Differential Equations

AP Calculus BC

Modeling Situations with Differential Equations

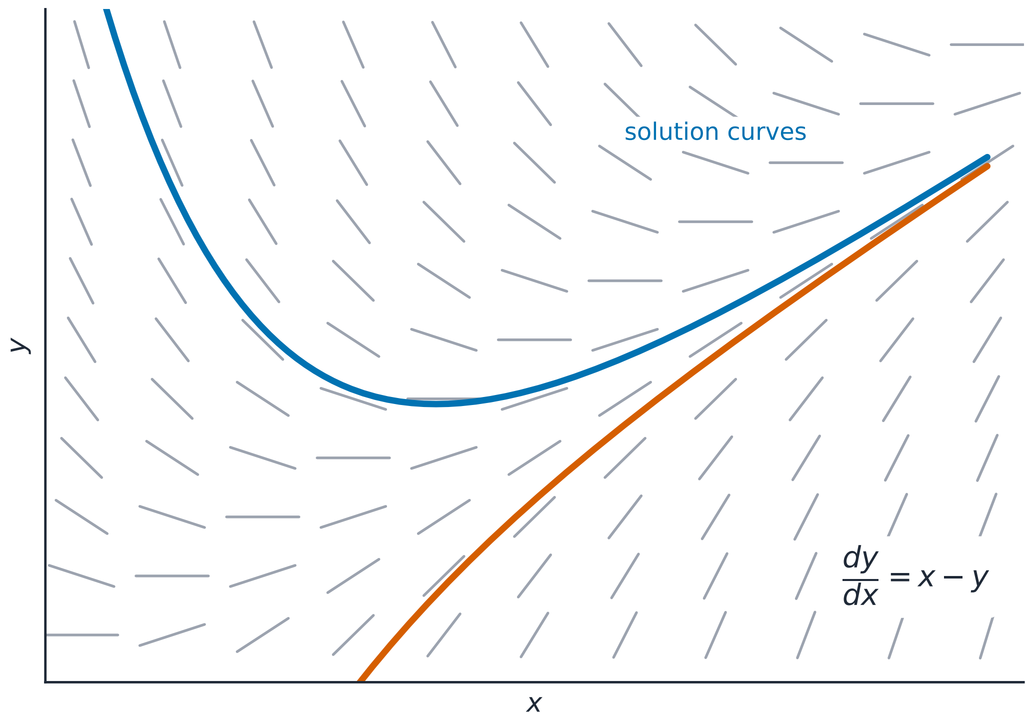
A **differential equation** 微分方程 relates a function to its **derivatives**. Many real situations are described by a rate: "the population grows at a rate proportional to its size" becomes $\frac{dP}{dt} = kP$. Setting up the equation from a verbal description –identifying what changes and what it is proportional to –is the first skill.

Verifying Solutions for Differential Equations

A **solution** is a function that satisfies the equation. To **verify** a proposed solution, differentiate it and substitute into the equation, checking that both sides agree. A **general solution** contains a constant C; a **particular solution** fixes C from a condition.

Sketching Slope Fields

A **slope field** 斜率场 draws a short line segment at many points, each with the slope $\frac{dy}{dx}$ the equation gives there. It pictures the family of solution curves without solving. To sketch one, evaluate the right-hand side at each grid point and draw a segment of that slope.



A slope field shows the gradient everywhere; solution curves follow it

Reasoning Using Slope Fields

A solution curve **follows the segments** like a boat following a current. From a slope field you can sketch the particular solution through a given point, describe long-run behavior, and locate where solutions level off (slopes near zero) –reasoning about solutions purely from the picture.

Approximating Solutions Using Euler's Method

Euler's method 欧拉方法 approximates a solution numerically by stepping along the slope field. Starting from a known point, take a small step Δx and update:

$$y_{\text{new}} = y_{\text{old}} + \frac{dy}{dx} \cdot \Delta x.$$

Repeat for each step. Smaller steps give a better approximation. (This is a BC-only technique.)

Exam skill: be able to carry out two or three Euler steps by hand from a table, and know that Euler's method **under- or over-estimates** depending on the solution's concavity.

Worked example. Approximate $y(1)$ for $\frac{dy}{dx} = x + y$, $y(0) = 1$, with step $\Delta x = 0.5$. Step 1: slope at $(0, 1)$ is $0+1 = 1$, so $y(0.5) \approx 1+1(0.5) = 1.5$. Step 2: slope at $(0.5, 1.5)$ is $0.5 + 1.5 = 2$, so $y(1) \approx 1.5 + 2(0.5) = 2.5$.

Finding General Solutions Using Separation of Variables

A **separable** 可分离 differential equation can be written with all the y's on one side and all the x's on the other, then integrated:

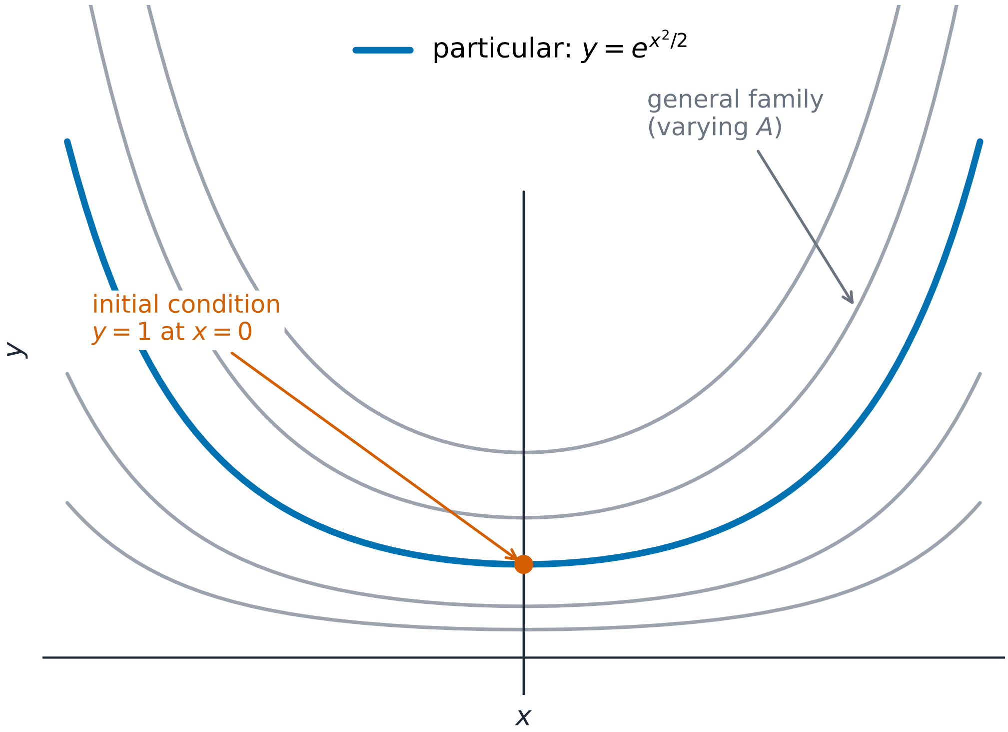
$$\frac{dy}{dx} = g(x)h(y) \;\Rightarrow\; \int \frac{dy}{h(y)} = \int g(x)\,dx.$$

This produces the general solution (with a $+C$), the main analytic method for solving differential equations in this course.

Worked example. Solve $\frac{dy}{dx} = xy$ with $y(0) = 2$. Separating, $\int \frac{dy}{y} = \int x\,dx$ gives $\ln|y| = \frac{x^2}{2} + C$, so $y = Ae^{x^2/2}$. The condition $y(0) = 2$ gives $A = 2$, so $y = 2e^{x^2/2}$.

Finding Particular Solutions Using Initial Conditions

An **initial condition** 初始条件 (a known point, e.g. $y(0) = 5$) pins down the constant C. Solve for the general solution, substitute the condition to find C, then write the particular solution. Watch the domain –a particular solution is valid only on the interval containing the initial point.



The constant gives a family of curves; an initial condition picks out one

Exponential Models with Differential Equations

The equation $\dfrac{dy}{dt} = ky$ says the rate of change is **proportional** to the amount –giving **exponential growth or decay** 指数增长. Separating variables yields

$$y = y_0 e^{kt},$$

with $k > 0$ for growth and $k < 0$ for decay. This models unrestricted population growth, radioactive decay, and continuously compounded interest.

Logistic Models with Differential Equations

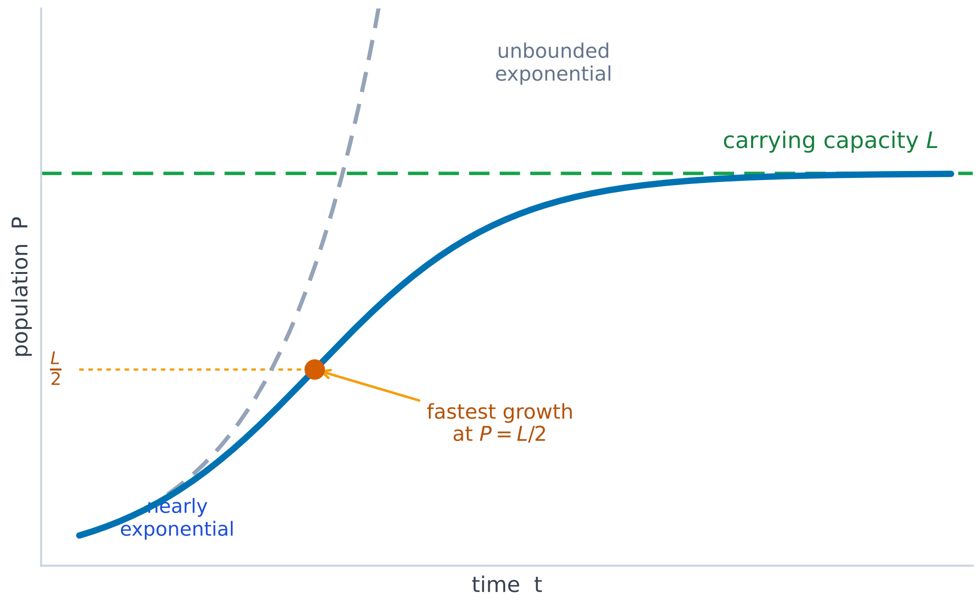
Real growth is limited by resources, so the **logistic model** 逻辑斯蒂模型 adds a **carrying capacity** 环境容纳量 L:

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{L}\right).$$

Growth is nearly exponential when P is small, slows as P approaches L, and stops at $P = L$. Key BC facts: the population **levels off at** L ($\lim_{t\to\infty} P = L$), and it grows **fastest when** $P = \frac{L}{2}$ (the inflection point of the S-shaped curve). You are expected to read L and the fastest-growth value directly from the equation.

Worked example. For $\dfrac{dP}{dt} = 0.05\,P\left(1 - \dfrac{P}{2000}\right)$, the carrying capacity is $L = 2000$

(the population levels off there), and growth is fastest when $P = \dfrac{L}{2} = 1000$ –both read straight off the equation, no solving needed.



The logistic model grows fastest at P=L/2 and levels off at the carrying capacity L

Exam tips

- Solve a **separable** equation by getting all y on one side and all x on the other, then integrating both sides (add $+C$ once).
- Use the initial condition to find C (a particular solution).
- Sketch or read a **slope field**: the little segments show $\frac{dy}{dx}$ at each point, and a solution curve follows them.
- Recognise **exponential** models $\frac{dy}{dt} = ky \Rightarrow y = Ce^{kt}$ (growth/decay).
- A differential equation gives the slope —you must integrate to recover the function.