisfying this condition is N=54: the omitted term has index 55.

Count from the actual start. If the series begins at index m, N included terms end at $K = m + N - 1$, and the omitted index is $m + N$. For $\sum_{n=3}^{\infty}(-1)^{n-3}/n^2$, four terms end at index 6. Their sum is 73/1200, the omitted magnitude is 1/49 at index 7, and the next term is positive. Hence $73/1200 \le S \le 73/1200 + 1/49$.

Include rounding error. For a reported rounded value r,

$$|S - r| \le |S - S_N| + |S_N - r|.$$

For example, rounding 5/6 to 0.833 adds 1/3000 to its truncation bound 1/4. Round an enclosure outward: $[0.583333, 0.833334]$ contains the exact interval $[7/12, 5/6]$; inward rounding is not generally safe.

An eventual tail can suffice. For magnitudes $b_n = n/(n^2+9)$, the derivative $(9-x^2)/(x^2+9)^2$ is non-positive for x≥3 and $b_n \to 0$. After two included terms, the omitted tail starts at index 3 and meets the conditions, giving bound $b_3 = 1/6$. Earlier increases do not invalidate this particular tail argument.

> **Try it.** For magnitudes $2/(n+1)^2$ starting at n=1, find the smallest counts satisfying next-term conditions for error at most 0.005 and strictly below 0.005. Then give the omitted index and enclosure for $\sum_{n=4}^{\infty}(-1)^{n-4}/n$ after five terms.

■ Error-bound worked answers

The shifted next magnitude is $2/(N+2)^2$. Its non-strict condition gives N≥18; its strict condition gives N≥19. In the second series, the last included index is 8 and the omitted index is 9. The five-term sum is 167/840, the bound is 1/9 and the next sign is negative, so $221/2520 \le S \le 167/840$. Combining truncation bound 0.004 with rounding to three decimal places gives $0.004 + 0.0005 = 0.0045$.

> **Worked example.** Estimate $S = \displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$ with four terms, and bound the error.

$$S_4 = 1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} \approx 0.7986.$$

The terms $\frac{1}{n^2}$ are positive, decreasing and tend to 0, so $|S - S_4| \le b_5 = \dfrac{1}{25} = 0.04$.

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(The true sum is $\frac{\pi^2}{12} \approx 0.8225$; the actual error, about 0.024, is inside the bound.) Use the **first omitted term**, b_5, not the last term used.

Finding Taylor Polynomial Approximations of Functions

A **Taylor polynomial** 泰勒多项式 uses **derivatives** 导数 evaluated at a **center** 中心 a to build a finite approximation. Through order n, it is

$$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k.$$

It matches $f(a)$ and the derivatives through order n at a. The coefficient of $(x-a)^k$ is $f^{(k)}(a)/k!$. Use derivative values at the center, not expressions still depending on x. Centered at $a = 0$, this is a **Maclaurin polynomial** 麦克劳林多项式. Its degree can be below n if a coefficient is zero.

> **Worked example: use a derivative table.** At $a = 1$, suppose the derivative values through order 3 are $2, -3, 8, 12$. Divide by $0!, 1!, 2!, 3!$ to get coefficients $2, -3, 4, 2$. Thus

$$P_3(x) = 2 - 3(x-1) + 4(x-1)^2 + 2(x-1)^3,$$
$$P_3(0.9) = 2 + 0.3 + 0.04 - 0.002 = 2.338.$$

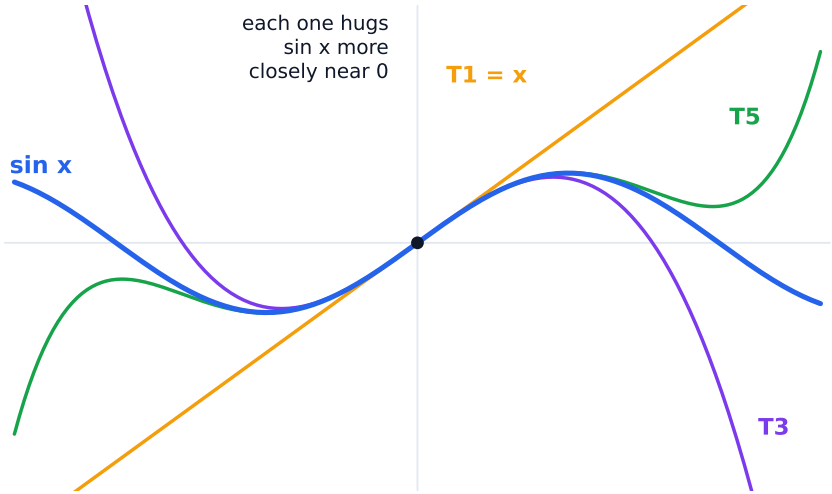
The change is $x - a = -0.1$, so the odd powers keep their negative sign. This gives $f(0.9) \approx 2.338$. A finite table alone does not supply a numerical error guarantee.

> **Worked example: Maclaurin for the exponential.** Every derivative of e^x at 0 equals 1. Divide these values by their factorials to get $P_2(x) = 1 + x + x^2/2$. Hence $e^{0.1} \approx P_2(0.1) = 1.105$. The polynomial value is exact for P_2, while its use for the exponential is an approximation.

> **Worked example: a useful nonzero center.** For $\ln x$ about 1, the derivative values are $0, 1, -1, 2$ through order 3. Let $h = x - 1$. Then

$$P_3(x) = h - \frac{h^2}{2} + \frac{h^3}{3}.$$

At $x = 1.1$, use $h = 0.1$, giving $\ln(1.1) \approx 0.0953333\ldots$. Using $h = 1.1$ would evaluate at the wrong shifted input.



The curves compare $\sin x$ *with* $T_1 = x$, $T_3 = x - x^3/6$ *and* $T_5 = x - x^3/6 + x^5/120$.

Order and degree. For cosine about 0, $P_2 = P_3 = 1 - x^2/2$ has degree 2 and matches derivatives through order 3.

> **Worked example: a higher order can be worse.** For e^x at -2, $P_0 = 1$ has error about 0.8647, while $P_1 = -1$ has error about 1.1353. More terms do not guarantee a smaller error at every point.

> **Try it.** At $a = -1$, suppose the derivative values through order 3 are $4, 2, -6, 24$. Build P_3 and estimate $f(-0.8)$.

Answer. Divide by factorials to get coefficients $4, 2, -3, 4$. Then

$$P_3(x) = 4 + 2(x+1) - 3(x+1)^2 + 4(x+1)^3.$$

At $x = -0.8$, the change is 0.2, giving $f(-0.8) \approx 4 + 0.4 - 0.12 + 0.032 = 4.312$.

> **Worked example.** $f(2) = 3$, $f'(2) = -1$, $f''(2) = 4$ and $f'''(2) = 12$. Write the third-**degree** 次数 Taylor polynomial about $x = 2$ and estimate $f(2.1)$.

Divide each derivative by its factorial: $\frac{4}{2!} = 2$ and $\frac{12}{3!} = 2$. So

$$P_3(x) = 3 - (x-2) + 2(x-2)^2 + 2(x-2)^3,$$

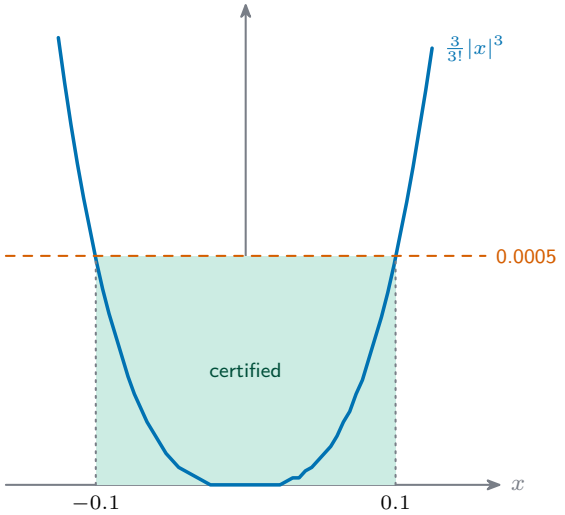
$$f(2.1) \approx P_3(2.1) = 3 - 0.1 + 2(0.01) + 2(0.001) = 2.922.$$

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Forgetting the factorials (writing $4(x-2)^2$ instead of $\frac{4}{2!}(x-2)^2$) is the most common error.

Lagrange Error Bound

The **Lagrange error bound** 拉格朗日误差界 gives an upper limit for the absolute error of a Taylor approximation. Let P_n include the terms through order n about a and write $R_n(x) = f(x) - P_n(x)$. If the derivatives through order $n+1$ are continuous on the closed interval joining a and x, and $|f^{(n+1)}(t)| \le M$ throughout that interval, then



Solve bound at most tolerance for the certified interval.

$$|R_n(x)| \le B_n(x) = \frac{M}{(n+1)!}|x-a|^{n+1}.$$

The constant M may be any justified upper bound; it need not be the exact maximum. Check the whole interval, not only the center or target. When $x < a$, use the interval $[x, a]$ and the positive distance $|x - a|$.

> **Worked example: certify a value.** For e^x at $x = 0.1$, $P_2 = 1 + x + x^2/2$ gives 1.105. On $[0, 0.1]$, the third derivative satisfies $e^t \le e^{0.1} < 1.2$, so choose $M = 1.2$. Then

$$|R_2(0.1)| \le \frac{1.2}{3!}(0.1)^3 = 0.0002.$$

This certifies error at most 0.001. It does not say that the actual error equals 0.0002.

The value $M = 1$ would be too small, since $e^t > 1$ for $t > 0$.

> **Worked example: a uniform interval.** On $[-1, 1]$, every derivative of e^x is at most $e < 3$. For the same P_2, choose the fixed bound $M = 3$. Requiring error at most 0.0005 gives
>
> $$\frac{3}{3!}|x|^3 \le 0.0005 \quad \Longrightarrow \quad |x| \le 0.1.$$
>
> Thus $[-0.1, 0.1]$ is certified by this bound and lies inside the interval where M is valid. It is not necessarily the largest interval where the actual approximation meets the tolerance.

Compare two bounds. For $\cos(0.2)$, $P_2 = 1 - x^2/2 = 0.98$. The third derivative has magnitude at most 1, giving $B_2 = 0.2^3/3! = 0.001333\ldots$. This does not certify tolerance 0.001, but it does not prove failure. The cosine series at this point alternates with decreasing term sizes tending to 0. Its first omitted nonzero term gives the sharper bound

$$|R_2(0.2)| \le \frac{0.2^4}{4!} = 0.000066666\ldots < 0.001.$$

Here $P_2 = P_3$ because the cubic coefficient is zero. The fourth-derivative Lagrange bound with $n = 3$ and $M = 1$ gives the same sharper estimate. Taylor order can be greater than the polynomial's actual degree.

> **Try it.** Suppose $|f^{(4)}(t)| \le 9$ on $[1.8, 2]$, with continuous derivatives through order 4. Bound the error of its third-order Taylor approximation about 2 at $x = 1.8$. Does it meet tolerance 0.001?

Answer. Use distance 0.2 and denominator 4!. The bound is $9(0.2)^4/24 = 0.0006 \le 0.001$, so the tolerance is certified.

Order is not a guarantee of a smaller bound. Recheck both the derivative bound and distance when increasing the order. For sine at $x = 4$, centered at 0, $M = 1$ works for every derivative. The order-0 bound is 4, but the order-1 bound is $4^2/2! = 8$. A larger factorial alone does not settle the comparison.

> **Worked example.** Continue the example above: suppose $|f^{(4)}(x)| \le 30$ for $2 \le x \le 2.1$. Then, with $n = 3$,
>
> $$|f(2.1) - P_3(2.1)| \le \frac{30}{4!}|2.1 - 2|^4 = \frac{30}{24}(0.0001) = 0.000125.$$
>
> So $P_3(2.1) = 2.922$ is within 0.000125 of $f(2.1)$. Use the next derivative after the polynomial's degree: a degree-3 polynomial uses $f^{(4)}$ and 4!.

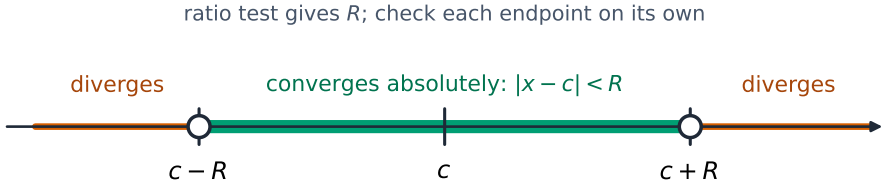
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Radius and Interval of Convergence of Power Series

A **power series** 幂级数 is a sum of terms $c_n(x - a)^n$, where the numbers c_n do not depend on x. Its **radius of convergence** 收敛半径 R measures a distance from the center a. For $0 < R < \infty$, it converges absolutely for $|x - a| < R$ and diverges for $|x - a| > R$. Test the two **endpoints** 端点 separately to finish the **interval of convergence** 收敛区间. The center itself is always included; positive-power terms vanish there.

Use the ratio test on the complete terms when its limit is available. Keep the factor involving x, solve for a distance, and check the center directly if the ratio expression is undefined there. A limit of 1 is inconclusive, not proof of divergence.

> **Worked example: a shifted center.** For $\displaystyle\sum_{n=1}^{\infty} \frac{(x-2)^n}{3^n n}$, the absolute ratio, away from the center, is
>
> $$\left|\frac{u_{n+1}}{u_n}\right| = \frac{|x-2|}{3}\frac{n}{n+1} \longrightarrow \frac{|x-2|}{3}.$$
>
> Thus $|x - 2| < 3$, or $-1 < x < 5$. The center is 2 and $R = 3$, not the right boundary 5 or the width 6. At $x = 2$, every term is 0. At $x = -1$, the original series becomes $\sum(-1)^n/n$, which converges by the alternating-series test: $1/n$ decreases to 0. At $x = 5$, it becomes the divergent harmonic series $\sum 1/n$. The full interval is $[-1, 5)$.



A finite positive radius gives the open interior. The left and right boundary series decide which ends are included.

The same radius can give different brackets. With $t = (x - 2)/3$ and $n \ge 1$, these families all have center 2, radius 3 and absolute ratio limit $|t|$.

Terms	Left boundary, $t = -1$	Right boundary, $t = 1$	Interval
t^n	Terms do not tend to 0: diverges	Terms are 1: diverges	$(-1, 5)$
t^n/n	Alternating harmonic: converges	Harmonic: diverges	$[-1, 5)$
t^n/n^2	Absolute convergence	p-series, $p = 2$: converges	$[-1, 5]$
$(-1)^n t^n/n$	Harmonic: diverges	Alternating harmonic: converges	$(-1, 5]$

Worked example: reverse the included end. For $\sum_{n=1}^{\infty}(-1)^n(x+1)^n/(2^n n)$, the absolute ratio limit is $|x+1|/2$. Hence $a=-1$, $R=2$ and the candidate bounds are -3 and 1. At -3, the two factors $(-1)^n$ cancel and give the divergent harmonic series. At 1, the alternating harmonic series converges. The interval is $(-3,1]$.

Special radii. For $\sum_{n=1}^{\infty} n!(x-2)^n$, the absolute ratio $(n+1)|x-2|$ grows without bound for every $x \neq 2$. Only $x=2$ converges, so $R=0$ and the set is $\{2\}$. For $\sum_{n=1}^{\infty}(x-2)^n/n!$, the ratio $|x-2|/(n+1)$ tends to 0 for every fixed real x, so $R=\infty$ and the interval is $(-\infty,\infty)$. These cases do not have two distinct finite boundary values to test.

Worked example: the radius stays the same. On the open interval, a positive-radius power series is the Taylor series of its sum function. Differentiating or integrating preserves its radius but may change the boundary results. The series $\sum_{n=1}^{\infty} x^n/n$ has interval $[-1,1)$. Differentiating gives $\sum_{n=1}^{\infty} x^{n-1}$, with interval $(-1,1)$: the terms do not tend to 0 at either end. Integrating from 0 to x gives $\sum_{n=1}^{\infty} x^{n+1}/[n(n+1)]$, with interval $[-1,1]$: the absolute boundary terms are at most $1/n^2$, so comparison applies. All three radii are 1.

Try it. Find the center, radius and interval for $\sum_{n=1}^{\infty}(x+2)^n/(4^n n^2)$, with a reason for each boundary result.

Answer. The ratio limit is $|x+2|/4$, so the center is -2, the radius is 4 and the candidate bounds are -6 and 2. Both boundary series converge absolutely by the p-series test with $p=2$. The interval is $[-6,2]$.

Finding Taylor or Maclaurin Series for a Function

Extending the Taylor-polynomial formula to infinitely many terms gives a **Taylor (or Maclaurin) series**. At a center a, its formula is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

A finite Taylor polynomial is a partial sum. To represent $f(x)$ exactly, the difference $f(x)-P_N(x)$ must tend to 0 as N grows. Convergence of the series alone does not establish equality to the original function. For the four standard identities below, the intervals where equality holds are known.

Derive the cosine pattern. The derivatives of $\cos x$ repeat every four steps. Their values at 0 are $1,0,-1,0,1,\ldots$. Dividing each by the factorial of its order gives coefficients $1,0,-1/2!,0,1/4!,\ldots$. The odd powers have zero coefficients; the even powers alternate signs.

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$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \quad (|x|<1)$$

These sine, cosine and exponential identities hold for all real x.

Use radians for sine and cosine. The geometric identity requires $-1<x<1$: at either endpoint its terms do not tend to 0. Its coefficients are 1, without factorial denominators.

Worked example: change the center. For e^x centered at 1, every derivative value is e. Thus

$$\begin{aligned} e^x &= \sum_{n=0}^{\infty} \frac{e}{n!}(x-1)^n \\ &= e + e(x-1) + \frac{e}{2!}(x-1)^2 + \cdots \end{aligned}$$

This identity holds for every real x. At $x=1$, all positive powers vanish and leave the correct value e. Using x^n instead would put the expansion at the wrong center.

Terms and degree. The first three nonzero cosine terms give $1-x^2/2+x^4/24$. This is degree 4, not degree 3. The first three nonzero sine terms have highest power 5. A finite table of derivative values determines a finite Taylor polynomial; without a pattern, it does not determine all higher terms.

Try it. Use $h=x-\pi/2$ and $\sin(\pi/2+h)=\cos(h)$ to find the Taylor series for $\sin x$ centered at $\pi/2$.

Answer. Replace h in the cosine identity:

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n (x-\pi/2)^{2n}}{(2n)!}.$$

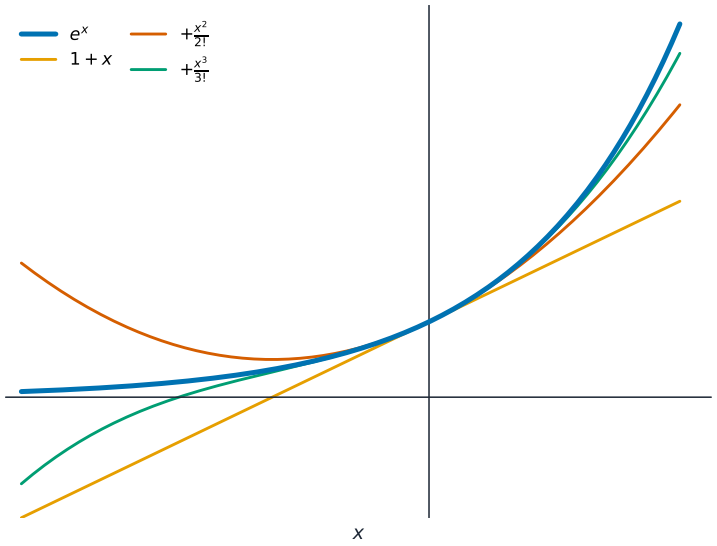
It is valid for every real x. The first three nonzero terms are $1-(x-\pi/2)^2/2!+(x-\pi/2)^4/4!$. The powers are even powers of the shifted input, although the sine Maclaurin series uses odd powers of x.

New series come from **manipulating** these – substituting, differentiating, integrating, or multiplying.

Worked example. Find the Maclaurin series for e^{x^2}. Substitute x^2 for x in $e^x = \sum \dfrac{x^n}{n!}$:

$$e^{x^2} = \sum_{n=0}^{\infty} \frac{(x^2)^n}{n!} = 1 + x^2 + \frac{x^4}{2!} + \frac{x^6}{3!} + \cdots$$

which converges for all x. This substitution trick is far faster than differentiating e^{x^2} six times.



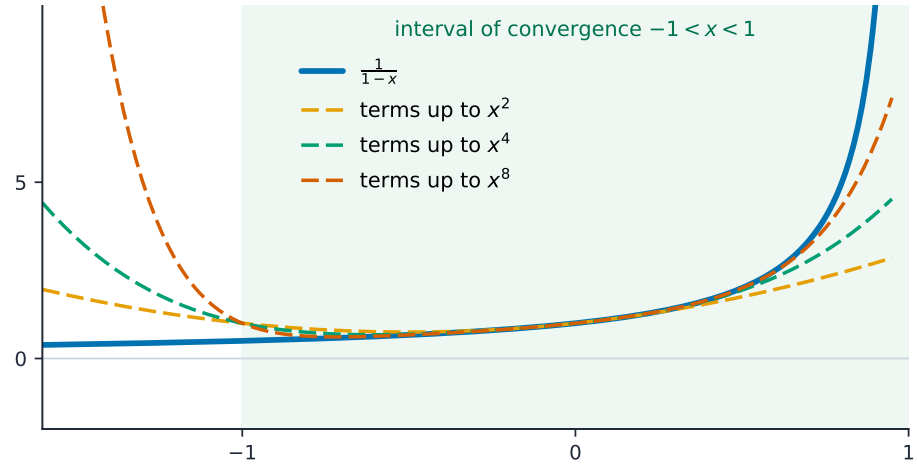
The linear, quadratic and cubic sums share the exponential function's value at the center.

Representing Functions as Power Series

Because a power series can be **differentiated and integrated term by term** (within its radius), you can build new series from known ones – e.g. integrate the geometric series for $\dfrac{1}{1-x}$ to get the series for $-\ln(1-x) = \sum_{n\ge 1} \dfrac{x^n}{n}$, or substitute $-x^2$ to get the series for $\dfrac{1}{1+x^2}$. Representing a function as a power series lets you approximate values and integrals that have no elementary antiderivative.

Exam skill: the BC series free-response usually asks you to derive a new Maclaurin series from a known one, find its interval of convergence, and use the alternating-series or Lagrange bound to estimate the error – the capstone skills of the course.

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Partial sums of $1 + x + x^2 + \ldots$ match $1/(1-x)$ only for $-1 < x < 1$

Exam tips

- Test a series for convergence with the right tool: **geometric** ($|r| < 1$, sum $\frac{a}{1-r}$), nth-term, ratio, integral, comparison, or alternating-series test.
- A **geometric** series with nonzero first term converges exactly when $|r| < 1$. Handle the all-zero series directly.
- Build a **Taylor/Maclaurin** series to approximate a function; more terms give a better fit **near the centre**.
- Know the standard Maclaurin series for e^x, $\sin x$, $\cos x$, and $\frac{1}{1-x}$.
- Find the **radius/interval of convergence** with the ratio test, then check the endpoints separately.