Applications of Integration

AP Calculus BC

Finding the Average Value of a Function on an Interval

The **average value** 平均值 of f over $[a, b]$ is the integral divided by the width:

$$f_{\text{avg}} = \frac{1}{b-a} \int_a^b f(x)\, dx.$$

It is the constant height a rectangle would need to have the same area as the region under f. Do not confuse this with the average **rate of change** (which uses the derivative).

Worked example. The average value of $f(x) = x^2$ on $[0, 3]$ is $\frac{1}{3} \int_0^3 x^2\, dx = \frac{1}{3} \left[\frac{x^3}{3} \right]_0^3 = \frac{1}{3}(9) = 3$.

Connecting Position, Velocity, and Acceleration Using Integrals

For straight-line motion, integration reverses differentiation:

$$v(t) = \int a(t)\, dt, \qquad s(t) = \int v(t)\, dt.$$

Two key distinctions: **displacement** 位移 is $\int_a^b v\, dt$ (net change in position), while **total distance** 总路程 is $\int_a^b |v|\, dt$ (splitting where v changes sign). Speed is $|v|$.

Using Accumulation Functions and Definite Integrals in Applied Contexts

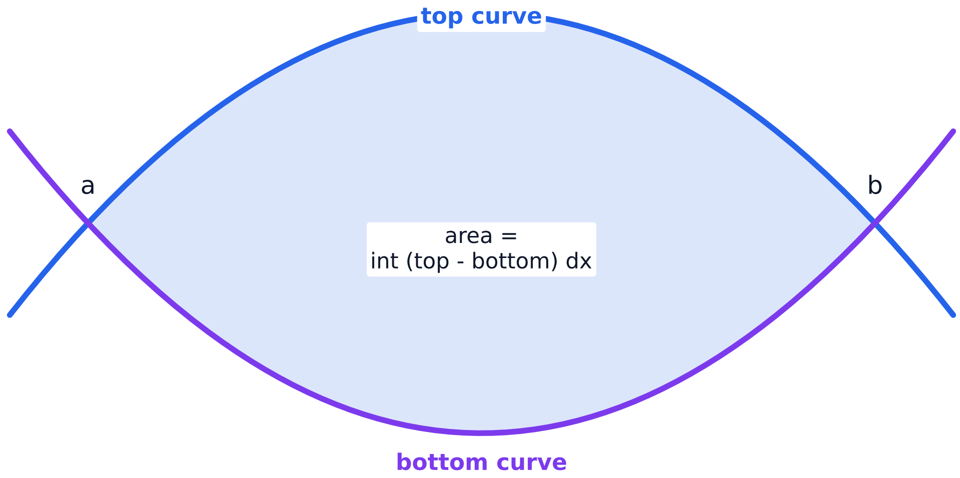
When a rate is given (flow rate, sales per day), the definite integral gives the **accumulated total**, and $\int_a^b R(t)\, dt$ carries the **units** of R times time. A common setup: initial amount $+ \int (\text{rate in} - \text{rate out})\, dt$ gives the amount at a later time. Always interpret the answer **in context, with units**.

Finding the Area Between Curves Expressed as Functions of x

The area between $y = f(x)$ (top) and $y = g(x)$ (bottom) from a to b is

$$\int_a^b \big(f(x) - g(x)\big)\, dx.$$

Find the intersection points for the limits, and always subtract **top minus bottom**.



The area between two curves is the integral of top minus bottom

Worked example. Between $y = x$ and $y = x^2$ (crossing at $x = 0, 1$, with $y = x$ on top), the area is $\int_0^1 (x - x^2)\, dx = \left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 = \frac{1}{6}$.

Finding the Area Between Curves Expressed as Functions of y

When curves are easier to describe as $x = f(y)$, integrate with respect to y instead, using **right minus left**:

$$\int_c^d \big(f_{\text{right}}(y) - g_{\text{left}}(y)\big)\, dy.$$

Choosing to integrate in y can avoid splitting the region into several pieces.

Finding the Area Between Curves That Intersect More Than Twice

If two curves cross several times, the top and bottom **switch**. Split the region at each intersection and integrate each piece with the correct top-minus-bottom (or use $\int |f - g|$), then add the pieces.

Volumes with Cross Sections: Squares and Rectangles

If a solid's **cross sections** 横截面 perpendicular to the x-axis are squares or rectangles, integrate their area. With side length equal to the distance between two curves, a square cross section gives

$$V = \int_a^b \big(f(x) - g(x)\big)^2 \, dx.$$

The method is always "**integrate the cross-sectional area**."

Volumes with Cross Sections: Triangles and Semicircles

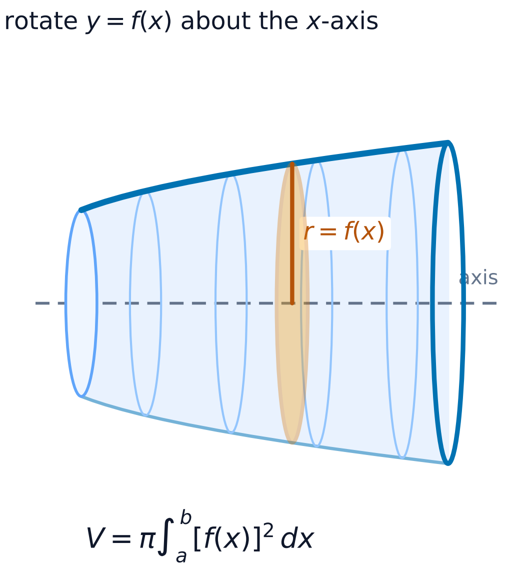
Same idea, different area formula: for equilateral-triangle cross sections use $A = \frac{\sqrt{3}}{4}s^2$, and for semicircular ones $A = \frac{\pi}{8}s^2$ (with s the distance between the curves). Substitute the area formula and integrate.

Volume with Disc Method: Revolving Around the x- or y-Axis

Revolving a region around an axis makes a solid whose cross sections are **discs**. The **disc method** 圆盘法 integrates $\pi(\text{radius})^2$:

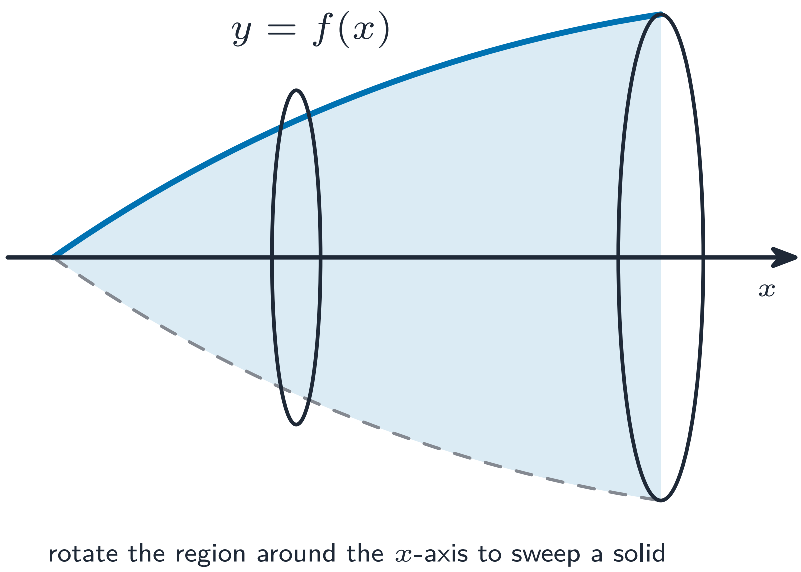
$$V = \pi \int_a^b \big(R(x)\big)^2 \, dx,$$

where the radius R is the distance from the curve to the axis. Use dy when revolving around the y-axis.



The disc method: rotating y=f(x) about the axis sweeps out disks of radius f(x)

Worked example. Revolving the region under $y = \sqrt{x}$ from 0 to 4 about the x-axis gives discs of radius $\sqrt{x}$: $V = \pi \int_0^4 (\sqrt{x})^2\, dx = \pi \int_0^4 x\, dx = 8\pi$.



Rotating a region about an axis sweeps out a solid of revolution

Volume with Disc Method: Revolving Around Other Axes

When the axis of revolution is a horizontal or vertical line like $y = k$ (not an axis), the **radius adjusts**: $R = |f(x) - k|$. Set up the radius as the distance from the curve to that line, then integrate πR^2 as before.

Volume with Washer Method: Revolving Around the x- or y-Axis

If the region does not touch the axis, revolving leaves a hole, so cross sections are **washers** (rings). The **washer method** 垫圈法 subtracts the inner disc:

$$V = \pi \int_a^b \left(R_{\text{outer}}^2 - R_{\text{inner}}^2 \right) dx.$$

Identify the outer and inner radii as distances from each curve to the axis.

Volume with Washer Method: Revolving Around Other Axes

As with discs, revolving around a line $y = k$ or $x = k$ shifts both radii –each becomes the distance from its curve to that line. Sketch the region and the axis, label R_{outer} and R_{inner}, then integrate the difference of squares.

The Arc Length of a Smooth Curve and Distance Traveled

The **arc length** 弧长 of $y = f(x)$ from a to b is

$$L = \int_a^b \sqrt{1 + \left(f'(x)\right)^2}\, dx.$$

This BC-only formula comes from summing tiny hypotenuses $\sqrt{dx^2 + dy^2}$. The same idea gives the distance a particle travels along a curved path.

Worked example. Find the arc length of $y = \frac{2}{3}x^{3/2}$ from $x = 0$ to $x = 3$. Here $f'(x) = x^{1/2}$, so $1 + (f')^2 = 1 + x$ and

$$L = \int_0^3 \sqrt{1+x}\, dx = \left[\tfrac{2}{3}(1+x)^{3/2}\right]_0^3 = \tfrac{2}{3}(8-1) = \tfrac{14}{3}.$$

Exam tips

- Area between curves is $\int(\text{top} - \text{bottom})\, dx$ —find the intersection points for the limits and keep top minus bottom.

- For a **volume of revolution**, add up disc/washer cross-sections of area πr^2 (or $\pi(R^2 - r^2)$).
- The **average value** of f on $[a, b]$ is $\frac{1}{b-a} \int_a^b f \, dx$.
- Accumulated change is $\int$ of a rate: total = initial value + $\int_a^b (\text{rate}) \, dt$.
- Integration means "adding up infinitely many tiny pieces" —set up the integrand as one thin slice.