Course Companion

AP Calculus BC

Every topic handout in one volume

全部专题讲义合集

exercises.lol

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Limits and Continuity

AP Calculus BC

Introducing Calculus: Can Change Occur at an Instant?

Calculus is the mathematics of **change** 变化 and of **accumulation** 累积. It answers two big questions: how fast is something changing *right now*, and how much has piled up *so far*? Unit 1 builds the one tool both questions rest on –the **limit** 极限.

Start with a puzzle. A car's speedometer reads 60 km/h. What does that mean *at a single instant* 瞬间? Speed is distance over time. But at one instant no time passes and no distance is covered, so the fraction looks like $\frac{0}{0}$ –undefined.

- The **average rate of change** 平均变化率 uses a whole *interval* 区间: the change in one quantity divided by the change in another. It divides by zero, and so is undefined, when the change in the input would be zero.
- The **instantaneous rate of change** 瞬时变化率 is what we want *at a point*. It is the value the average rate **approaches** 趋近 as the interval shrinks toward zero length.

The clever move is not to plug in zero (undefined), but to watch what the average rate *approaches* as the interval gets smaller and smaller. That approaching value is a **limit**. So calculus lets us describe change at an instant –as a limit of average rates over ever-shorter intervals. This one idea powers the **derivative** 导数 (Unit 2) and, run in reverse, the **integral** 积分 (Unit 6). Everything else in this unit defines limits carefully and computes them reliably.

Defining Limits and Using Limit Notation

A smooth bridge curve: limits describe the value a graph approaches as we zoom in

Given a function f, the limit of $f(x)$ as x approaches c is a real number R if $f(x)$ can be made **arbitrarily** 任意地 close to R by taking x **sufficiently** 足够 close to c –**but not equal to** c. We write

$$\lim_{x \to c} f(x) = R$$

and read it: "the limit of $f(x)$, as x approaches c, equals R."

The last words are the heart of a limit: it describes the **behavior** 行为 of f *near* c, not the value *at* c. The function may be undefined at c, or defined but equal to something else –the limit does not care.

A limit can be shown in three ways: **graphically** 用图象, **numerically** 用数值 (a table), and **analytically** 用解析式 (algebra). Learning to move between these representations is a core skill.

*(Note: the epsilon-delta definition of a limit is **not** tested on the AP Exam, so this handout does not use it.)*

Estimating Limit Values from Graphs

A graph is often the fastest way to read a limit. To find $\lim_{x \to c} f(x)$, run your finger along the curve toward $x = c$ from each side and ask: what height is the curve heading for?

- Trace from the **left** (inputs smaller than c): this gives the **left-hand limit** 左极限, $\lim_{x \to c^-} f(x)$.

- Trace from the **right** (inputs larger than c): this gives the **right-hand limit** 右极限, $\lim_{x \to c^+} f(x)$.
- These are the **one-sided limits** 单侧极限. If both head to the *same* height R, then the two-sided limit exists and $\lim_{x \to c} f(x) = R$.

Crucially, **ignore the point itself**. Graphs mark the difference between the limit and the value:

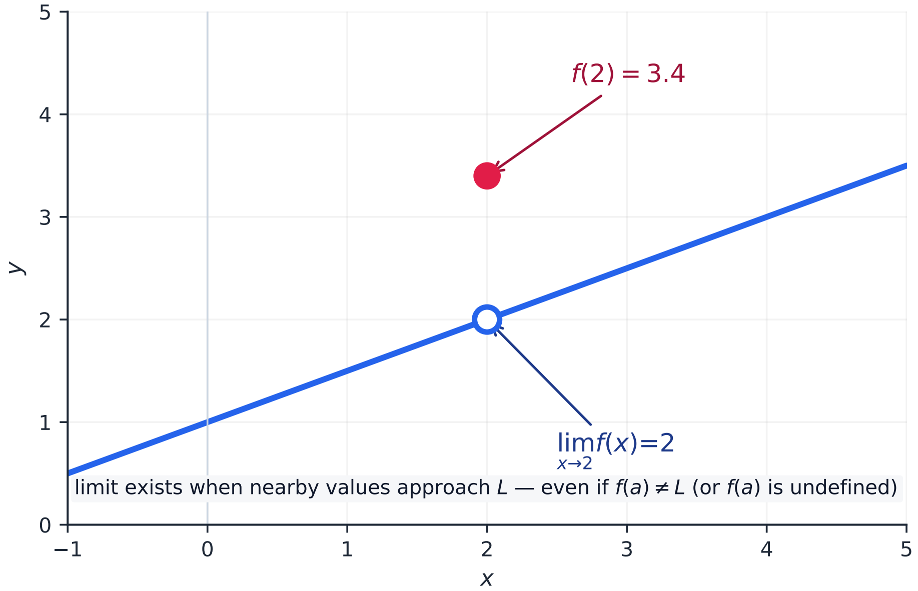
- An **open circle** 空心圆 marks a height the curve *approaches* but does not reach – a "hole" 空洞.
- A **closed circle** 实心圆 marks the actual value $f(c)$.

So a curve may approach $R = 3$ from both sides (limit is 3) while a filled dot sits at height 5 (value $f(c) = 5$). The limit is 3; the two need not match.

A limit **does not exist** (often written **DNE**) when the two sides disagree (a **jump** 跳跃), when the function is **unbounded** 无界 (grows without limit), or when it **oscillates** 振荡 forever near c. For example:

$$\lim_{x \to 0} \frac{1}{x^2} = \infty, \qquad \lim_{x \to 0} \frac{|x|}{x} \text{ DNE}, \qquad \lim_{x \to 0} \sin\left(\frac{1}{x}\right) \text{ DNE}.$$

Watch the **scale** 比例 of a graph: a zoomed-out picture can hide important behavior near a point, so confirm with algebra when you can.



The open circle is the height the curve approaches (the limit); the filled dot is the actual value $f(2)$ – they need not agree.

Estimating Limit Values from Tables

When you have data or a formula but no picture, a **table 表格** of values estimates a limit numerically. Choose inputs that creep toward c from both sides and watch the outputs.

For example, to estimate $\lim_{x\to 2} \dfrac{x^2-4}{x-2}$ (which is $\frac{0}{0}$ at $x=2$):

x	1.9	1.99	1.999	$\to 2 \leftarrow$	2.001	2.01	2.1
$f(x)$	3.9	3.99	3.999	?	4.001	4.01	4.1

Both sides march toward 4, so we estimate the limit is 4. A table only *suggests* a value –it is a numerical estimate, not a proof.

Determining Limits Using Algebraic Properties of Limits

Most limits are found **analytically** using **limit theorems 极限定理**. If $\lim_{x\to c} f(x)$ and $\lim_{x\to c} g(x)$ both exist, the limit of a combination is the same combination of the limits:

- **Sum / difference:** $\lim\limits_{x\to c} \big[f(x) \pm g(x)\big] = \lim\limits_{x\to c} f(x) \pm \lim\limits_{x\to c} g(x)$
- **Product:** $\lim\limits_{x\to c} \big[f(x)\,g(x)\big] = \lim\limits_{x\to c} f(x) \cdot \lim\limits_{x\to c} g(x)$
- **Quotient:** $\lim\limits_{x\to c} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)}$, provided the bottom limit is not 0.
- **Composite 复合函数:** if g is continuous at $\lim_{x\to c} f(x)$, then $\lim\limits_{x\to c} g\big(f(x)\big) = g\Big(\lim\limits_{x\to c} f(x)\Big)$.

The practical rule: for a function built from polynomials, roots, and the like, first try **direct substitution 直接代入**–put $x=c$ in. If you get a real number, that is the limit. One-sided limits obey the same theorems, read from one direction only.

Determining Limits Using Algebraic Manipulation

Direct substitution sometimes gives the **indeterminate form 未定式** $\frac{0}{0}$. This does **not** mean the limit fails –it means you must rewrite the function into an **equivalent form** 等价形式 that removes the trouble, then substitute. Three standard moves:

- **Factor and cancel 因式分解并约分** a **rational function 有理函数**. Example: $\lim_{x\to 2}\frac{x^2-4}{x-2}=\lim_{x\to 2}\frac{(x-2)(x+2)}{x-2}=\lim_{x\to 2}(x+2)=4$.
- Multiply by the **conjugate 共轭** to simplify a **radical 根式**. Example: $\lim_{x\to 0}\frac{\sqrt{x+1}-1}{x}=\lim_{x\to 0}\frac{x}{x\left(\sqrt{x+1}+1\right)}=\frac{1}{2}$.
- **Use alternate forms** of trigonometric functions (identities) to simplify.

The cancelled factor is why the original graph had a hole: the two functions agree everywhere except at $x=c$, so they share the same limit there.

Selecting Procedures for Determining Limits

This is a skill topic, not new content: choose the right tool for the limit in front of you.

1. **Try direct substitution first.** A real answer means you are done.
2. Getting $\frac{0}{0}$? **Rewrite** –factor and cancel, or use the conjugate, or a trig identity – then substitute.
3. A **non-zero number over** 0 (like $\frac{5}{0}$)? The limit is infinite or DNE –check the sign from each side (see vertical asymptotes below).
4. As $x\to\pm\infty$? Compare the **dominant 主导** terms (see limits at infinity).
5. Trapped between two functions? The **squeeze theorem** may apply.

Determining Limits Using the Squeeze Theorem

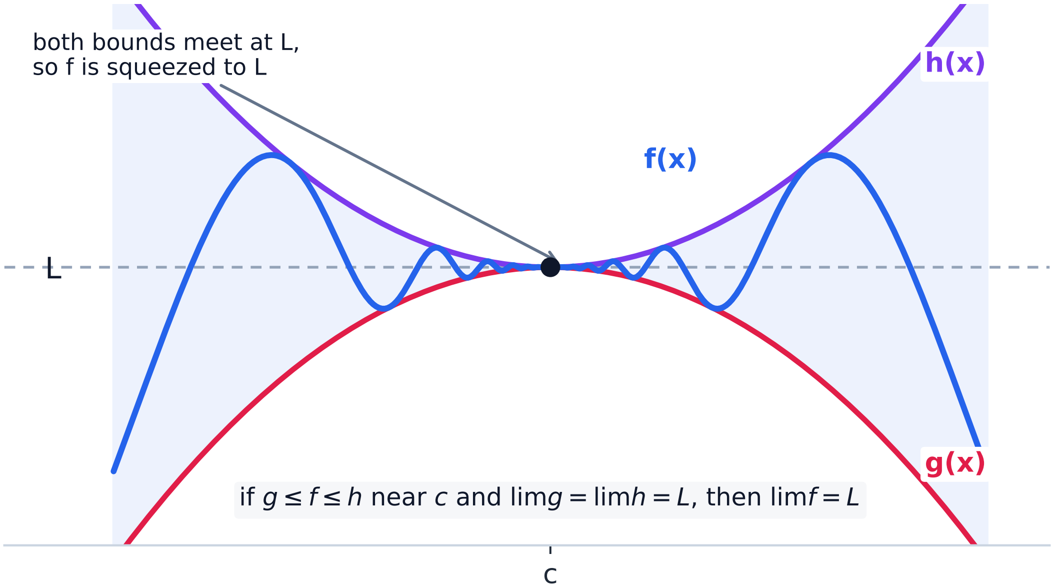
The **squeeze theorem 夹逼定理** (also called the sandwich theorem) finds a limit by trapping the function between two others. If $g(x)\le f(x)\le h(x)$ near c, and

$$\lim_{x\to c} g(x)=\lim_{x\to c} h(x)=L,$$

then f is squeezed to the same place: $\lim_{x\to c} f(x)=L$.

The two famous results proved this way, both used throughout calculus, are:

$$\lim_{x\to 0}\frac{\sin x}{x}=1 \qquad \text{and} \qquad \lim_{x\to 0}\frac{1-\cos x}{x}=0.$$



The Squeeze Theorem traps x squared sin(1/x) between minus x squared and x squared

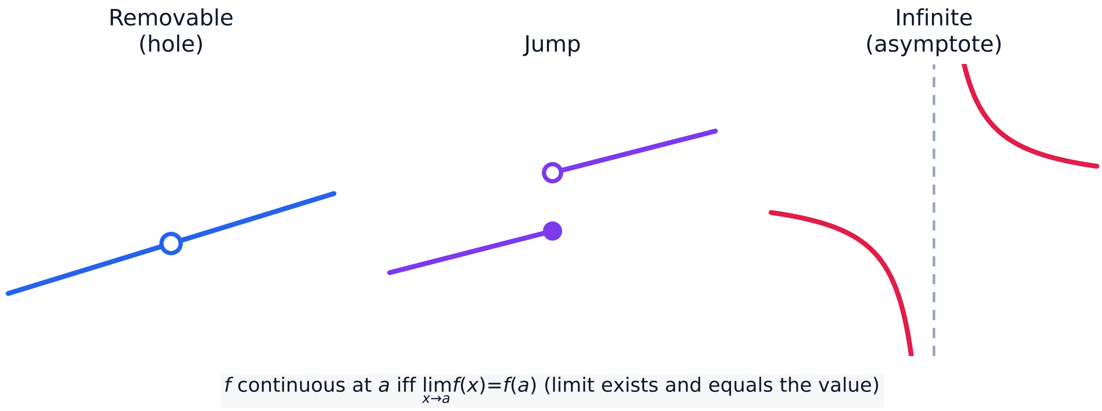
Connecting Multiple Representations of Limits

Another skill topic: the same limit lives in a graph, a table, and an algebraic form, and you should be able to translate between them. A graph shows the shape and any holes or jumps; a table gives numerical evidence; algebra gives an exact value and a reason. Strong answers use one representation to *confirm* another.

Exploring Types of Discontinuities

A function is **discontinuous** 间断 at c when its graph "breaks" there. There are three types:

- **Removable discontinuity** 可去间断–a single hole. The two-sided limit exists, but the point is missing or misplaced.
- **Jump discontinuity** 跳跃间断–the two one-sided limits exist but disagree, so the curve jumps.
- **Infinite discontinuity** 无穷间断–the function blows up to $\pm\infty$ at a **vertical asymptote** 垂直渐近线.



The three types of discontinuity: removable, jump, and infinite

Defining Continuity at a Point

Continuity is defined by a three-part test. A function f is **continuous** 连续 at $x = c$ exactly when all three hold:

$$\boxed{f(c) \text{ exists} \quad \text{and} \quad \lim_{x\to c} f(x) \text{ exists} \quad \text{and} \quad \lim_{x\to c} f(x) = f(c)}$$

In words: the point is there, the limit is there, and the two agree. If any one fails, f is discontinuous at c. This test is the backbone of nearly every continuity question, so learn it as a checklist.

Confirming Continuity over an Interval

A function is **continuous on an interval** 在区间上连续 if it is continuous at *every* point of that interval. You rarely check point by point, because whole families are continuous on their domains:

> **Polynomial, rational, power, exponential 指数, logarithmic 对数, and trigonometric 三角 functions are continuous at every point of their domains.**

So a rational function is continuous everywhere *except* where its denominator is zero; $\ln x$ is continuous for $x > 0$; and so on. Knowing this lets you declare continuity quickly and correctly.

Removing Discontinuities

If the **limit exists** at a hole, the discontinuity is **removable**: redefine the function at that one point to equal the limit, and the graph is repaired. Formally, set the missing value to $\lim_{x\to c} f(x)$.

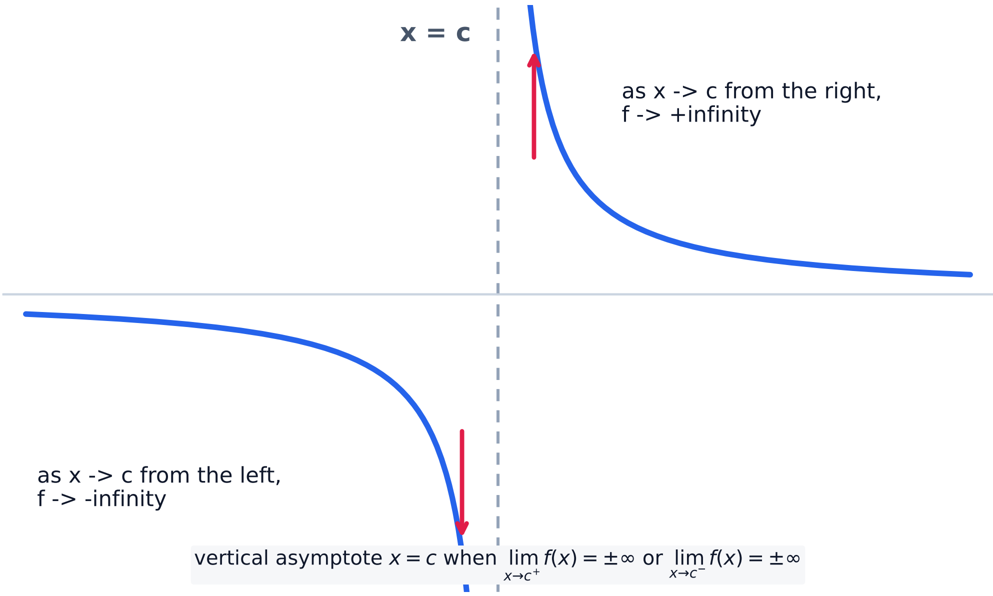
For a **piecewise-defined function** 分段函数, continuity at a boundary $x = c$ needs the two pieces to *meet*: the left piece's value, the right piece's value, and $f(c)$ must all be equal. This is a common exam setup –you solve for a **parameter** 参数 (an unknown constant) that makes the pieces match:

$$\lim_{x\to c^-} f(x) = \lim_{x\to c^+} f(x) = f(c).$$

Connecting Infinite Limits and Vertical Asymptotes

The idea of a limit extends to **infinite limits** 无穷极限. When a function grows without bound near $x = c$, we write $\lim_{x \to c} f(x) = \pm\infty$. This describes a **vertical asymptote** at $x = c$: the graph hugs the vertical line $x = c$ and shoots off toward $\pm\infty$.

This happens where a **non-zero number is divided by something approaching** 0, such as at a zero of a denominator that does not cancel. Always check each side separately –the two sides can shoot **opposite** ways (one to $+\infty$, one to $-\infty$).



Near a vertical asymptote the graph hugs the line $x = c$, and the two sides can shoot to opposite infinities.

Connecting Limits at Infinity and Horizontal Asymptotes

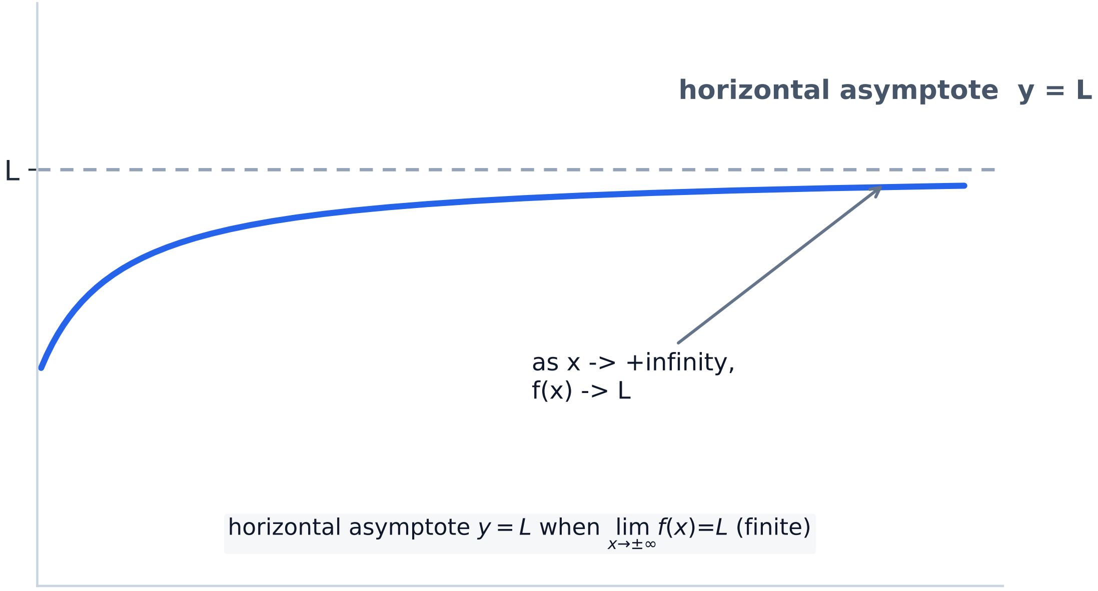
We can also let the **input** grow: **limits at infinity** 无穷远处的极限 describe the **end behavior** 末端行为 of a function as $x \to \pm\infty$. If the outputs settle toward a finite value L, then $y = L$ is a **horizontal asymptote** 水平渐近线.

For a rational function, compare the **degrees** 次数 of the top and bottom:

- top degree $<$ bottom degree $\Rightarrow$ limit is 0 (asymptote $y = 0$);

- top degree = bottom degree ⇒ limit is the **ratio of the leading coefficients** 首项系数之比;
- top degree > bottom degree ⇒ the function is unbounded (no horizontal asymptote).

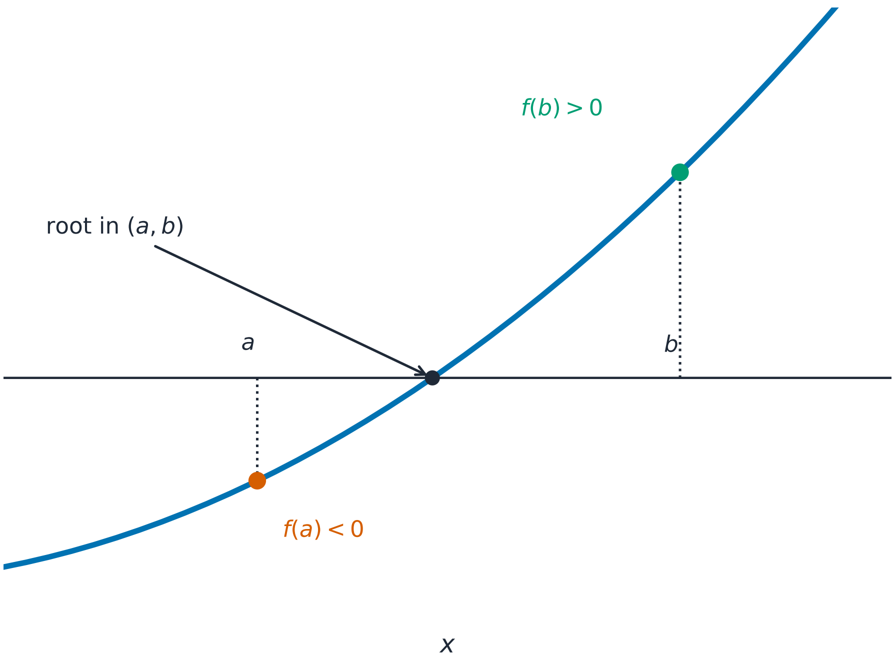
More generally, we compare the **relative magnitudes** 相对大小 (relative growth rates) of functions: far out, an exponential beats any polynomial, and a polynomial beats any logarithm. On the exam, "as $t \to \infty$, which quantity is larger/where does the rate settle?" is answered with a limit at infinity.



A limit at infinity produces a horizontal asymptote

Working with the Intermediate Value Theorem (IVT)

The **Intermediate Value Theorem** 介值定理 is an **existence theorem** 存在性定理–it guarantees a value exists without telling you where:



Opposite signs of f(a) and f(b) trap a root between a and b

> If f is **continuous** on the closed interval $[a, b]$, and d is any number **between** $f(a)$ and $f(b)$, then there is **at least one** number c in (a, b) with $f(c) = d$.

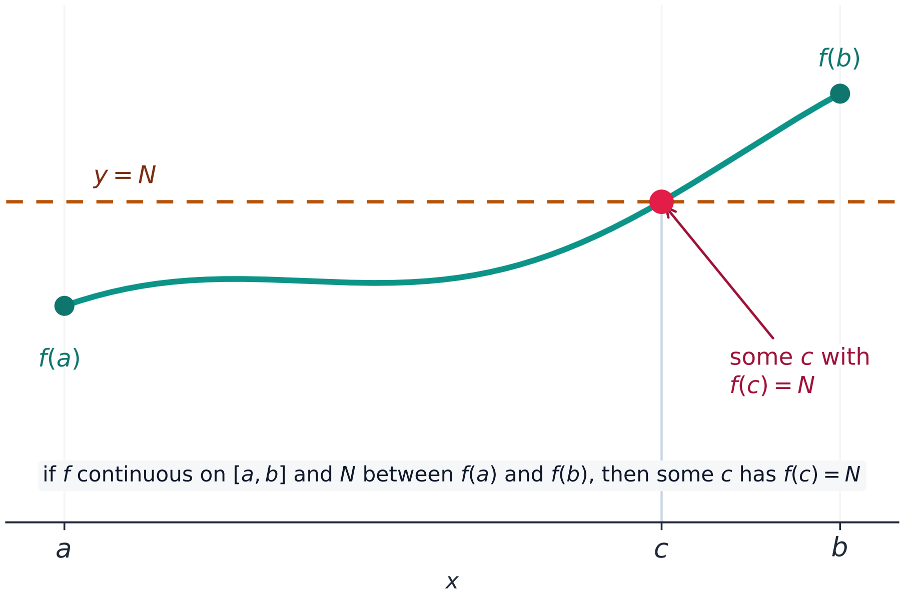
An unbroken curve cannot skip a height between its endpoints –it must pass through every one.

Exam skill –how to justify with the IVT. These questions appear almost every year (for example, "Must there be a value c with $R(c) = 155$?" or "Is there a time when $r'(t) = -6$?"). A full-credit justification has three moves:

1. **State continuity.** Say the function is continuous on $[a, b]$ (often because it is differentiable, or given continuous).
2. **Show d is trapped.** Compute the two endpoint values and show the target d lies **between** them, e.g. $f(a) < d < f(b)$.
3. **Conclude by name.** "By the Intermediate Value Theorem, there is a c in (a, b) with $f(c) = d$."

Skipping the continuity statement, or not showing d is between the endpoints, loses the point –the theorem *requires* both conditions.

Worked example. Evaluate $\lim_{x\to\infty} \dfrac{3x^2-5}{2x^2+x}$. Divide top and bottom by the highest power, x^2: $\dfrac{3-5/x^2}{2+1/x} \to \dfrac{3-0}{2+0} = \dfrac{3}{2}$. Because the limit is a finite number, the line $y = \frac{3}{2}$ is a **horizontal asymptote** of the graph.



A continuous curve from $(a, f(a))$ to $(b, f(b))$ must cross every height d in between at least once.

Exam tips

- A limit describes what $f(x)$ **approaches**, which need not equal $f(a)$ —the two-sided limit exists only if both sides agree.
- Try direct substitution first; for a $\frac{0}{0}$ form, **factor and cancel** or rationalise before substituting.
- A function is **continuous** at a when the limit exists, $f(a)$ is defined, and they are equal.
- Use the **Intermediate Value Theorem** to guarantee a root: a continuous function that changes sign on $[a, b]$ takes every value between.
- Read horizontal asymptotes from end behaviour (limits at $\pm\infty$) and vertical asymptotes where the denominator (not the numerator) is zero.

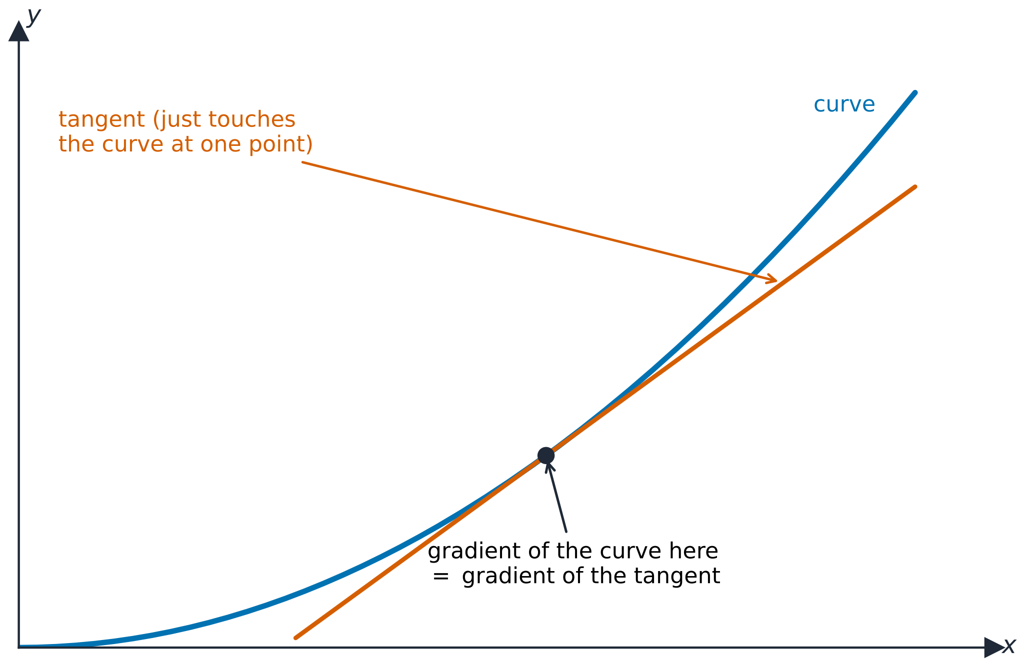
Differentiation: Definition and Fundamental Properties

AP Calculus BC

Average and Instantaneous Rates of Change at a Point

A roller coaster on a lift hill: the derivative measures instantaneous rate of change

Unit 1 built the limit. Unit 2 uses it to define the **derivative** 导数–the exact rate of change at a point.



The instantaneous rate of change is the gradient of the tangent at a point

Over an interval, the average rate of change is a **difference quotient** 差商. Two equivalent forms appear:

$$\frac{f(a+h)-f(a)}{h} \qquad \text{and} \qquad \frac{f(x)-f(a)}{x-a}.$$

The first uses a step of size h from a; the second uses two points x and a. Both compute $\dfrac{\text{change in output}}{\text{change in input}}$ over the interval.

The **instantaneous** 瞬时 rate of change at $x = a$ is what the difference quotient approaches as the interval shrinks to zero. This limit *is* the derivative at a, written $f'(a)$:

$$f'(a) = \lim_{h \to 0} \frac{f(a+h)-f(a)}{h} = \lim_{x \to a} \frac{f(x)-f(a)}{x-a},$$

provided the limit exists.

Defining the Derivative and Reading Its Notation

Let the point a vary and the derivative becomes a **new function**:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h}.$$

This is the **definition of the derivative** (sometimes called differentiating "by first principles" 用定义求导). Its value at each x is the instantaneous rate of change there.

Common **notations** 记号 for the derivative of $y = f(x)$ are:

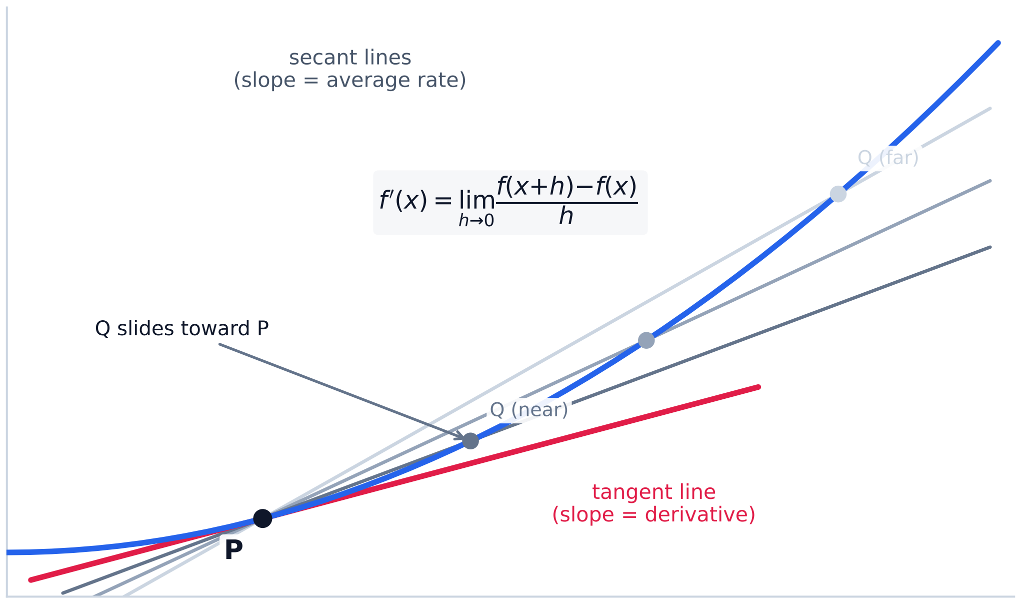
$$\frac{dy}{dx}, \qquad f'(x), \qquad y'.$$

The derivative can be represented graphically, numerically, analytically, and verbally –be ready to move between them.

Geometric meaning. The derivative at a point is the **slope** 斜率 of the **tangent line** 切线 to the graph there. So the tangent line at $x = a$ passes through $\big(a, f(a)\big)$ with slope $f'(a)$:

$$y - f(a) = f'(a)\,(x - a).$$

Writing this line is a routine exam task, so keep the point-slope form ready.



Secant slopes approach the tangent slope: the derivative is the limit of average rates

Estimating a Derivative at a Point

You do not always have a formula. When a function is given by a **table** 表格 or a graph, estimate the derivative $f'(a)$ with a difference quotient over a **small interval around** a. A table with values on both sides of a gives the best estimate:

$$f'(a) \approx \frac{f(b) - f(c)}{b - c}, \qquad \text{where } c < a < b \text{ are the closest table inputs.}$$

Technology (a calculator) can also estimate a derivative at a point.

Exam skill (appears almost every year). Questions such as ”Approximate $M'(7.5)$ using the average rate of change of M over the interval $5 \le t \le 10$” ask for exactly this difference quotient. Show the setup:

$$M'(7.5) \approx \frac{M(10) - M(5)}{10 - 5}.$$

Full credit needs the numbers plugged in **and** the correct **units** 单位 (output units per input unit), since these come from real-world models.

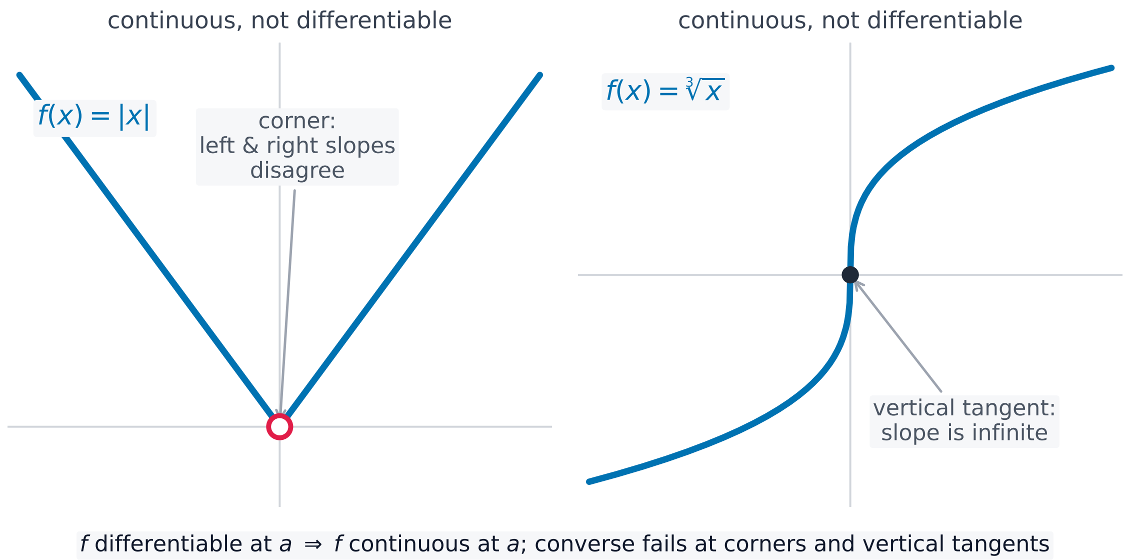
Differentiability and Continuity: When a Derivative Exists

Differentiability is stronger than continuity. The key relationship:

> **If f is differentiable 可导 at a point, then f is continuous 连续 there.**

So differentiability *implies* continuity. The reverse is false: a continuous function can fail to be differentiable. Two ways this happens:

- A **corner** 尖点: the left and right difference-quotient limits disagree, as with $f(x) = |x|$ at $x = 0$.
- A **vertical tangent** 垂直切线: the slope is infinite (no real number), as with $f(x) = \sqrt[3]{x}$ at $x = 0$.



Two ways a continuous function is not differentiable: a corner and a vertical tangent

Also, a point outside the domain of f cannot be in the domain of f'. Use the contrapositive on the exam: **if f is not continuous at a, then f is not differentiable at a.**

The Power Rule

From here we use **rules** instead of the limit definition each time. The **power rule** 幂法则 handles any power of x:

$$\frac{d}{dx}\,x^r = r\,x^{\,r-1} \qquad \text{for any real } r.$$

It works for whole-number powers, negative powers ($\frac{1}{x} = x^{-1}$), and roots ($\sqrt{x} = x^{1/2}$) –rewrite as a power first, then apply the rule.

Constant, Sum, Difference, and Constant Multiple Rules

These rules let you differentiate term by term:

- **Constant:** $\dfrac{d}{dx}\,k = 0$ (a constant does not change).
- **Constant multiple** 常数倍: $\dfrac{d}{dx}\big[k\,f(x)\big] = k\,f'(x)$.
- **Sum / difference:** $\dfrac{d}{dx}\big[f(x) \pm g(x)\big] = f'(x) \pm g'(x)$.

Combined with the power rule, they differentiate any **polynomial** 多项式 term by term. Example:

$$\frac{d}{dx}\big(4x^3 - 5x + 7\big) = 12x^2 - 5.$$

Derivatives of cos x, sin x, e^x, and ln x

Learn these four building-block derivatives by heart:

$$\frac{d}{dx}\sin x = \cos x, \qquad \frac{d}{dx}\cos x = -\sin x,$$

$$\frac{d}{dx}e^x = e^x, \qquad \frac{d}{dx}\ln x = \frac{1}{x} \quad (x > 0).$$

Note the minus sign on the derivative of cosine, and that e^x is its own derivative.

A limit that is really a derivative (LIM-3.A.1). Sometimes a limit is secretly the definition of a known derivative. If you recognize

$$\lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$$

for a function f whose derivative you know, just evaluate $f'(a)$. For example, $\displaystyle\lim_{h\to 0} \frac{\sin\left(\frac{\pi}{2}+h\right)-1}{h} = \frac{d}{dx}\sin x\bigg|_{x=\pi/2} = \cos\frac{\pi}{2} = 0.$

The Product Rule

A product of two functions is **not** differentiated by multiplying the derivatives. Use the **product rule** 乘积法则:

$$\frac{d}{dx}[u\,v] = u'v + uv'.$$

"Derivative of the first times the second, plus the first times the derivative of the second." Example:

$$\frac{d}{dx}\left(x^2 e^x\right) = 2x\,e^x + x^2 e^x.$$

Exam questions often build a new function from given pieces, e.g. $k'(x) = \big(f(x)\big)^2 g(x)$, and ask you to combine rules while reading values from a table.

The Quotient Rule

For a quotient, use the **quotient rule** 商法则:

$$\frac{d}{dx}\left[\frac{u}{v}\right] = \frac{u'v-uv'}{v^2}.$$

"Bottom times derivative of top, minus top times derivative of bottom, all over bottom squared." The order matters because of the minus sign, so write the numerator carefully. Example:

$$\frac{d}{dx}\left(\frac{x}{\cos x}\right) = \frac{1\cdot\cos x - x\cdot(-\sin x)}{\cos^2 x} = \frac{\cos x + x\sin x}{\cos^2 x}.$$

Derivatives of Tangent, Cotangent, Secant, and Cosecant

The remaining trigonometric derivatives are not memorized separately –you rewrite them with **identities** 恒等式 and apply the quotient (or product) rule. For instance, $\tan x = \dfrac{\sin x}{\cos x}$, so the quotient rule gives

$$\frac{d}{dx}\tan x = \frac{\cos x \cos x - \sin x(-\sin x)}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x.$$

The same method (writing $\cot x = \frac{\cos x}{\sin x}$, $\sec x = \frac{1}{\cos x}$, $\csc x = \frac{1}{\sin x}$) gives $-\csc^2 x$, $\sec x \tan x$, and $-\csc x \cot x$.

Higher-order derivatives. Differentiating f' again gives the **second derivative** 二阶导数 $f''(x)$ (or $\frac{d^2y}{dx^2}$) –the rate of change of the rate of change. An exam part like "Find $k''(3)$" just means differentiate twice, then substitute. You can also estimate a second derivative from a table by applying the average-rate-of-change method to the f' values.

Worked example. Differentiate $g(x) = \dfrac{\sin x}{x}$ with the **quotient rule** $\left(\frac{u}{v}\right)' = \dfrac{u'v - uv'}{v^2}$: take $u = \sin x$, $v = x$, giving $g'(x) = \dfrac{x\cos x - \sin x}{x^2}$. Keep the order $u'v - uv'$ in the numerator —swapping the terms flips the sign and loses the mark.

Exam tips

- The derivative is the **slope of the tangent** —the limit of the secant slope $\frac{f(x+h)-f(x)}{h}$ as $h \to 0$.
- Memorise the rules: power, product, quotient, and the derivatives of sin, cos, e^x, and $\ln x$.
- **Differentiability implies continuity**, but not the reverse (a corner or cusp is continuous yet not differentiable).
- Distinguish average rate of change (secant slope over an interval) from instantaneous rate (the derivative at a point).
- Give a tangent-line equation as $y - f(a) = f'(a)(x - a)$.

Differentiation: Composite, Implicit, and Inverse Functions

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The Chain Rule

A mountain path: composite functions nest rates of change (chain rule)

Unit 2 differentiated single functions. Unit 3 differentiates functions built *inside* other functions. The **chain rule** 链式法则 differentiates a **composite function** 复合函数 $f\big(g(x)\big)$:

$$\frac{d}{dx}\,f\big(g(x)\big) = f'\big(g(x)\big)\cdot g'(x).$$

"Derivative of the outer function (leaving the inside alone), times the derivative of the inside." The inner derivative $g'(x)$ is the piece students forget, so always ask "what is the inside, and what is its derivative?" Example:

$$\frac{d}{dx}\sin(x^2) = \cos(x^2)\cdot 2x.$$

In Leibniz notation, with $y = f(u)$ and $u = g(x)$, the rule reads $\dfrac{dy}{dx} = \dfrac{dy}{du}\cdot\dfrac{du}{dx}$ –the intermediate du appears to "cancel." Exam questions often give a table for f, g, f', g' and ask for $h'(a)$ where $h(x) = f\big(g(x)\big)$; evaluate $f'\big(g(a)\big)\cdot g'(a)$ by reading values.

Worked example. Differentiate $h(x) = (2x^2+1)^5$. The outer function is "(something)^5" and the inner is $2x^2+1$:

$$h'(x) = 5(2x^2+1)^4 \cdot 4x = 20x(2x^2+1)^4.$$

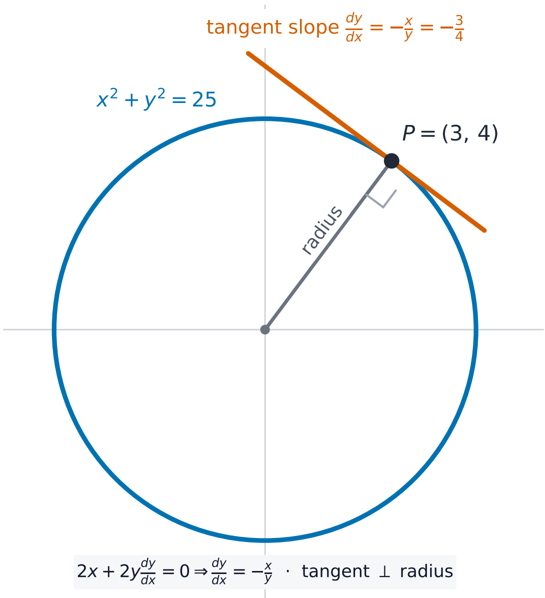
Implicit Differentiation

Some curves are defined **implicitly** –by an equation in x and y that is not solved for y, such as $x^2+y^2=25$. **Implicit differentiation** 隐函数求导 finds $\frac{dy}{dx}$ without solving for y first. It is just the chain rule, treating y as a function of x.

The method: differentiate **both sides** with respect to x; every time you differentiate a y-term, multiply by $\frac{dy}{dx}$ (the chain rule); then solve algebraically for $\frac{dy}{dx}$. For $x^2+y^2=25$:

$$2x + 2y\frac{dy}{dx} = 0 \implies \frac{dy}{dx} = -\frac{x}{y}.$$

Worked example. Find the tangent to $x^2+y^2=25$ at $(3,4)$. Here $\frac{dy}{dx} = -\frac{3}{4}$, so the tangent line is $y-4 = -\frac{3}{4}(x-3)$ –perpendicular to the radius, as geometry predicts.



Implicit differentiation gives the tangent to a circle, perpendicular to the radius

Exam skill –"Show that $\frac{dy}{dx} = \ldots$". This exact prompt appears most years (e.g. "Show that $\frac{dy}{dx} = \frac{2y}{y^2-2x}$"). Because the target is given, you must show **every**

algebra step cleanly: differentiate both sides, use the product/chain rules on mixed xy terms, collect all $\dfrac{dy}{dx}$ terms on one side, factor, and divide. A correct final line that skips the algebra earns little. Follow-up parts then ask for a tangent line, or where the tangent is horizontal ($\frac{dy}{dx} = 0$, so the numerator is 0) or vertical (the denominator is 0).

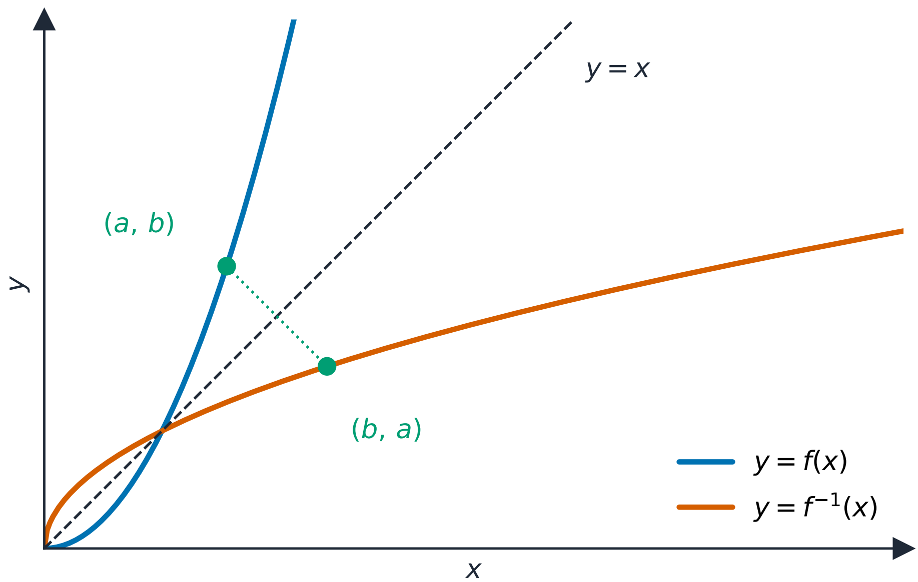
Differentiating Inverse Functions

If g is the **inverse function** 反函数 of f (so $f(g(x)) = x$), the chain rule links their derivatives:

$$g'(x) = \frac{1}{f'\big(g(x)\big)}, \qquad \text{provided } f'\big(g(x)\big) \neq 0.$$

In words: the derivative of the inverse at a point is the **reciprocal** 倒数 of the derivative of the original function at the *matching* point. A common exam setup gives a table and a point (a, b) on f (so (b, a) is on g), then asks for $g'(b) = \dfrac{1}{f'(a)}$.

Worked example. If $f(2) = 5$ and $f'(2) = 3$, and g is the inverse of f, then $(5, 2)$ lies on g and $g'(5) = \dfrac{1}{f'(2)} = \dfrac{1}{3}$.



The inverse function is the mirror image of the function in the line $y = x$

Differentiating Inverse Trigonometric Functions

The same idea gives the derivatives of the **inverse trigonometric functions** 反三角函数. The three you should know:

$$\frac{d}{dx}\arcsin x = \frac{1}{\sqrt{1-x^2}}, \quad \frac{d}{dx}\arctan x = \frac{1}{1+x^2}, \quad \frac{d}{dx}\operatorname{arcsec} x = \frac{1}{|x|\sqrt{x^2-1}}.$$

Combine these with the chain rule when the input is itself a function, e.g. $\frac{d}{dx}\arctan(3x) = \frac{3}{1+9x^2}$.

Selecting Procedures for Calculating Derivatives

A skill topic: real derivatives mix several rules, so read the structure of the expression first, from the outside in.

- Is it a **sum**? Differentiate term by term.
- A **product** or **quotient**? Apply that rule, and expect to use the chain rule inside.
- A **composite** (something inside something)? Chain rule.
- Given **implicitly**? Implicit differentiation.

Name the outermost operation, apply its rule, and recurse inward. Neatness prevents the sign and bookkeeping errors that cost marks.

Calculating Higher-Order Derivatives

Differentiating f' produces the **second derivative** 二阶导数 f''; repeating gives higher-order derivatives. The notations:

$$f''(x) = \frac{d^2y}{dx^2} = y'', \qquad \text{and in general} \qquad f^{(n)}(x) = \frac{d^ny}{dx^n}.$$

The second derivative measures how the *slope* is changing; it drives **concavity** 凹凸性 and acceleration in later units. To find f'' implicitly, differentiate the expression for $\frac{dy}{dx}$ again (with the quotient and chain rules), then substitute $\frac{dy}{dx}$ back in.

Exam tips

- Use the **chain rule** for composite functions —differentiate the outside, then multiply by the derivative of the inside (the most-forgotten factor).
- For **implicit** differentiation, differentiate both sides with respect to x and attach $\frac{dy}{dx}$ each time y is differentiated, then solve.

- Get the **second derivative** by differentiating twice (velocity $\to$ acceleration).
- Combine rules carefully in layered expressions (chain inside product, etc.).
- The inverse function's graph is the reflection in $y = x$; its slope is the reciprocal of the original's at the matching point.

Contextual Applications of Differentiation

AP Calculus BC

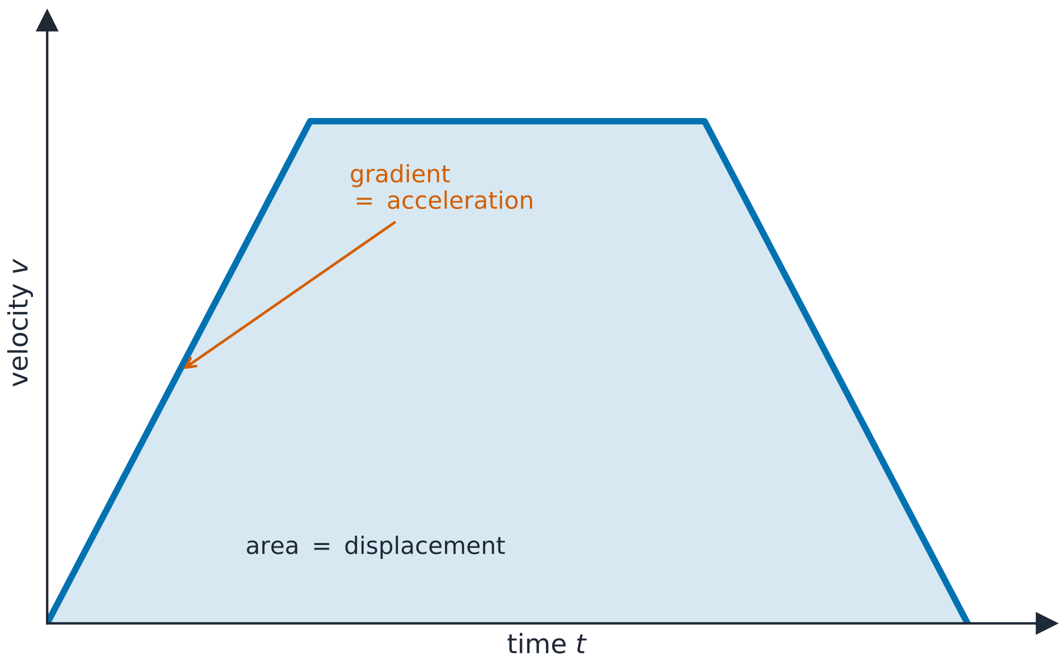
Interpreting the Meaning of the Derivative in Context

Once you can compute derivatives, you use them to describe the real world. The derivative $f'(x)$ **is** the instantaneous rate of change of f with respect to its input. Reading and reporting this rate correctly is a graded skill.

Units matter. The unit of $f'(x)$ is the unit of f divided by the unit of x. If $C(t)$ is a number of acres and t is in weeks, then $C'(t)$ is in acres per week. On the exam, "Using correct units, interpret the meaning of $g'(140)$" wants a full sentence: the *value*, the *quantity*, the *rate word* "per", and the *moment*. For example: "$g'(140) = 2.3$ means that at $x = 140$, the quantity is increasing at about 2.3 units per unit of x."

Straight-Line Motion: Position, Velocity, and Acceleration

For a particle moving on a line, three functions of time are linked by differentiation:



On a velocity-time graph, the area is displacement and the gradient is acceleration

- **position** 位置 $s(t)$;
- **velocity** 速度 $v(t) = s'(t)$ –signed; its sign gives direction;
- **acceleration** 加速度 $a(t) = v'(t) = s''(t)$.

Key readings (frequent exam parts):

- The particle is **at rest** 静止 when $v(t) = 0$.
- It moves **right/up** when $v(t) > 0$ and **left/down** when $v(t) < 0$; it **changes direction** where v changes sign.
- **Speed** 速率 is $|v(t)|$. Speed is *increasing* when v and a have the **same sign** (the particle is speeding up), and *decreasing* when they have **opposite signs**.

Distinguish carefully between velocity (has direction) and speed (does not) –the exam tests this exact difference.

Worked example. A particle moves with $s(t) = t^3 - 6t^2 + 9t$. Then $v(t) = 3(t-1)(t-3)$, so it is at rest at $t = 1$ and $t = 3$ and changes direction at each. At $t = 2$, $v = -3 < 0$ and $a(2) = 6(2) - 12 = 0$; just after, $a > 0$ while $v < 0$, so the particle is **slowing down** there.

Rates of Change in Applied Contexts Other Than Motion

The same derivative idea models any changing quantity: a draining tank, a spreading population, a cooling cup. Whenever a problem says ”the rate at which...”, it is describing a derivative. Read the units to know which quantity's rate you have, then interpret in context.

Introduction to Related Rates

A roller coaster: related rates connect how fast linked quantities change together

In a **related rates** 相关变化率 problem, several quantities change together over time, and you know some rates but want another. The engine is the chain rule: differentiate a relationship **with respect to time** t. Every variable becomes a function of t, so each derivative picks up a ” $/\,dt$” factor. Product and quotient rules may also be needed.

A rising balloon: related rates link how fast radius, volume and height change together

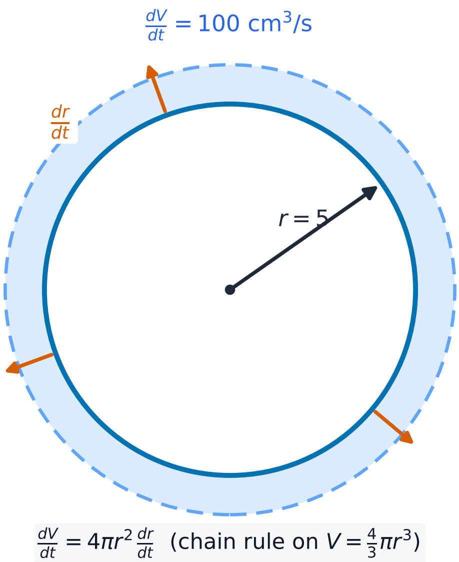
Solving Related Rates Problems

A reliable procedure –and a full-credit template on the exam:

1. **Name the variables** and write down the given rates and the unknown rate (e.g. „$\frac{dh}{dt} = -2$ cm/day, find $\frac{dV}{dt}$”).
2. **Write an equation** relating the quantities (often a geometric or volume formula).
3. **Differentiate both sides with respect to** t (chain rule) –*before* substituting numbers.
4. **Substitute** the known values at the instant of interest, and solve for the unknown rate.
5. State the answer with **units** and the correct **sign** (a decreasing quantity has a negative rate).

Substituting numbers too early is the classic error: differentiate the *general* relationship first, then plug in.

Worked example. A spherical balloon's volume grows at $\frac{dV}{dt} = 100$ cm^3/s. From $V = \frac{4}{3}\pi r^3$, differentiate first: $\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$. At $r = 5$, $100 = 4\pi(25)\frac{dr}{dt}$, so $\frac{dr}{dt} = \frac{1}{\pi} \approx$ 0.32 cm/s.



An inflating balloon links dV/dt and dr/dt through the chain rule

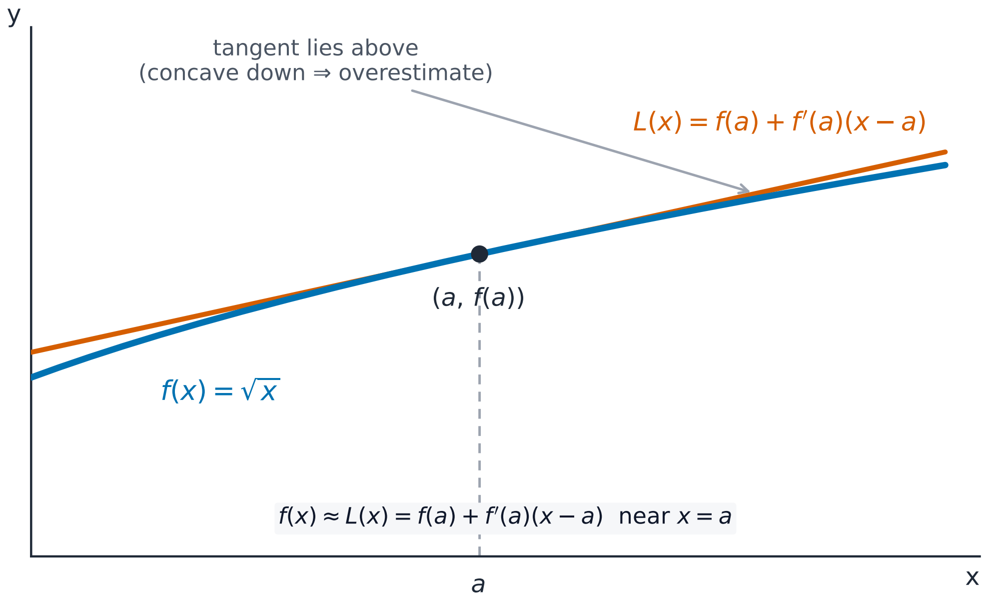
Approximating Values Using Local Linearity and Linearization

Near a point of tangency, a smooth curve looks like its tangent line –this is **local linearity** 局部线性. So the tangent line gives a **linear approximation** 线性近似 (linearization) of the function near that point:

$$f(x) \approx L(x) = f(a) + f'(a)(x - a).$$

Use it to estimate f at an x close to a.

Over- or underestimate? The answer follows from **concavity** 凹凸性. If the graph is **concave up** near a (it curves above its tangent), the tangent-line value is an **underestimate** 低估. If it is **concave down**, the tangent line lies above the curve, giving an **overestimate** 高估. Exam parts test this reasoning, so justify with the sign of f''.



The tangent line is a local linear approximation; concavity fixes over- or under-estimate

Using L'Hospital's Rule for Indeterminate Forms

When direct substitution in a quotient of limits gives the **indeterminate form 未定式** $\dfrac{0}{0}$ or $\dfrac{\infty}{\infty}$, you may use **L'Hospital's Rule 洛必达法则**:

$$\lim_{x\to c}\frac{f(x)}{g(x)}=\lim_{x\to c}\frac{f'(x)}{g'(x)},$$

provided the right-hand limit exists. **Differentiate the top and bottom separately** (this is *not* the quotient rule), then try the limit again. First confirm the form really is $\frac{0}{0}$ or $\frac{\infty}{\infty}$ –applying the rule to any other form is a mistake.

Worked example. $\displaystyle\lim_{x\to 0}\frac{\sin x}{x}$ gives $\frac{0}{0}$, so differentiate top and bottom: $\displaystyle\lim_{x\to 0}\frac{\cos x}{1}=1$. And $\displaystyle\lim_{x\to 0}\frac{e^{2x}-1}{x}$ is also $\frac{0}{0}$; it becomes $\displaystyle\lim_{x\to 0}\frac{2e^{2x}}{1}=2$.

Exam tips

- In motion problems: velocity is the derivative of position, acceleration the derivative of velocity; **speed increases when velocity and acceleration share a sign**.
- For **related rates**, differentiate the relating equation with respect to **time**, then substitute the given values last.
- Use the tangent line for a **linear approximation** near a known point; it is accurate only close by.
- Read the sign of a rate: positive means the quantities move together, negative means opposite.
- Always state units and interpret the answer in context.

Analytical Applications of Differentiation

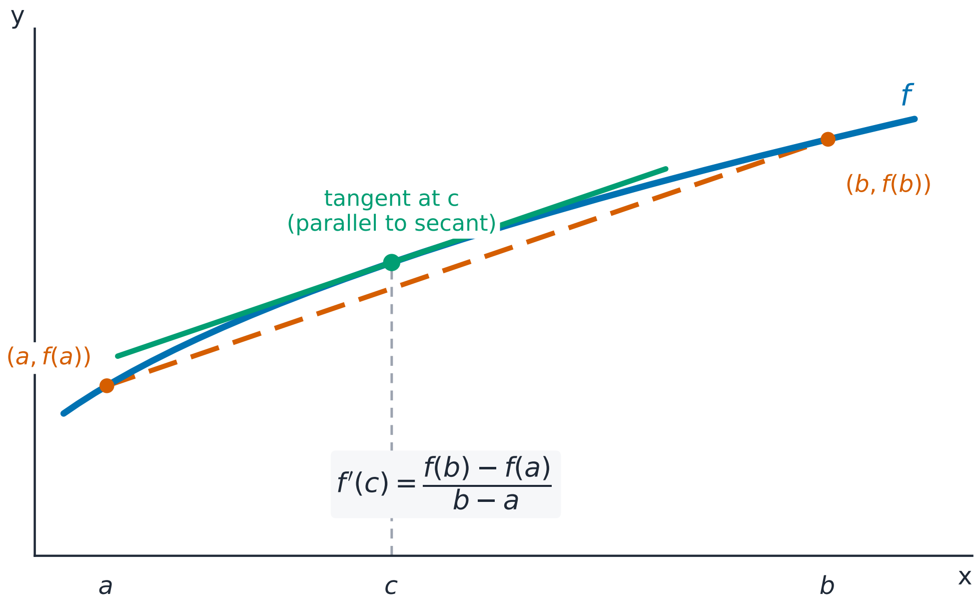
AP Calculus BC

Using the Mean Value Theorem

The **Mean Value Theorem** 中值定理 (MVT) links the average rate of change to an instantaneous one:

> If f is **continuous** on $[a, b]$ and **differentiable** on (a, b), then there is at least one point c in (a, b) where
>
> $$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

In words: somewhere inside the interval, the instantaneous rate equals the average rate. Geometrically, some tangent line is parallel to the line joining the endpoints.



The Mean Value Theorem: some tangent is parallel to the secant over the interval

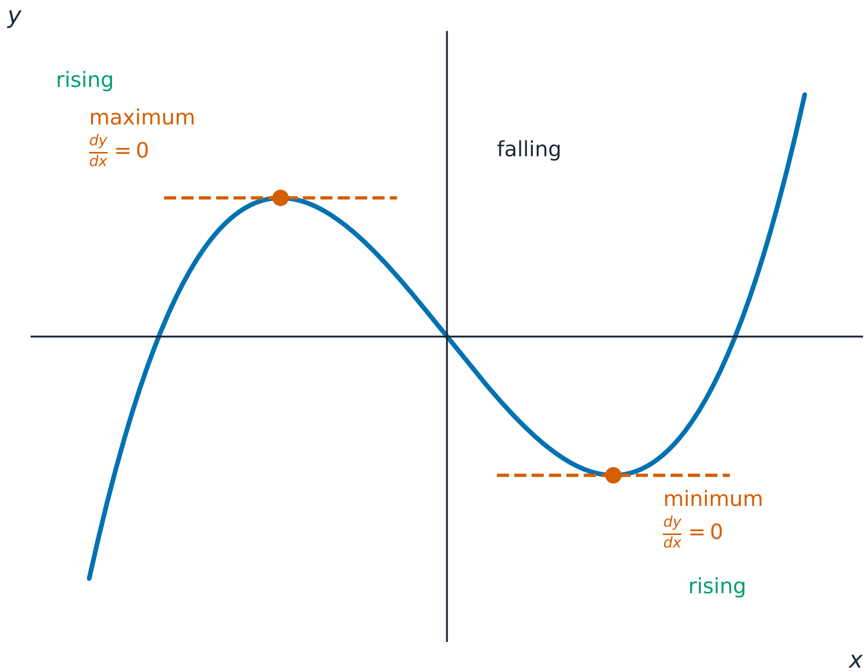
Exam skill. Like the IVT, the MVT is an existence theorem, and questions ask you to justify. Full credit needs: (1) state f is continuous on $[a, b]$ and differentiable on (a, b); (2) compute the average rate $\frac{f(b)-f(a)}{b-a}$; (3) conclude "by the MVT there is a c in (a, b) with $f'(c)$ equal to that value." Both hypotheses must be named.

Worked example. For $f(x) = x^2$ on $[1, 3]$ the average rate is $\dfrac{9-1}{2} = 4$; setting $f'(c) = 2c = 4$ gives $c = 2$, which lies in $(1, 3)$ –the guaranteed point.

Extreme Values, Global vs Local Extrema, and Critical Points

A mountain peak: extrema and critical points mark highest and lowest values

The **Extreme Value Theorem** 极值定理 (EVT) guarantees extremes exist: a function **continuous** on a closed interval $[a, b]$ attains **both** an absolute maximum and an absolute minimum on it.



At a maximum or minimum the derivative is zero

A **critical point** 临界点 is an interior point where $f'(x) = 0$ **or** $f'(x)$ does not exist. All **local (relative) extrema** 局部极值 occur at critical points –but **not every critical point is an extremum**. So critical points are the *candidates*; you must test each.

Where a Function Increases or Decreases

The **first derivative** tells you where f rises or falls:

- $f'(x) > 0$ on an interval $\Rightarrow f$ is **increasing** 递增 there;
- $f'(x) < 0 \Rightarrow f$ is **decreasing** 递减.

On the exam, "find the intervals where f is increasing" means: find the critical points, then test the sign of f' between them, and **justify with the sign of** f' (a stated reason, not just an interval).

The First Derivative Test for Local Extrema

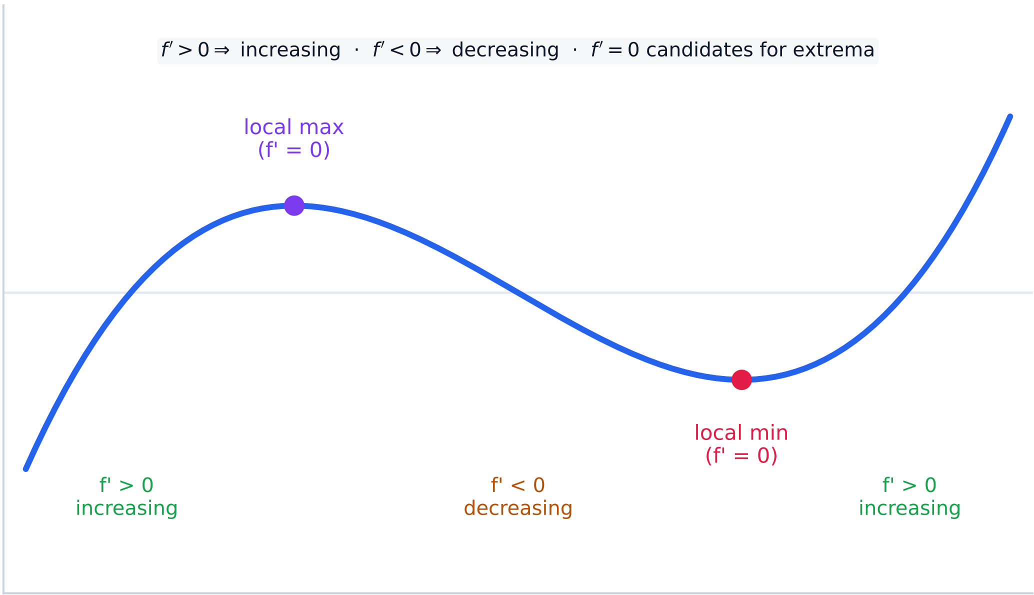
To classify a critical point $x = c$ as a local max, local min, or neither, check how f' changes sign there:

- f' changes $+$ **to** $-$ at $c \Rightarrow$ **local maximum** 极大值;

- f' changes $-$ **to** $+$ at $c \Rightarrow$ **local minimum 极小值**;
- f' does **not** change sign $\Rightarrow$ neither.

Always state the sign change as your justification.

Worked example. For $f(x) = x^3 - 3x^2$, $f'(x) = 3x(x-2)$ is zero at $x = 0, 2$. Signs give $+, -, +$, so $x = 0$ is a **local maximum** ($f = 0$) and $x = 2$ a **local minimum** ($f = -4$).



Where $f' > 0$ the graph rises and where $f' < 0$ it falls; a local max sits where f' turns $+$ to $-$, a local min where it turns $-$ to $+$.

The Candidates Test for Absolute Extrema

On a **closed interval**, absolute (global) extrema occur only at **critical points or endpoints**. The **candidates test**:

1. List all critical points in $[a, b]$ **and** the two endpoints.
2. Evaluate f at each candidate.
3. The largest output is the absolute maximum; the smallest is the absolute minimum.

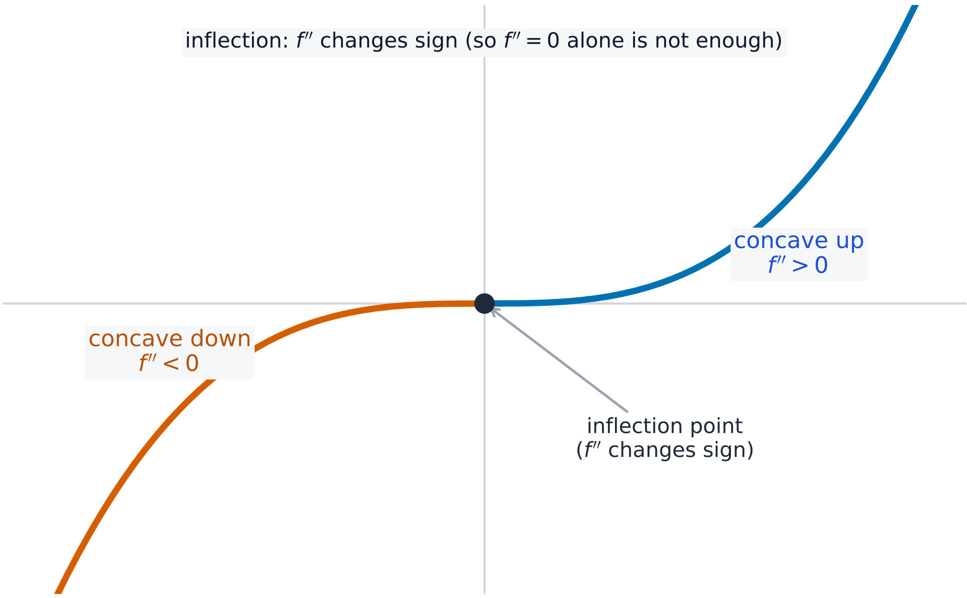
Show the table of values –the comparison is the argument.

Concavity and Points of Inflection

The **second derivative** describes bending:

- $f'' > 0 \Rightarrow f$ is **concave up** 上凹 (curving like a cup; f' is increasing);
- $f'' < 0 \Rightarrow f$ is **concave down** 下凹 (f' is decreasing).

A **point of inflection** 拐点 is where concavity **changes**, i.e. where f'' changes sign (not merely where $f'' = 0$). Report its x-coordinate and justify with the sign change of f''.



Concavity comes from the sign of the second derivative; it flips at an inflection point

The Second Derivative Test for Extrema

An alternative way to classify a critical point c where $f'(c) = 0$:

- $f''(c) > 0 \Rightarrow$ concave up $\Rightarrow$ **local minimum**;
- $f''(c) < 0 \Rightarrow$ concave down $\Rightarrow$ **local maximum**;
- $f''(c) = 0 \Rightarrow$ the test is **inconclusive** –fall back on the first derivative test.

Special case: if a continuous function has only **one** critical point on an interval and it is a local extremum, that point is also the **absolute** extremum there.

Sketching a Function and Its Derivative

Key features of f, f', and f'' mirror each other. To sketch or read graphs:

- f increasing $\Leftrightarrow$ f' above the axis; f has a local max $\Leftrightarrow$ f' crosses from $+$ to $-$.
- f concave up $\Leftrightarrow$ f' increasing $\Leftrightarrow$ f'' above the axis; f has an inflection point $\Leftrightarrow$ f' has a local extremum $\Leftrightarrow$ f'' crosses zero.

Connecting f, f', and f''

This is the skill of reading one graph to describe another. A very common exam setup gives the graph of f' and asks about f: where is f increasing (where $f' > 0$), where are f's extrema (where f' crosses zero, with a sign change), where is f concave up (where f' is increasing). Answer questions about f using the *height and slope* of the f' graph.

Introduction to Optimization Problems

Optimization 最优化 uses the derivative to find the largest or smallest value of a quantity on an interval. It is the candidates/derivative-test machinery applied to a real goal.

Fenced enclosures: optimization finds the dimensions that maximise area for a fixed fence length

Solving Optimization Problems

A dependable procedure:

1. Write the quantity to optimize as a function of **one** variable (use a constraint equation to eliminate extras).
2. State the interval of allowed inputs.
3. Find critical points ($f' = 0$ or undefined) and test them (first- or second-derivative test, or candidates test if the interval is closed).
4. Answer the question asked, **with units and interpretation** in context –the maximum *area*, the minimum *cost*, etc.

Worked example. With 100 m of fence for a rectangular pen against a wall (only three sides fenced), let the ends be x and the far side $y = 100 - 2x$. The area $A(x) = x(100 - 2x) = 100x - 2x^2$ has $A'(x) = 100 - 4x = 0$ at $x = 25$; since $A'' = -4 < 0$ this is the maximum, giving $y = 50$ and $A = 1250\text{ m}^2$.

Exploring Behaviors of Implicit Relations

All of this extends to **implicitly defined** relations. A critical point of an implicit relation is where $\dfrac{dy}{dx} = 0$ (horizontal tangent) or is undefined (vertical tangent). Because $\dfrac{dy}{dx}$ is usually a relation in x **and** y, and the second derivative involves x, y, and $\dfrac{dy}{dx}$, substitute your first-derivative expression back in when finding $\dfrac{d^2y}{dx^2}$, then reason about concavity from its sign.

Exam tips

- $f' > 0$ means increasing, $f' < 0$ decreasing; candidates for **extrema** are where $f' = 0$ or is undefined.
- Classify a critical point with the **first-derivative sign change** or the **second-derivative test** ($f'' > 0$ minimum, $f'' < 0$ maximum).
- $f'' > 0$ is concave up, $f'' < 0$ concave down; a **point of inflection** is where concavity changes (f'' changes sign).
- For an **absolute** extremum on a closed interval, also check the endpoints.
- Justify every conclusion by citing the sign of f' or f'' —the exam demands the reasoning, not just the answer.

Integration and Accumulation of Change

AP Calculus BC

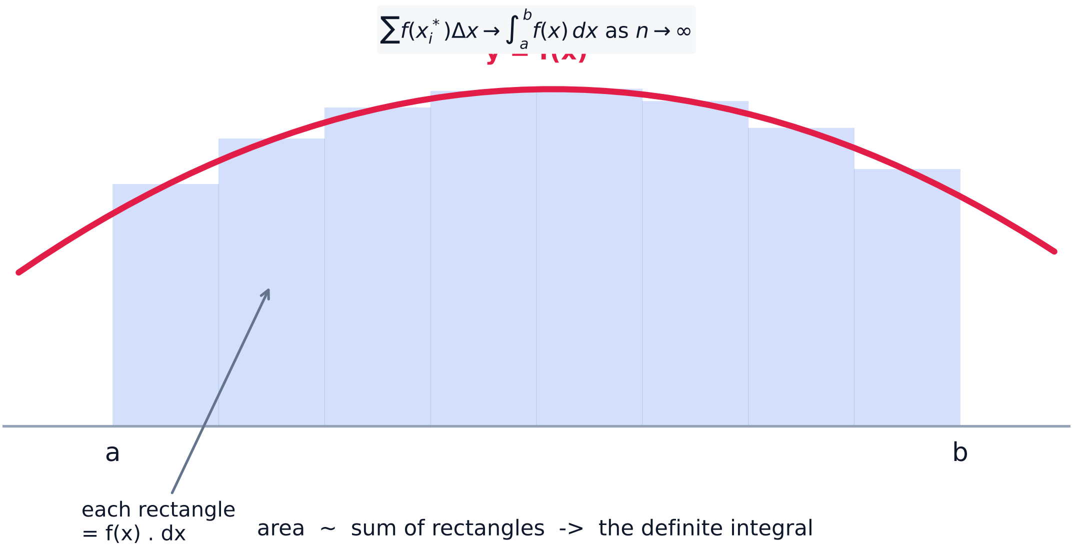
Exploring Accumulations of Change

A water tank: integration accumulates change —total volume from a rate of flow

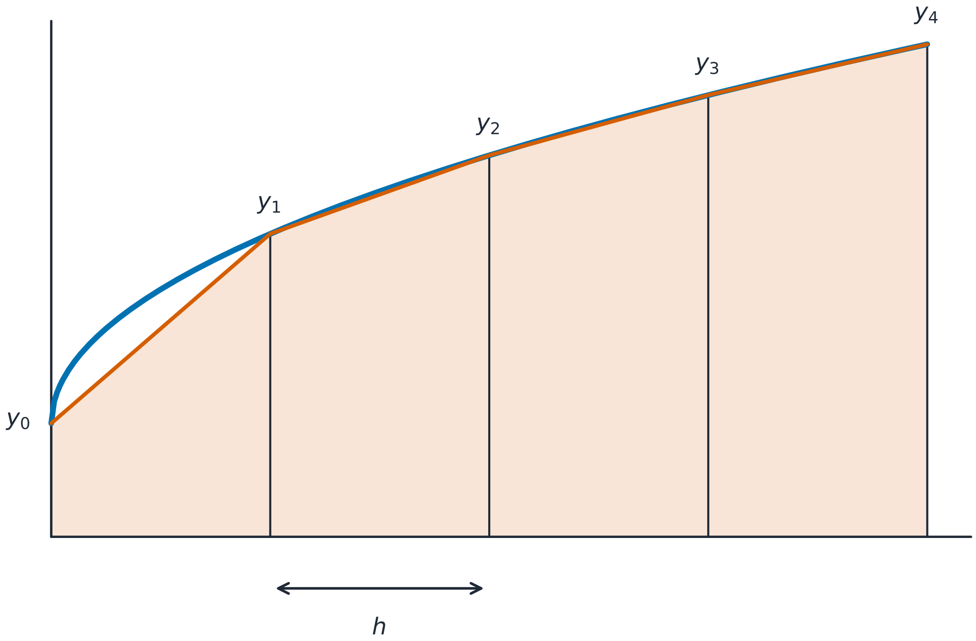
Where a derivative measures a **rate**, an **integral** 积分 measures an **accumulation** 累积–a total built up from a rate. If a rate of change acts over an interval, the **area** between its graph and the axis gives the net accumulated change. This ”area = total change” idea is the foundation of integral calculus.

Approximating Areas with Riemann Sums

A **Riemann sum** 黎曼和 estimates the area under a curve by adding the areas of thin rectangles. Split $[a, b]$ into subintervals and use the function's height at the **left** endpoint, **right** endpoint, or **midpoint** of each. A **trapezoidal sum** 梯形法 uses trapezoids instead, averaging the two endpoint heights –usually more accurate. With more, thinner rectangles the estimate improves.



A Riemann sum approximates the area under a curve with rectangles



Strips of width h approximate the area under a curve

Exam skill: be able to compute left, right, midpoint, and trapezoidal estimates from a table or graph, and state whether each **over- or under-estimates** based on whether the function is increasing/decreasing or concave up/down.

Riemann Sums, Summation Notation, and Definite Integral Notation

Writing a Riemann sum with **summation notation** $\sum_{k=1}^{n} f(x_k)\,\Delta x$ and letting the number of rectangles grow without bound gives the exact area –the **definite integral** 定积分:

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k)\,\Delta x.$$

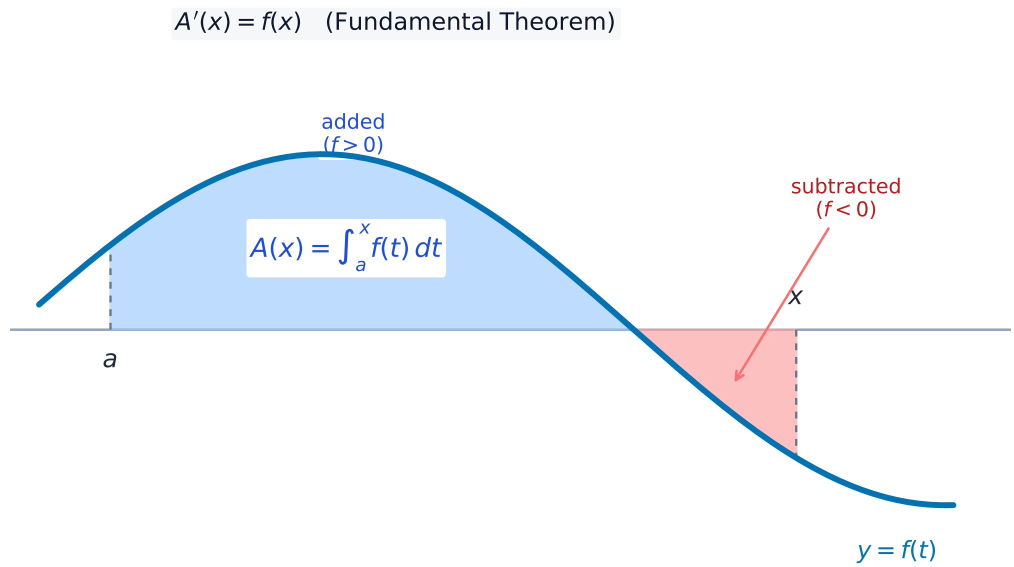
The integral is the **limit of Riemann sums**; a and b are the limits of integration.

The Fundamental Theorem of Calculus and Accumulation Functions

An **accumulation function** 累积函数 $g(x) = \int_a^x f(t)\,dt$ gives the accumulated area from a up to x. The **Fundamental Theorem of Calculus (FTC)** 微积分基本定理 says its derivative is the integrand:

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x).$$

Differentiation and integration are **inverse** operations. With a variable upper limit and the chain rule, $\dfrac{d}{dx}\int_a^{u(x)} f(t)\,dt = f(u(x))\,u'(x)$.



The accumulation function adds signed area; the FTC says its derivative is f

Interpreting the Behavior of Accumulation Functions

Because $g'(x) = f(x)$, the graph of f tells you everything about g: g **increases** where $f > 0$, **decreases** where $f < 0$, has **extrema** where f crosses zero, and is **concave up** where f is increasing. Reading these connections off a graph of f is a classic free-response task.

Applying Properties of Definite Integrals

Definite integrals obey useful rules: reversing the limits negates the value ($\int_b^a = -\int_a^b$), an integral over a zero-width interval is 0, they **add over adjacent intervals** ($\int_a^c = \int_a^b + \int_b^c$), and constants factor out. Use these to combine or split given integral values.

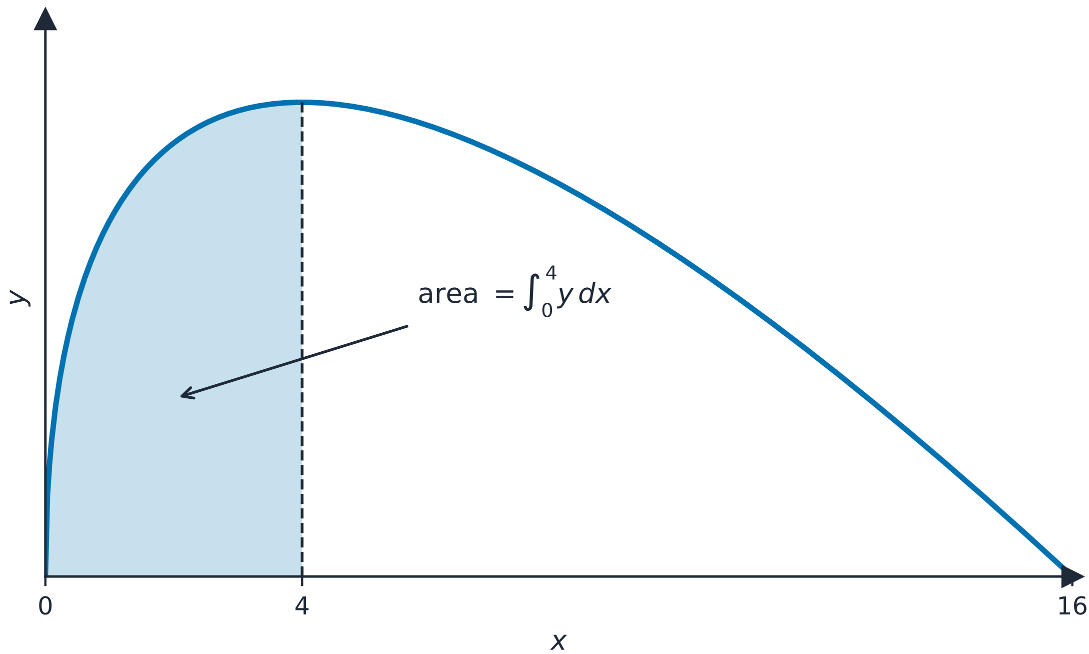
The Fundamental Theorem of Calculus and Definite Integrals

The **evaluation** form of the FTC computes a definite integral from an **antiderivative** 原函数 F (where $F' = f$):

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

So integrating a rate of change over $[a, b]$ gives the **net change** in the quantity –the single most-used result in the course.

Worked example. $\displaystyle\int_1^3 (2x+1)\,dx$: an antiderivative is $F(x) = x^2 + x$, so the value is $F(3) - F(1) = 12 - 2 = 10$.



A definite integral is the signed area between the curve and the x-axis

Finding Antiderivatives and Indefinite Integrals

An **indefinite integral** 不定积分 $\int f(x)\,dx = F(x) + C$ is the family of all antiderivatives (hence the **constant of integration** C). Reverse each derivative rule: the **power rule** becomes $\int x^n\,dx = \dfrac{x^{n+1}}{n+1} + C$ (for $n \neq -1$), with $\int \frac{1}{x}\,dx = \ln|x| + C$, and the antiderivatives of e^x, $\sin x$, $\cos x$, and $\sec^2 x$ come straight from their derivatives.

Integrating Using Substitution

u-substitution 换元积分 reverses the chain rule: choose $u = g(x)$ so that $g'(x)$ also appears, turning $\int f(g(x))g'(x)\,dx$ into $\int f(u)\,du$. Remember to convert dx to du and, for a definite integral, either change the limits to u-values or convert back to x at the end.

Integrating Using Long Division and Completing the Square

When a rational integrand is "top-heavy" (numerator degree $\geq$ denominator degree), **long division** rewrites it as a polynomial plus a proper fraction you can integrate. **Completing the square** in a denominator turns it into a form like $u^2 + a^2$, leading to an **arctangent** antiderivative $\frac{1}{a}\arctan\frac{u}{a} + C$.

Integrating Using Integration by Parts

Integration by parts 分部积分 reverses the product rule:

$$\int u\,dv = uv - \int v\,du.$$

Choose u to simplify when differentiated and dv to be easy to integrate (the **LIATE** guide: logs, inverse-trig, algebraic, trig, exponential). It handles products like $\int xe^x\,dx$ and $\int x\ln x\,dx$, sometimes applied twice.

Worked example. For $\int xe^x\,dx$, choose $u = x$ $(du = dx)$ and $dv = e^x\,dx$ $(v = e^x)$:

$$\int xe^x\,dx = xe^x - \int e^x\,dx = xe^x - e^x + C = e^x(x-1) + C.$$

Integrating Using Linear Partial Fractions

Partial fractions 部分分式 split a rational function with a factorable denominator into a sum of simpler fractions:

$$\frac{1}{(x-a)(x-b)} = \frac{A}{x-a} + \frac{B}{x-b},$$

each of which integrates to a **logarithm**. This technique is essential for the logistic differential equation in the next unit.

Evaluating Improper Integrals

An **improper integral** 反常积分 has an **infinite limit** of integration or an **infinite discontinuity** in the integrand. Evaluate it as a **limit**: $\int_a^\infty f\,dx = \lim_{b\to\infty}\int_a^b f\,dx$. If the limit is a finite number the integral **converges** 收敛; otherwise it **diverges** 发散.

Worked example. $\displaystyle\int_1^\infty \frac{1}{x^2}\,dx = \lim_{b\to\infty}\left[-\frac{1}{x}\right]_1^b = \lim_{b\to\infty}\left(1-\frac{1}{b}\right) = 1$, so it **converges** to 1. By contrast $\int_1^\infty \frac{1}{x}\,dx$ gives $\lim_{b\to\infty}\ln b = \infty$ and **diverges** –the same integrand-shape can go either way.

Selecting Techniques for Antidifferentiation

The BC exam expects you to **recognize which method fits**: basic rules, substitution (a chain-rule pattern), by parts (a product), partial fractions (a factorable rational), or long division/completing the square. Being able to look at an integral and pick the right tool quickly is itself a tested skill.

Exam tips

- Integration is antidifferentiation; use the **power rule** $\int x^n\,dx = \frac{x^{n+1}}{n+1} + C$ and don't forget the $+C$.
- The **Fundamental Theorem** links the two: $\int_a^b f'(x)\,dx = f(b) - f(a)$, and $\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$.
- Approximate a definite integral with **Riemann sums** or the trapezoidal rule from a table of values.
- A definite integral is a signed area (below the axis counts negative); split at sign changes for total area.
- Use **u-substitution** and remember to change the limits (or back-substitute) accordingly.

Differential Equations

AP Calculus BC

Modeling Situations with Differential Equations

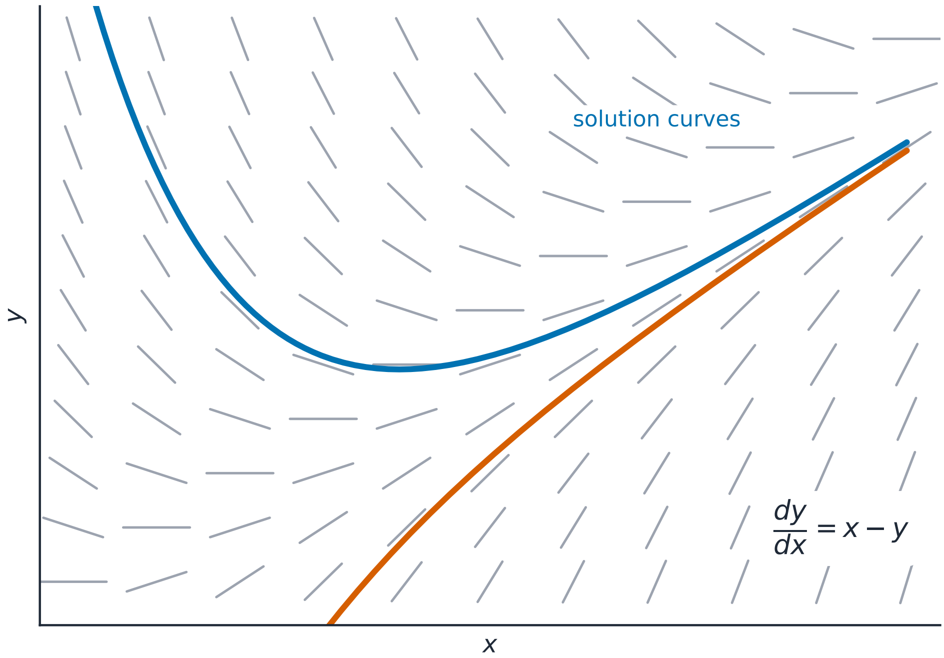
A **differential equation** 微分方程 relates a function to its **derivatives**. Many real situations are described by a rate: "the population grows at a rate proportional to its size" becomes $\frac{dP}{dt} = kP$. Setting up the equation from a verbal description – identifying what changes and what it is proportional to –is the first skill.

Verifying Solutions for Differential Equations

A **solution** is a function that satisfies the equation. To **verify** a proposed solution, differentiate it and substitute into the equation, checking that both sides agree. A **general solution** contains a constant C; a **particular solution** fixes C from a condition.

Sketching Slope Fields

A **slope field** 斜率场 draws a short line segment at many points, each with the slope $\dfrac{dy}{dx}$ the equation gives there. It pictures the family of solution curves without solving. To sketch one, evaluate the right-hand side at each grid point and draw a segment of that slope.



A slope field shows the gradient everywhere; solution curves follow it

Reasoning Using Slope Fields

A solution curve **follows the segments** like a boat following a current. From a slope field you can sketch the particular solution through a given point, describe long-run behavior, and locate where solutions level off (slopes near zero) –reasoning about solutions purely from the picture.

Approximating Solutions Using Euler's Method

Euler's method 欧拉方法 approximates a solution numerically by stepping along the slope field. Starting from a known point, take a small step Δx and update:

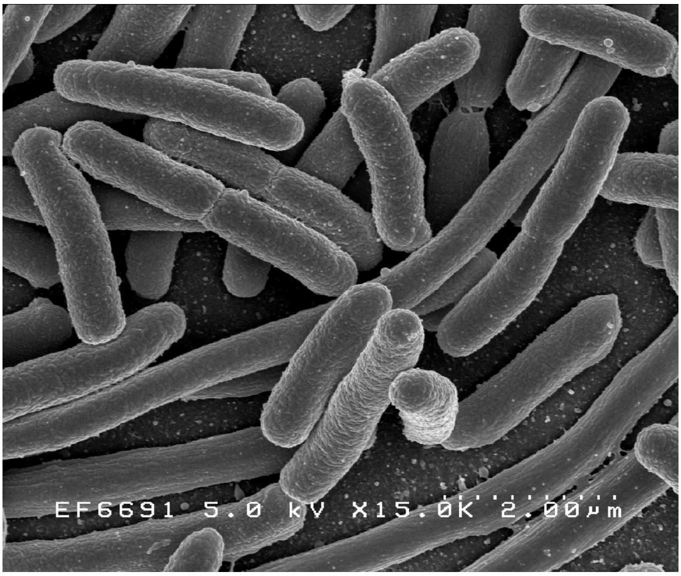
$$y_{\text{new}} = y_{\text{old}} + \frac{dy}{dx} \cdot \Delta x.$$

Repeat for each step. Smaller steps give a better approximation. (This is a BC-only technique.)

Exam skill: be able to carry out two or three Euler steps by hand from a table, and know that Euler's method **under- or over-estimates** depending on the solution's concavity.

Worked example. Approximate $y(1)$ for $\dfrac{dy}{dx} = x + y$, $y(0) = 1$, with step $\Delta x = 0.5$. Step 1: slope at $(0, 1)$ is $0 + 1 = 1$, so $y(0.5) \approx 1 + 1(0.5) = 1.5$. Step 2: slope at $(0.5, 1.5)$ is $0.5 + 1.5 = 2$, so $y(1) \approx 1.5 + 2(0.5) = 2.5$.

Finding General Solutions Using Separation of Variables



E. coli under a microscope: differential equations model exponential growth

A **separable** 可分离 differential equation can be written with all the y's on one side and all the x's on the other, then integrated:

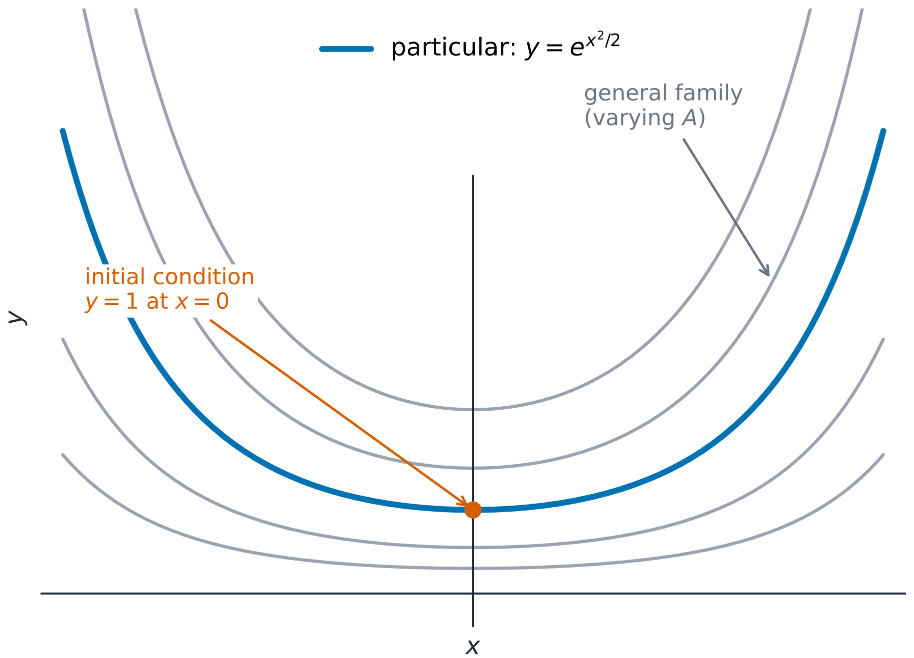
$$\frac{dy}{dx} = g(x)h(y) \;\Rightarrow\; \int \frac{dy}{h(y)} = \int g(x)\,dx.$$

This produces the general solution (with a $+C$), the main analytic method for solving differential equations in this course.

Worked example. Solve $\dfrac{dy}{dx} = xy$ with $y(0) = 2$. Separating, $\int \frac{dy}{y} = \int x\,dx$ gives $\ln|y| = \frac{x^2}{2} + C$, so $y = Ae^{x^2/2}$. The condition $y(0) = 2$ gives $A = 2$, so $y = 2e^{x^2/2}$.

Finding Particular Solutions Using Initial Conditions

An **initial condition** 初始条件 (a known point, e.g. $y(0) = 5$) pins down the constant C. Solve for the general solution, substitute the condition to find C, then write the particular solution. Watch the domain –a particular solution is valid only on the interval containing the initial point.



The constant gives a family of curves; an initial condition picks out one

Exponential Models with Differential Equations

The equation $\frac{dy}{dt} = ky$ says the rate of change is **proportional** to the amount –giving **exponential growth or decay** 指数增长. Separating variables yields

$$y = y_0 e^{kt},$$

with $k > 0$ for growth and $k < 0$ for decay. This models unrestricted population growth, radioactive decay, and continuously compounded interest.

A cooling cup of coffee: Newton's law of cooling is a classic differential equation model

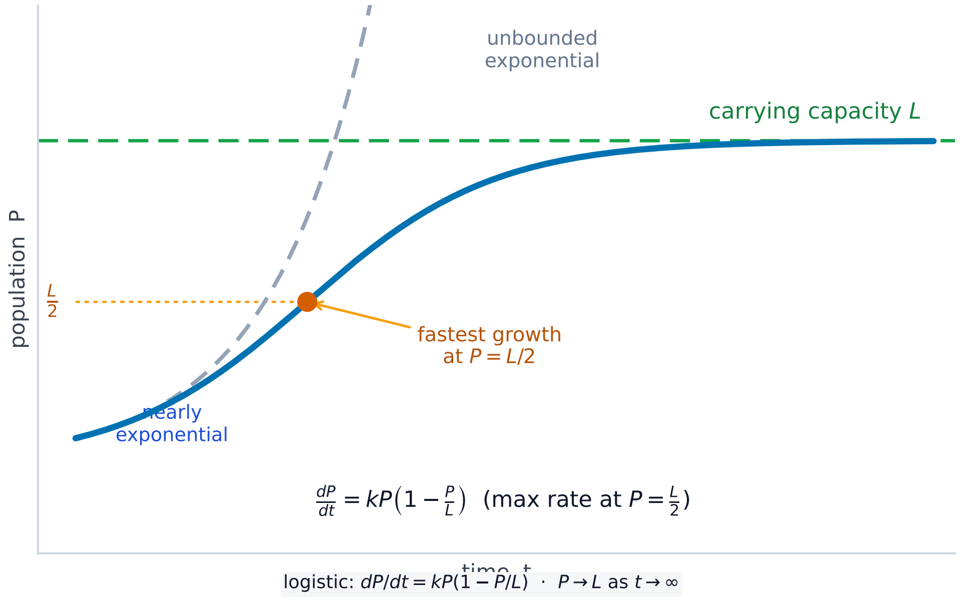
Logistic Models with Differential Equations

Real growth is limited by resources, so the **logistic model** 逻辑斯蒂模型 adds a **carrying capacity** 环境容纳量 L:

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{L}\right).$$

Growth is nearly exponential when P is small, slows as P approaches L, and stops at $P = L$. Key BC facts: the population **levels off at** L ($\lim_{t\to\infty} P = L$), and it grows **fastest when** $P = \frac{L}{2}$ (the inflection point of the S-shaped curve). You are expected to read L and the fastest-growth value directly from the equation.

Worked example. For $\frac{dP}{dt} = 0.05\,P\left(1 - \frac{P}{2000}\right)$, the carrying capacity is $L = 2000$ (the population levels off there), and growth is fastest when $P = \frac{L}{2} = 1000$ –both read straight off the equation, no solving needed.



The logistic model grows fastest at P=L/2 and levels off at the carrying capacity L

Exam tips

- Solve a **separable** equation by getting all y on one side and all x on the other, then integrating both sides (add $+C$ once).
- Use the initial condition to find C (a particular solution).
- Sketch or read a **slope field**: the little segments show $\frac{dy}{dx}$ at each point, and a solution curve follows them.
- Recognise **exponential** models $\frac{dy}{dt} = ky \Rightarrow y = Ce^{kt}$ (growth/decay).
- A differential equation gives the slope —you must integrate to recover the function.

Applications of Integration

AP Calculus BC

Finding the Average Value of a Function on an Interval

The **average value** 平均值 of f over $[a, b]$ is the integral divided by the width:

$$f_{\text{avg}} = \frac{1}{b-a} \int_a^b f(x)\, dx.$$

It is the constant height a rectangle would need to have the same area as the region under f. Do not confuse this with the average **rate of change** (which uses the derivative).

Worked example. The average value of $f(x) = x^2$ on $[0, 3]$ is $\dfrac{1}{3} \displaystyle\int_0^3 x^2\, dx = \dfrac{1}{3} \left[\dfrac{x^3}{3} \right]_0^3 = \dfrac{1}{3}(9) = 3.$

Connecting Position, Velocity, and Acceleration Using Integrals

For straight-line motion, integration reverses differentiation:

$$v(t) = \int a(t)\, dt, \qquad s(t) = \int v(t)\, dt.$$

Two key distinctions: **displacement** 位移 is $\int_a^b v\, dt$ (net change in position), while **total distance** 总路程 is $\int_a^b |v|\, dt$ (splitting where v changes sign). Speed is $|v|$.

Using Accumulation Functions and Definite Integrals in Applied Contexts

When a rate is given (flow rate, sales per day), the definite integral gives the **accumulated total**, and $\int_a^b R(t)\, dt$ carries the **units** of R times time. A common setup: initial amount $+ \int(\text{rate in} - \text{rate out})\, dt$ gives the amount at a later time. Always interpret the answer **in context, with units**.

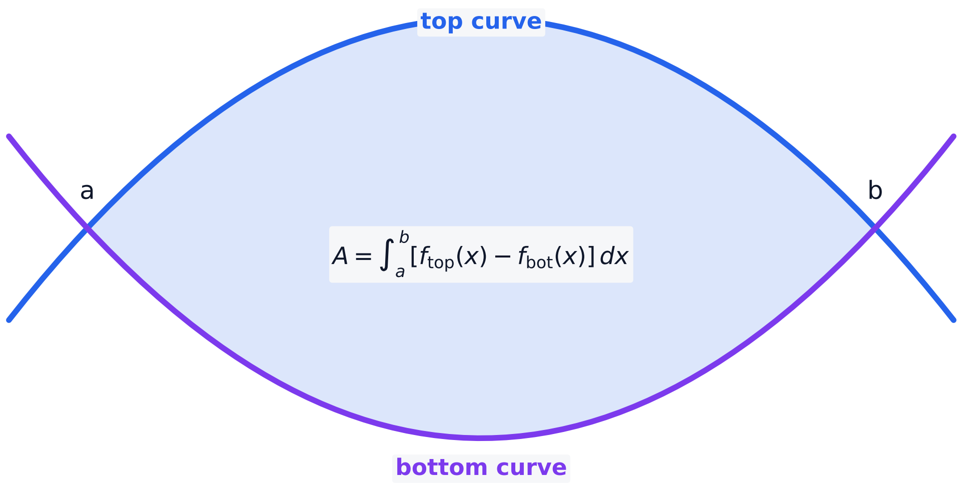
Finding the Area Between Curves Expressed as Functions of x

A potter shaping a vessel: volumes of revolution rotate a plane region about an axis

The area between $y = f(x)$ (top) and $y = g(x)$ (bottom) from a to b is

$$\int_a^b \big(f(x) - g(x)\big)\, dx.$$

Find the intersection points for the limits, and always subtract **top minus bottom**.



The area between two curves is the integral of top minus bottom

Worked example. Between $y = x$ and $y = x^2$ (crossing at $x = 0, 1$, with $y = x$ on top), the area is $\displaystyle\int_0^1 (x - x^2)\,dx = \left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 = \frac{1}{6}$.

Finding the Area Between Curves Expressed as Functions of y

When curves are easier to describe as $x = f(y)$, integrate with respect to y instead, using **right minus left**:

$$\int_c^d \big(f_{\text{right}}(y) - g_{\text{left}}(y)\big)\,dy.$$

Choosing to integrate in y can avoid splitting the region into several pieces.

Finding the Area Between Curves That Intersect More Than Twice

If two curves cross several times, the top and bottom **switch**. Split the region at each intersection and integrate each piece with the correct top-minus-bottom (or use $\int |f - g|$), then add the pieces.

Volumes with Cross Sections: Squares and Rectangles

If a solid's **cross sections** 横截面 perpendicular to the x-axis are squares or rectangles, integrate their area. With side length equal to the distance between two curves, a square cross section gives

$$V = \int_a^b \big(f(x) - g(x)\big)^2\, dx.$$

The method is always "**integrate the cross-sectional area**."

Volumes with Cross Sections: Triangles and Semicircles

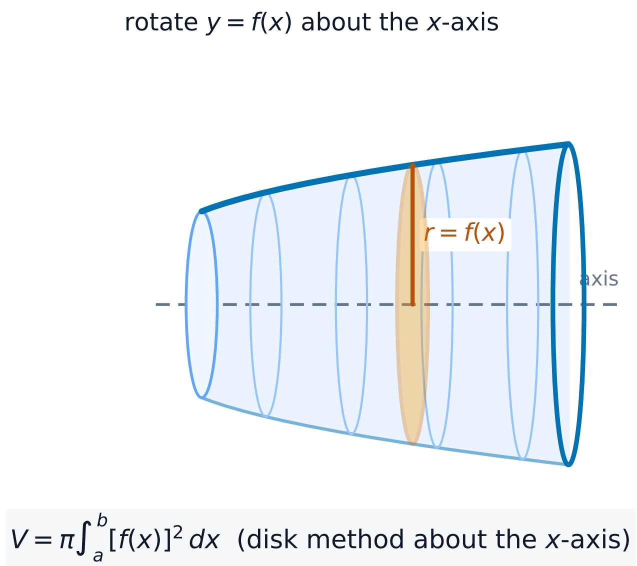
Same idea, different area formula: for equilateral-triangle cross sections use $A = \frac{\sqrt{3}}{4}s^2$, and for semicircular ones $A = \frac{\pi}{8}s^2$ (with s the distance between the curves). Substitute the area formula and integrate.

Volume with Disc Method: Revolving Around the x- or y-Axis

Revolving a region around an axis makes a solid whose cross sections are **discs**. The **disc method** 圆盘法 integrates $\pi(\text{radius})^2$:

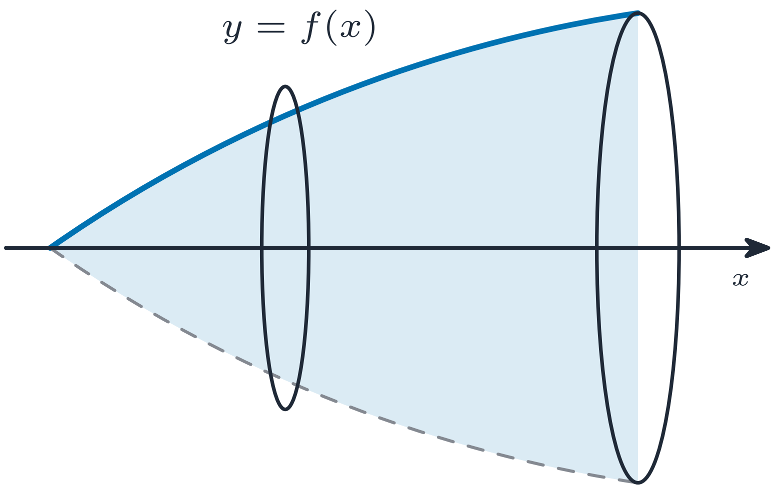
$$V = \pi \int_a^b \big(R(x)\big)^2\, dx,$$

where the radius R is the distance from the curve to the axis. Use dy when revolving around the y-axis.



The disc method: rotating y=f(x) about the axis sweeps out disks of radius f(x)

Worked example. Revolving the region under $y = \sqrt{x}$ from 0 to 4 about the x-axis gives discs of radius $\sqrt{x}$: $V = \pi \int_0^4 (\sqrt{x})^2\, dx = \pi \int_0^4 x\, dx = 8\pi$.



Rotating a region about an axis sweeps out a solid of revolution

Volume with Disc Method: Revolving Around Other Axes

When the axis of revolution is a horizontal or vertical line like $y = k$ (not an axis), the **radius adjusts**: $R = |f(x) - k|$. Set up the radius as the distance from the curve to that line, then integrate πR^2 as before.

Volume with Washer Method: Revolving Around the x- or y-Axis

If the region does not touch the axis, revolving leaves a hole, so cross sections are **washers** (rings). The **washer method** 垫圈法 subtracts the inner disc:

$$V = \pi \int_a^b \left(R_{\text{outer}}^2 - R_{\text{inner}}^2\right) dx.$$

Identify the outer and inner radii as distances from each curve to the axis.

Volume with Washer Method: Revolving Around Other Axes

As with discs, revolving around a line $y = k$ or $x = k$ shifts both radii –each becomes the distance from its curve to that line. Sketch the region and the axis, label R_{outer} and R_{inner}, then integrate the difference of squares.

The Arc Length of a Smooth Curve and Distance Traveled

The **arc length** 弧长 of $y = f(x)$ from a to b is

$$L = \int_a^b \sqrt{1 + \big(f'(x)\big)^2}\, dx.$$

This BC-only formula comes from summing tiny hypotenuses $\sqrt{dx^2 + dy^2}$. The same idea gives the distance a particle travels along a curved path.

Worked example. Find the arc length of $y = \frac{2}{3}x^{3/2}$ from $x = 0$ to $x = 3$. Here $f'(x) = x^{1/2}$, so $1 + (f')^2 = 1 + x$ and

$$L = \int_0^3 \sqrt{1+x}\, dx = \left[\tfrac{2}{3}(1+x)^{3/2}\right]_0^3 = \tfrac{2}{3}(8-1) = \tfrac{14}{3}.$$

The bridge's main cable hangs as a smooth curve; an integral gives its exact length

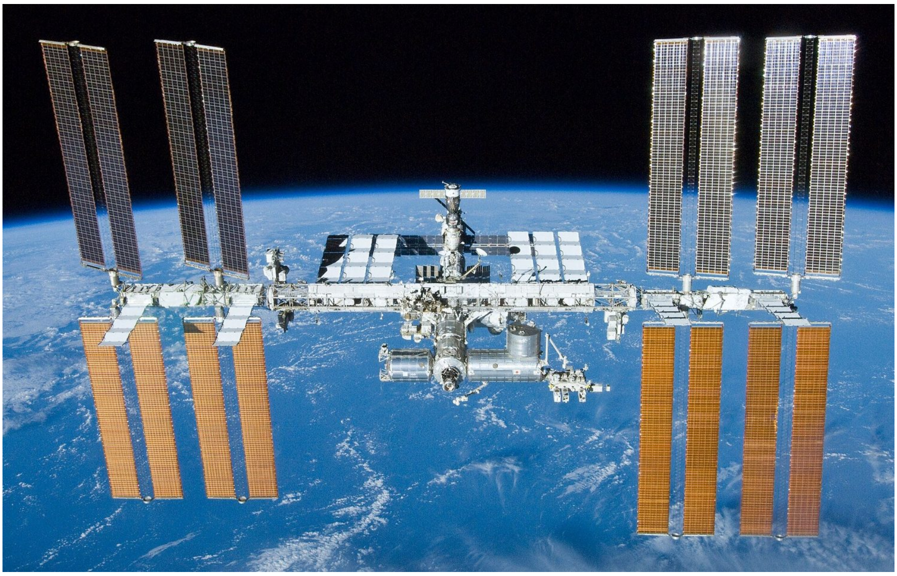
Exam tips

- Area between curves is $\int(\text{top} - \text{bottom})\,dx$ —find the intersection points for the limits and keep top minus bottom.
- For a **volume of revolution**, add up disc/washer cross-sections of area πr^2 (or $\pi(R^2 - r^2)$).
- The **average value** of f on $[a, b]$ is $\frac{1}{b-a}\int_a^b f\,dx$.
- Accumulated change is $\int$ of a rate: total = initial value + $\int_a^b(\text{rate})\,dt$.
- Integration means "adding up infinitely many tiny pieces" —set up the integrand as one thin slice.

Parametric Equations, Polar Coordinates, and Vector-Valued Functions

AP Calculus BC

Defining and Differentiating Parametric Equations



The ISS in orbit: parametric equations describe position as a function of time

Parametric equations 参数方程 give x and y each as functions of a **parameter** t (often time): $x = x(t)$, $y = y(t)$. They trace a curve that need not be a function of x. The slope of the curve is found with the chain rule:

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} \qquad (dx/dt \neq 0).$$

Worked example. For $x = t^2$, $y = t^3 - t$, find the slope at $t = 2$. Here $\frac{dx}{dt} = 2t$ and $\frac{dy}{dt} = 3t^2 - 1$, so $\frac{dy}{dx} = \frac{3t^2 - 1}{2t}$; at $t = 2$ this is $\frac{11}{4}$.

Second Derivatives of Parametric Equations

The second derivative is **not** $\dfrac{d^2y/dt^2}{d^2x/dt^2}$. Instead, differentiate the first derivative with respect to t, then divide by dx/dt again:

$$\frac{d^2y}{dx^2} = \frac{\dfrac{d}{dt}\left(\dfrac{dy}{dx}\right)}{dx/dt}.$$

Use it to test concavity of a parametric curve.

Finding Arc Lengths of Curves Given by Parametric Equations

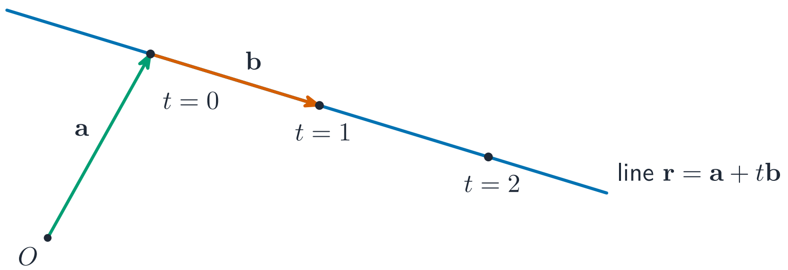
The length of a parametric curve for t from a to b is

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt.$$

This is the parametric version of arc length –summing tiny hypotenuses $\sqrt{dx^2 + dy^2}$ over the parameter.

Defining and Differentiating Vector-Valued Functions

A **vector-valued function** 向量值函数 $\vec{r}(t) = \langle x(t), y(t)\rangle$ gives a position vector for each t. Differentiate **component-wise**: the **velocity** is $\vec{v}(t) = \langle x'(t), y'(t)\rangle$ and the **acceleration** is $\vec{a}(t) = \langle x''(t), y''(t)\rangle$. The **speed** is the magnitude $|\vec{v}| = \sqrt{x'^2 + y'^2}$.



A vector-valued line: start at a, then slide by t lots of the direction b

Integrating Vector-Valued Functions

Integrate a vector function **component by component**. Given acceleration or velocity plus an initial condition, integrate each component and use the condition to find the constants –recovering velocity from acceleration, or position from velocity.

Solving Motion Problems Using Parametric and Vector-Valued Functions

For a particle moving in a plane: position is $\langle x(t), y(t)\rangle$, velocity and acceleration are its derivatives, speed is $|\vec{v}|$, and the **distance traveled** over $[a, b]$ is

$$\int_a^b \sqrt{x'(t)^2 + y'(t)^2}\, dt.$$

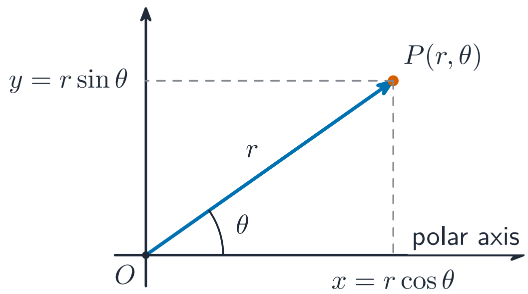
Exam skill: these plane-motion problems appear on the BC free-response nearly every year –be fluent finding speed, the position at a later time (initial point plus the integral of velocity), and total distance.

Worked example. A particle has position $\langle t^2,\ t^3 - t\rangle$. Its velocity is $\langle 2t,\ 3t^2 - 1\rangle$, so at $t = 1$ the velocity is $\langle 2,\ 2\rangle$ and the speed is $\sqrt{2^2 + 2^2} = 2\sqrt{2}$. Its position at $t = 2$ is $\langle 4,\ 6\rangle$.

Defining Polar Coordinates and Differentiating in Polar Form

Polar coordinates 极坐标 locate a point by its distance r from the origin and angle θ: convert with $x = r\cos\theta$, $y = r\sin\theta$. A polar curve $r = f(\theta)$ is a parametric curve in θ, so its slope is

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta}, \quad \text{with } x = r\cos\theta,\ y = r\sin\theta.$$



Polar coordinates give a point by its distance r and angle theta

Finding the Area of a Polar Region

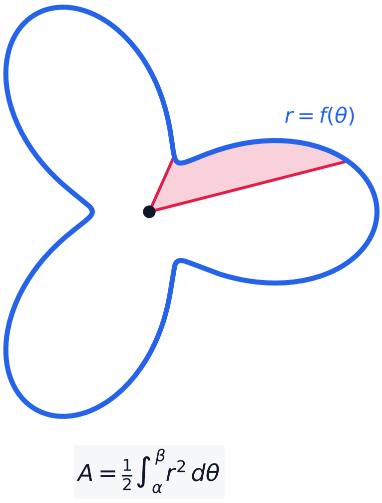
The area swept out by a polar curve $r = f(\theta)$ from α to β is

$$A = \frac{1}{2}\int_{\alpha}^{\beta} \big(f(\theta)\big)^2\, d\theta.$$

The region is a "fan" of thin triangular sectors; choosing the correct θ-limits (where the curve starts and finishes tracing the region) is the main challenge.

Worked example. Find the area of one petal of the rose $r = 2\sin(2\theta)$ (traced for θ from 0 to $\frac{\pi}{2}$). Using $\sin^2 u = \frac{1}{2}(1 - \cos 2u)$,

$$A = \frac{1}{2}\int_0^{\pi/2} (2\sin 2\theta)^2\, d\theta = \int_0^{\pi/2} (1 - \cos 4\theta)\, d\theta = \left[\theta - \tfrac{\sin 4\theta}{4}\right]_0^{\pi/2} = \frac{\pi}{2}.$$



The shaded region is a fan of thin sectors swept from α to β; each has area $\frac{1}{2}r^2\, d\theta$, so the total is $\frac{1}{2}\int_{\alpha}^{\beta} r^2\, d\theta$.

Finding the Area of the Region Bounded by Two Polar Curves

For the area between an outer curve r_1 and an inner curve r_2, subtract the sectors:

$$A = \frac{1}{2}\int_{\alpha}^{\beta} \left(r_1^2 - r_2^2\right) d\theta.$$

Find the **intersection angles** first (set $r_1 = r_2$), and be careful which curve is outer over each interval –they can swap.

Exam tips

- For **parametric** curves, $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$; speed is $\sqrt{(dx/dt)^2 + (dy/dt)^2}$.
- A **vector-valued** function carries the same information —differentiate/integrate it component by component.
- In **polar**, convert with $x = r\cos\theta$, $y = r\sin\theta$; area swept is $\frac{1}{2}\int r^2\, d\theta$.
- Watch the direction of tracing (the sign of dx/dt) and set correct θ-limits for polar area.
- Eliminate the parameter to recover the ordinary y-vs-x shape when it helps.

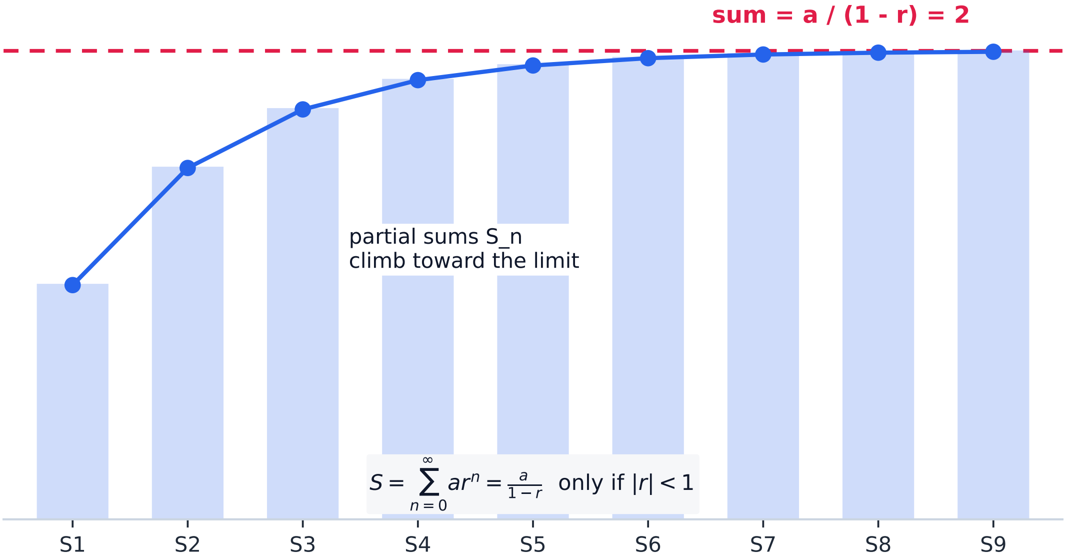
Infinite Sequences and Series

AP Calculus BC

Defining Convergent and Divergent Infinite Series

Nested matryoshka dolls: an infinite series keeps adding terms —some converge, some diverge

An **infinite series** 无穷级数 adds infinitely many terms, $\sum_{n=1}^{\infty} a_n$. Its value is defined as the limit of the **partial sums** 部分和 $S_N = a_1 + a_2 + \cdots + a_N$. If S_N approaches a finite number L, the series **converges** 收敛 to L; otherwise it **diverges** 发散. Every convergence question is really a question about the limit of the partial sums.



For the geometric series with $a = 1,\ r = \frac{1}{2}$, the partial sums $S_1, S_2, S_3, \ldots$ climb toward the limit $\dfrac{a}{1-r} = 2$ —that limit is the series' value.

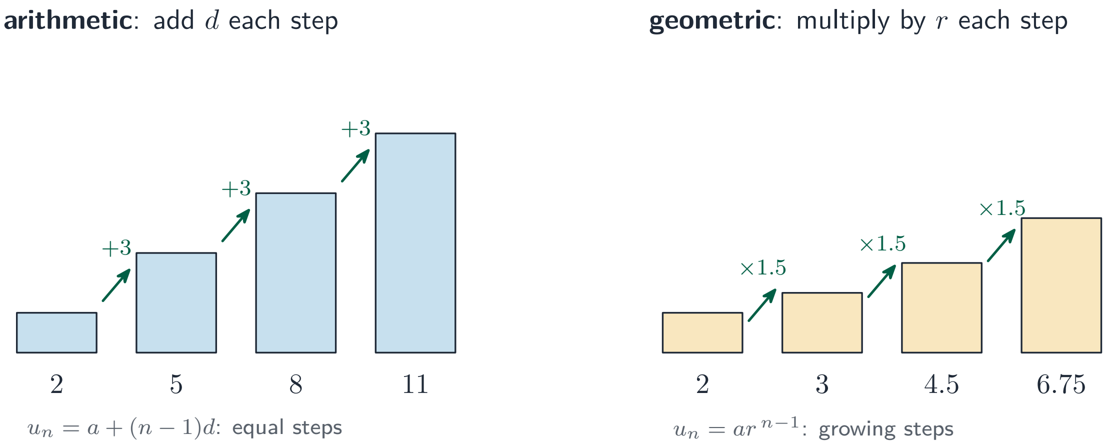
Working with Geometric Series

A **geometric series** 几何级数 $\sum ar^n$ has a constant **ratio** r between terms. It **converges exactly when** $|r| < 1$, and then

$$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}.$$

This is the one series whose sum you can find exactly, and it underlies power series later in the unit.

Worked example. Sum $3 + \frac{3}{2} + \frac{3}{4} + \frac{3}{8} + \cdots$. Here $a = 3$ and $r = \frac{1}{2}$ (with $|r| < 1$), so the sum is $\dfrac{a}{1-r} = \dfrac{3}{1-\frac{1}{2}} = 6$.



A geometric sequence multiplies by the same ratio at each step

The nth Term Test for Divergence

If the terms do not shrink to zero, the sum cannot settle: **if** $\lim_{n\to\infty} a_n \neq 0$**, the series diverges**. This is only a test for **divergence** –if the terms *do* go to zero, the test is inconclusive (the series may still diverge, like the harmonic series). Always check this quick test first.

Integral Test for Convergence

If $a_n = f(n)$ for a positive, decreasing, continuous f, then $\sum a_n$ and $\int_1^{\infty} f(x)\,dx$ **both converge or both diverge**. The **integral test** 积分判别法 turns a series question into an improper-integral question, and it is what proves the p-series rule below.

Harmonic Series and p-Series

A **p-series** $\sum \frac{1}{n^p}$ **converges if** $p > 1$ and **diverges if** $p \leq 1$. The special case $p = 1$, $\sum \frac{1}{n}$, is the **harmonic series** 调和级数–it **diverges** even though its terms go to zero (a famous, must-know fact). The p-series family is the standard yardstick for comparison tests.

Comparison Tests for Convergence

Compare an unfamiliar series to a known one (a p-series or geometric series):

- **Direct comparison** 直接比较: if $0 \leq a_n \leq b_n$ and $\sum b_n$ converges, so does $\sum a_n$; if $a_n \geq b_n \geq 0$ and $\sum b_n$ diverges, so does $\sum a_n$.
- **Limit comparison** 极限比较: if $\lim \frac{a_n}{b_n}$ is a **finite positive** number, the two series do the same thing. This is easier when the terms only *behave* like a known series.

Alternating Series Test for Convergence

An **alternating series** 交错级数 has terms that switch sign, $\sum(-1)^n b_n$. It **converges if** the b_n are positive, **decreasing**, and $\lim b_n = 0$. This lets series like $\sum \frac{(-1)^n}{n}$ converge even though the same terms without the signs (the harmonic series) diverge.

Ratio Test for Convergence

The **ratio test** 比值判别法 examines $L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$:

- $L < 1$: the series **converges absolutely**;
- $L > 1$: it **diverges**;
- $L = 1$: **inconclusive**.

It is the go-to test for series with **factorials** or n**th powers**, and it is exactly how you find the radius of convergence of a power series.

Worked example. Test $\sum \frac{n}{2^n}$. The ratio is $\left|\frac{a_{n+1}}{a_n}\right| = \frac{n+1}{2^{n+1}} \cdot \frac{2^n}{n} = \frac{n+1}{2n} \to \frac{1}{2} < 1$, so the series **converges**.

Determining Absolute or Conditional Convergence

A series **converges absolutely** 绝对收敛 if $\sum |a_n|$ converges. It **converges conditionally** 条件收敛 if $\sum a_n$ converges but $\sum |a_n|$ diverges (the classic example is $\sum \frac{(-1)^n}{n}$). Absolute convergence is the stronger property; conditional convergence relies on the cancellation of signs.

Alternating Series Error Bound

For a **convergent alternating series**, the error in stopping at the Nth partial sum is **no larger than the first omitted term**:

$$|S - S_N| \le b_{N+1}.$$

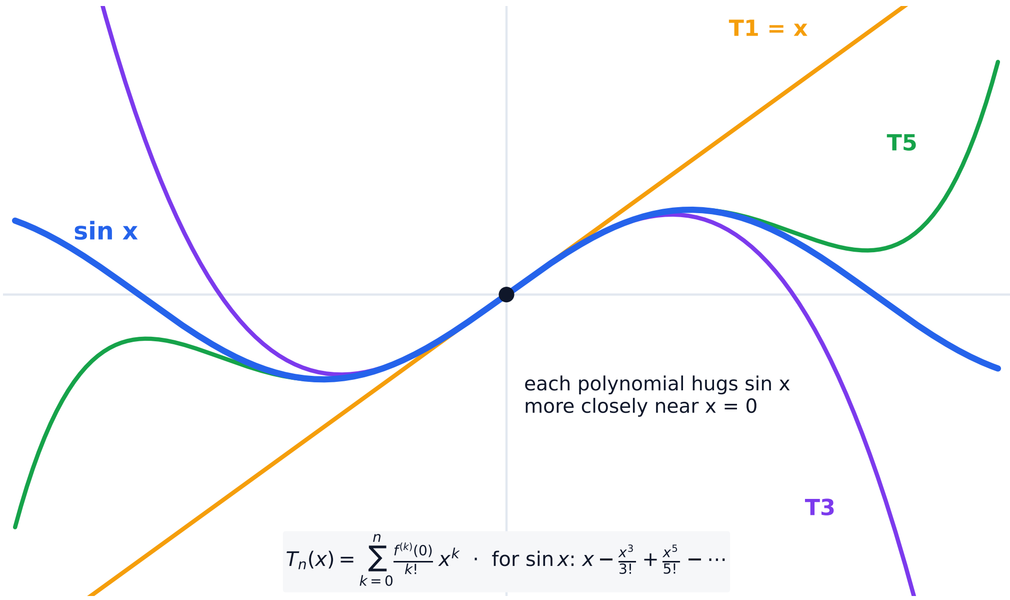
This simple, powerful bound lets you say how many terms guarantee a desired accuracy.

Finding Taylor Polynomial Approximations of Functions

A **Taylor polynomial** 泰勒多项式 approximates a function near a center $x = a$ using its derivatives there:

$$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k.$$

Each added term matches one more derivative, so the polynomial hugs the curve more closely near a. Centered at $a = 0$ it is a **Maclaurin polynomial**.



The Maclaurin polynomials of $\sin x$ *–*$T_1 = x$, T_3, T_5 *–each match one more derivative at* 0*, so each hugs* $\sin x$ *over a wider interval before peeling away.*

Exam skill: be able to build a Taylor polynomial from a table of derivative values and use it to estimate a function value.

Lagrange Error Bound

The **Lagrange error bound** 拉格朗日误差界 bounds how far a Taylor polynomial can be from the true value:

$$|R_n(x)| \le \frac{\max\left|f^{(n+1)}(z)\right|}{(n+1)!}\,|x-a|^{n+1}.$$

You bound the $(n+1)$th derivative on the interval, then compute –the standard way to prove a Taylor estimate is accurate enough.

Radius and Interval of Convergence of Power Series

A **power series 幂级数** $\sum c_n(x-a)^n$ converges for x within a **radius of convergence 收敛半径** R of the center a. Find R with the **ratio test**. Then test the two **endpoints** separately (the ratio test is inconclusive there) to state the full **interval of convergence 收敛区间**–including or excluding each endpoint.

Worked example. Find the radius of convergence of $\sum \frac{x^n}{n}$. The ratio test gives $\left|\frac{x^{n+1}}{n+1}\cdot\frac{n}{x^n}\right| = |x|\frac{n}{n+1} \to |x|$, which is < 1 when $|x| < 1$, so $R = 1$. Testing the endpoints, $x = -1$ gives the convergent alternating harmonic series and $x = 1$ the divergent harmonic series, so the interval is $[-1, 1)$.

Finding Taylor or Maclaurin Series for a Function

Extending a Taylor polynomial to infinitely many terms gives a **Taylor (or Maclaurin) series**. Memorize the key Maclaurin series:

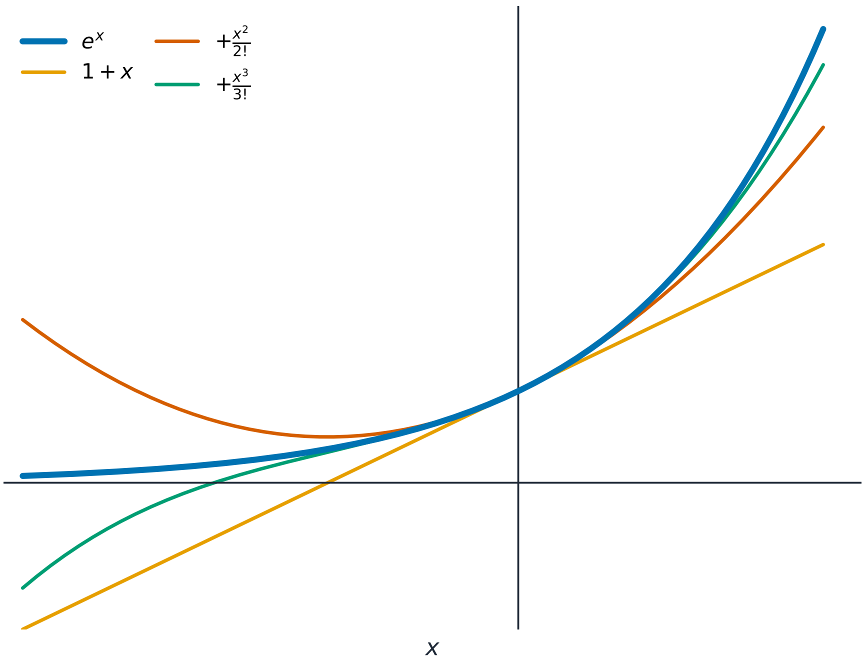
$$e^x = \sum \frac{x^n}{n!}, \quad \sin x = \sum \frac{(-1)^n x^{2n+1}}{(2n+1)!}, \quad \cos x = \sum \frac{(-1)^n x^{2n}}{(2n)!}, \quad \frac{1}{1-x} = \sum x^n.$$

New series come from **manipulating** these –substituting, differentiating, integrating, or multiplying.

Worked example. Find the Maclaurin series for e^{x^2}. Substitute x^2 for x in $e^x = \sum \frac{x^n}{n!}$:

$$e^{x^2} = \sum_{n=0}^{\infty} \frac{(x^2)^n}{n!} = 1 + x^2 + \frac{x^4}{2!} + \frac{x^6}{3!} + \cdots,$$

which converges for all x. This substitution trick is far faster than differentiating e^{x^2} six times.



Each extra Maclaurin term hugs the function over a wider range

Representing Functions as Power Series

Because a power series can be **differentiated and integrated term by term** (within its radius), you can build new series from known ones –e.g. integrate the geometric series for $\dfrac{1}{1-x}$ to get the series for $-\ln(1-x) = \sum_{n\geq 1} \dfrac{x^n}{n}$, or substitute $-x^2$ to get the series for $\dfrac{1}{1+x^2}$. Representing a function as a power series lets you approximate values and integrals that have no elementary antiderivative.

Exam skill: the BC series free-response usually asks you to derive a new Maclaurin series from a known one, find its interval of convergence, and use the alternating-series or Lagrange bound to estimate the error –the capstone skills of the course.

Exam tips

- Test a series for convergence with the right tool: **geometric** ($|r| < 1$, sum $\frac{a}{1-r}$), nth-term, ratio, integral, comparison, or alternating-series test.
- A **geometric** infinite sum converges only when $|r| < 1$; otherwise it diverges.
- Build a **Taylor/Maclaurin** series to approximate a function; more terms give a better fit **near the centre**.
- Know the standard Maclaurin series for e^x, $\sin x$, $\cos x$, and $\frac{1}{1-x}$.

- Find the **radius/interval of convergence** with the ratio test, then check the endpoints separately.