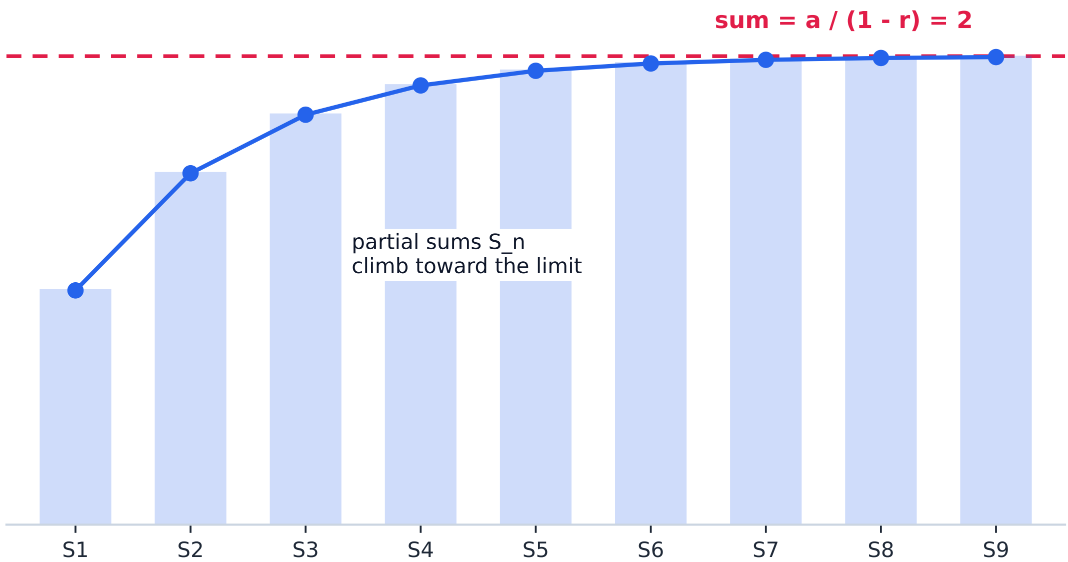
Infinite Sequences and Series

AP Calculus BC

Defining Convergent and Divergent Infinite Series

An **infinite series** 无穷级数 adds infinitely many terms, $\sum_{n=1}^{\infty} a_n$. Its value is defined as the limit of the **partial sums** 部分和 $S_N = a_1 + a_2 + \cdots + a_N$. If S_N approaches a finite number L, the series **converges** 收敛 to L; otherwise it **diverges** 发散. Every convergence question is really a question about the limit of the partial sums.



For the geometric series with $a = 1,\ r = \frac{1}{2}$, the partial sums $S_1, S_2, S_3, \ldots$ climb toward the limit $\dfrac{a}{1-r} = 2$ –that limit is the series' value.

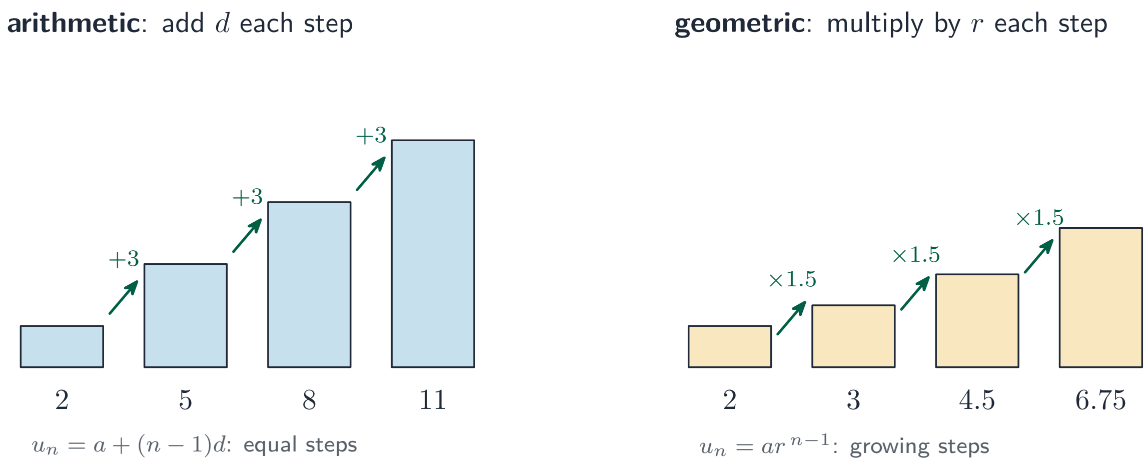
Working with Geometric Series

A **geometric series** 几何级数 $\sum ar^n$ has a constant **ratio** r between terms. It **converges exactly when** $|r| < 1$, and then

$$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}.$$

This is the one series whose sum you can find exactly, and it underlies power series later in the unit.

Worked example. Sum $3 + \frac{3}{2} + \frac{3}{4} + \frac{3}{8} + \cdots$. Here $a = 3$ and $r = \frac{1}{2}$ (with $|r| < 1$), so the sum is $\dfrac{a}{1-r} = \dfrac{3}{1-\frac{1}{2}} = 6$.



A geometric sequence multiplies by the same ratio at each step

The nth Term Test for Divergence

If the terms do not shrink to zero, the sum cannot settle: **if** $\lim_{n\to\infty} a_n \neq 0$**, the series diverges**. This is only a test for **divergence** –if the terms *do* go to zero, the test is inconclusive (the series may still diverge, like the harmonic series). Always check this quick test first.

Integral Test for Convergence

If $a_n = f(n)$ for a positive, decreasing, continuous f, then $\sum a_n$ and $\int_1^\infty f(x)\,dx$ **both converge or both diverge**. The **integral test** 积分判别法 turns a series question into an improper-integral question, and it is what proves the p-series rule below.

Harmonic Series and p-Series

A **p-series** $\sum \frac{1}{n^p}$ **converges if** $p > 1$ and **diverges if** $p \leq 1$. The special case $p = 1$, $\sum \frac{1}{n}$, is the **harmonic series** 调和级数–it **diverges** even though its terms go to zero (a famous, must-know fact). The p-series family is the standard yardstick for comparison tests.

Comparison Tests for Convergence

Compare an unfamiliar series to a known one (a p-series or geometric series):

- **Direct comparison** 直接比较: if $0 \leq a_n \leq b_n$ and $\sum b_n$ converges, so does $\sum a_n$; if $a_n \geq b_n \geq 0$ and $\sum b_n$ diverges, so does $\sum a_n$.
- **Limit comparison** 极限比较: if $\lim \frac{a_n}{b_n}$ is a **finite positive** number, the two series do the same thing. This is easier when the terms only *behave* like a known series.

Alternating Series Test for Convergence

An **alternating series** 交错级数 has terms that switch sign, $\sum(-1)^n b_n$. It **converges if** the b_n are positive, **decreasing**, and $\lim b_n = 0$. This lets series like $\sum \frac{(-1)^n}{n}$ converge even though the same terms without the signs (the harmonic series) diverge.

Ratio Test for Convergence

The **ratio test** 比值判别法 examines $L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$:

- $L < 1$: the series **converges absolutely**;
- $L > 1$: it **diverges**;
- $L = 1$: **inconclusive**.

It is the go-to test for series with **factorials** or n**th powers**, and it is exactly how you find the radius of convergence of a power series.

Worked example. Test $\sum \frac{n}{2^n}$. The ratio is $\left|\frac{a_{n+1}}{a_n}\right| = \frac{n+1}{2^{n+1}} \cdot \frac{2^n}{n} = \frac{n+1}{2n} \to \frac{1}{2} < 1$, so the series **converges**.

Determining Absolute or Conditional Convergence

A series **converges absolutely** 绝对收敛 if $\sum |a_n|$ converges. It **converges conditionally** 条件收敛 if $\sum a_n$ converges but $\sum |a_n|$ diverges (the classic example is $\sum \frac{(-1)^n}{n}$). Absolute convergence is the stronger property; conditional convergence relies on the cancellation of signs.

Alternating Series Error Bound

For a **convergent alternating series**, the error in stopping at the Nth partial sum is **no larger than the first omitted term**:

$$|S - S_N| \le b_{N+1}.$$

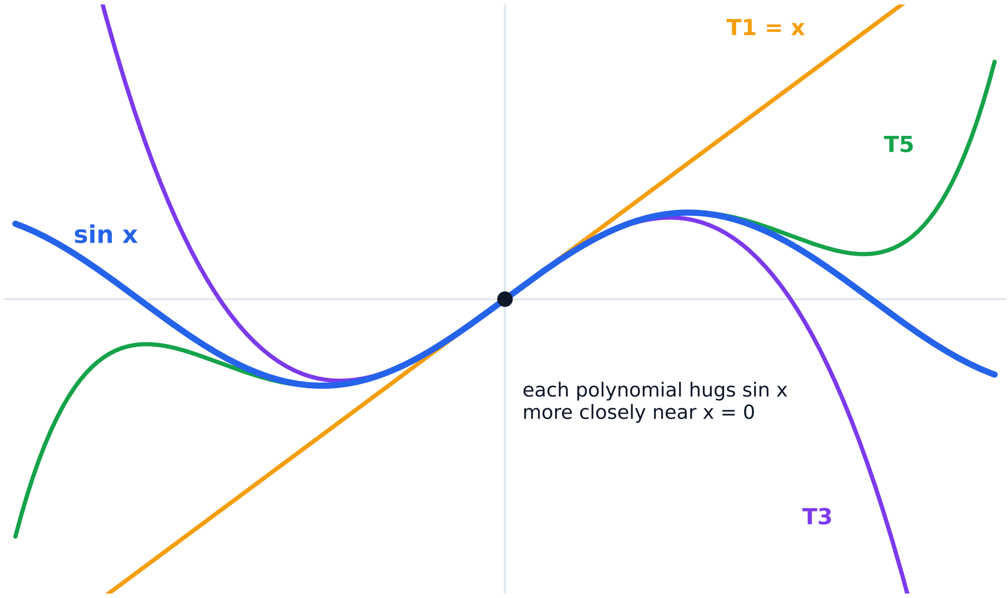
This simple, powerful bound lets you say how many terms guarantee a desired accuracy.

Finding Taylor Polynomial Approximations of Functions

A **Taylor polynomial** 泰勒多项式 approximates a function near a center $x = a$ using its derivatives there:

$$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k.$$

Each added term matches one more derivative, so the polynomial hugs the curve more closely near a. Centered at $a = 0$ it is a **Maclaurin polynomial**.



The Maclaurin polynomials of $\sin x$ *–*$T_1 = x$, T_3, T_5 *–each match one more derivative at* 0, *so each hugs* $\sin x$ *over a wider interval before peeling away.*

Exam skill: be able to build a Taylor polynomial from a table of derivative values and use it to estimate a function value.

Lagrange Error Bound

The **Lagrange error bound** 拉格朗日误差界 bounds how far a Taylor polynomial can be from the true value:

$$|R_n(x)| \le \frac{\max\left|f^{(n+1)}(z)\right|}{(n+1)!}\,|x-a|^{n+1}.$$

You bound the $(n+1)$th derivative on the interval, then compute –the standard way to prove a Taylor estimate is accurate enough.

Radius and Interval of Convergence of Power Series

A **power series** 幂级数 $\sum c_n(x-a)^n$ converges for x within a **radius of convergence** 收敛半径 R of the center a. Find R with the **ratio test**. Then test the two **endpoints** separately (the ratio test is inconclusive there) to state the full **interval of convergence** 收敛区间–including or excluding each endpoint.

Worked example. Find the radius of convergence of $\sum \frac{x^n}{n}$. The ratio test gives $\left|\frac{x^{n+1}}{n+1}\cdot\frac{n}{x^n}\right| = |x|\frac{n}{n+1} \to |x|$, which is < 1 when $|x| < 1$, so $R = 1$. Testing the endpoints, $x = -1$ gives the convergent alternating harmonic series and $x = 1$ the divergent harmonic series, so the interval is $[-1, 1)$.

Finding Taylor or Maclaurin Series for a Function

Extending a Taylor polynomial to infinitely many terms gives a **Taylor (or Maclaurin) series**. Memorize the key Maclaurin series:

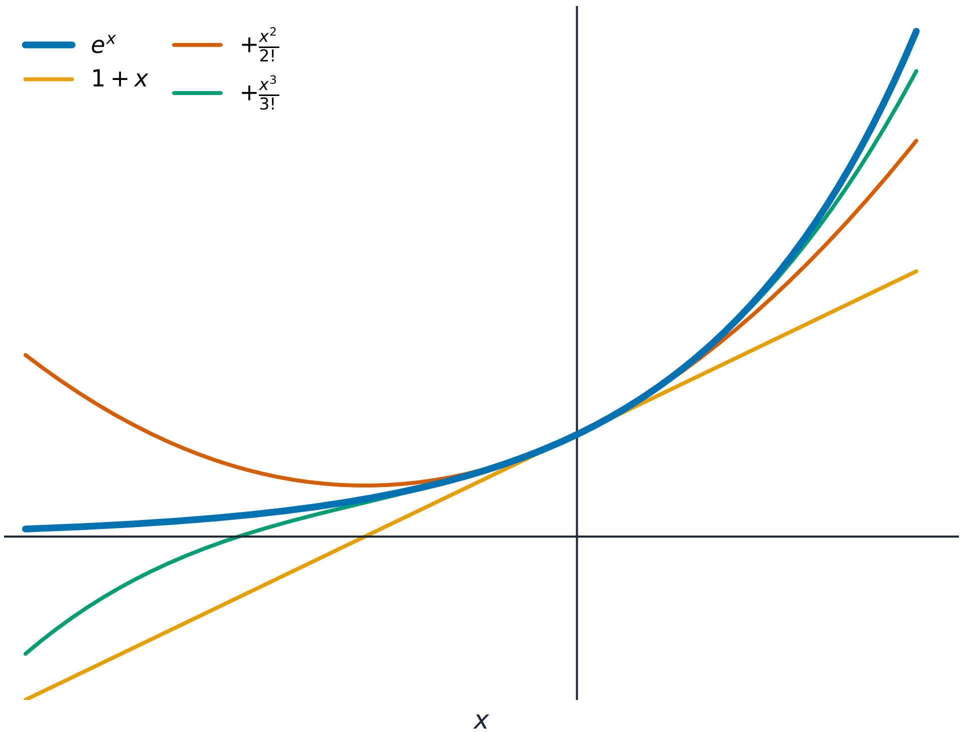
$$e^x = \sum \frac{x^n}{n!}, \quad \sin x = \sum \frac{(-1)^n x^{2n+1}}{(2n+1)!}, \quad \cos x = \sum \frac{(-1)^n x^{2n}}{(2n)!}, \quad \frac{1}{1-x} = \sum x^n.$$

New series come from **manipulating** these –substituting, differentiating, integrating, or multiplying.

Worked example. Find the Maclaurin series for e^{x^2}. Substitute x^2 for x in $e^x = \sum \frac{x^n}{n!}$:

$$e^{x^2} = \sum_{n=0}^{\infty} \frac{(x^2)^n}{n!} = 1 + x^2 + \frac{x^4}{2!} + \frac{x^6}{3!} + \cdots,$$

which converges for all x. This substitution trick is far faster than differentiating e^{x^2} six times.



Each extra Maclaurin term hugs the function over a wider range

Representing Functions as Power Series

Because a power series can be **differentiated and integrated term by term** (within its radius), you can build new series from known ones –e.g. integrate the geometric series for $\frac{1}{1-x}$ to get the series for $\ln(1-x)$, or substitute $-x^2$ to get the series for $\frac{1}{1+x^2}$.

Representing a function as a power series lets you approximate values and integrals that have no elementary antiderivative.

Exam skill: the BC series free-response usually asks you to derive a new Maclaurin series from a known one, find its interval of convergence, and use the alternating-series or Lagrange bound to estimate the error –the capstone skills of the course.

Exam tips

- Test a series for convergence with the right tool: **geometric** ($|r| < 1$, sum $\frac{a}{1-r}$), nth-term, ratio, integral, comparison, or alternating-series test.
- A **geometric** infinite sum converges only when $|r| < 1$; otherwise it diverges.
- Build a **Taylor/Maclaurin** series to approximate a function; more terms give a better fit **near the centre**.
- Know the standard Maclaurin series for e^x, $\sin x$, $\cos x$, and $\frac{1}{1-x}$.
- Find the **radius/interval of convergence** with the ratio test, then check the endpoints separately.