Integration and Accumulation of Change

AP Calculus BC

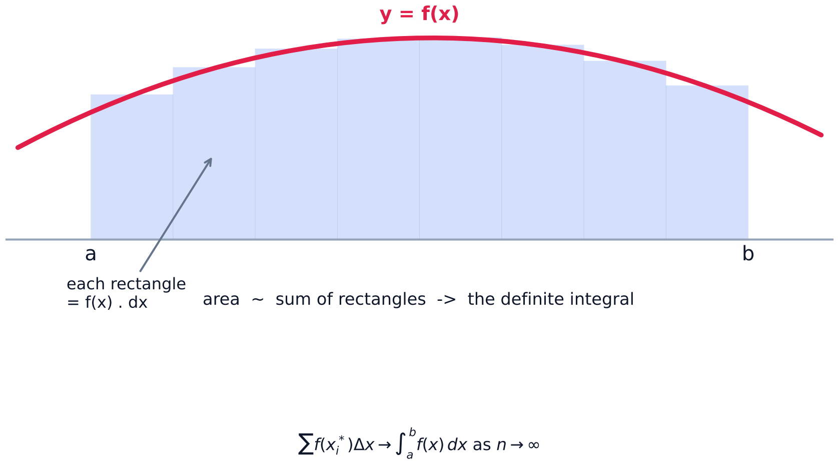
Exploring Accumulations of Change

A water tank: integration accumulates change — total volume from a rate of flow

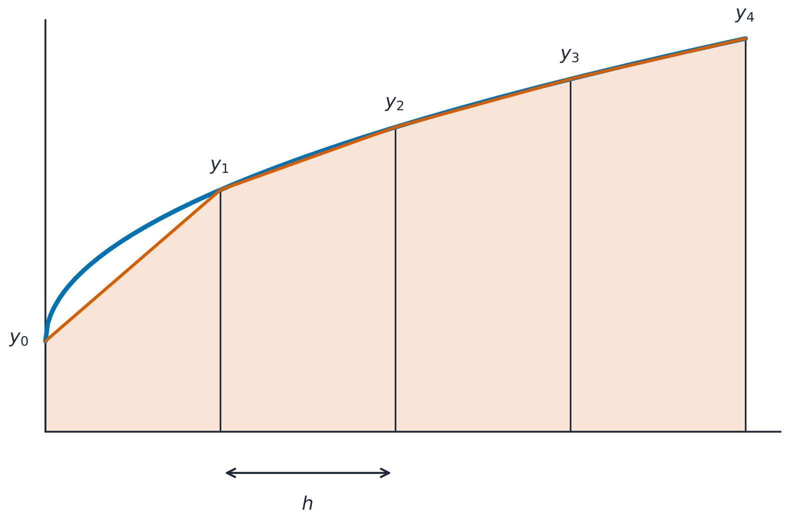
Where a derivative measures a **rate**, an **integral** 积分 measures an **accumulation** 累积 – a total built up from a rate. If a rate of change acts over an interval, the **area** between its graph and the axis gives the net accumulated change. This "area = total change" idea is the foundation of integral calculus.

Approximating Areas with Riemann Sums

A **Riemann sum** 黎曼和 estimates the area under a curve by adding the areas of thin rectangles. Split $[a, b]$ into subintervals and use the function's height at the **left** endpoint, **right** endpoint, or **midpoint** of each. A **trapezoidal sum** 梯形法 uses trapezoids instead, averaging the two endpoint heights – usually more accurate. With more, thinner rectangles the estimate improves.



A Riemann sum approximates the area under a curve with rectangles



Strips of width h approximate the area under a curve

Exam skill: be able to compute left, right, midpoint, and trapezoidal estimates from a table or graph, and state whether each **over- or under-estimates** based on whether the function is increasing/decreasing or concave up/down.

Riemann Sums, Summation Notation, and Definite Integral Notation

Writing a Riemann sum with **summation notation** $\sum_{k=1}^{n} f(x_k)\,\Delta x$ and letting the number of rectangles grow without bound gives the exact area – the **definite integral** 定积分:

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k)\,\Delta x.$$

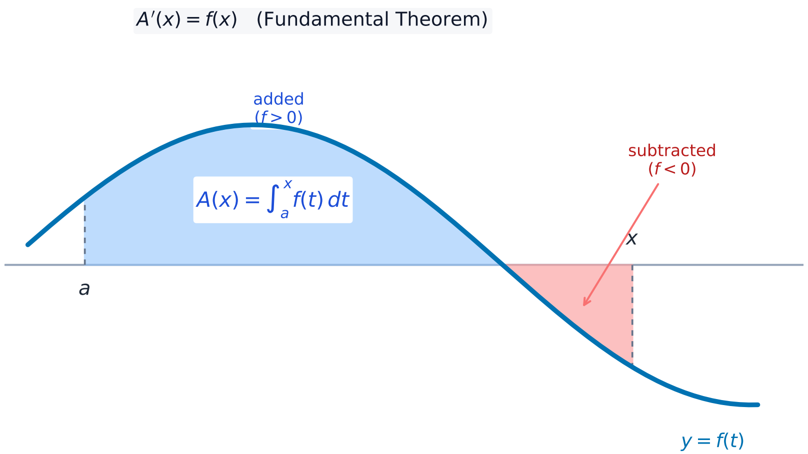
The integral is the **limit of Riemann sums**; a and b are the limits of integration.

The Fundamental Theorem of Calculus and Accumulation Functions

An **accumulation function** 累积函数 $g(x) = \int_a^x f(t)\,dt$ gives the accumulated area from a up to x. The **Fundamental Theorem of Calculus (FTC)** 微积分基本定理 says its derivative is the integrand:

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x).$$

Differentiation and integration are **inverse** operations. With a variable upper limit and the chain rule, $\dfrac{d}{dx}\int_a^{u(x)} f(t)\,dt = f(u(x))\,u'(x)$.



The accumulation function adds signed area; the FTC says its derivative is f

Interpreting the Behavior of Accumulation Functions

Because $g'(x) = f(x)$, the graph of f tells you everything about g: g **increases** where $f > 0$, **decreases** where $f < 0$, has **extrema** where f crosses zero, and is **concave up** where f is increasing. Reading these connections off a graph of f is a classic free-response task.

Applying Properties of Definite Integrals

Definite integrals obey useful rules: reversing the limits negates the value ($\int_b^a = -\int_a^b$), an integral over a zero-width interval is 0, they **add over adjacent intervals** ($\int_a^c = \int_a^b + \int_b^c$), and constants factor out. Use these to combine or split given integral values.

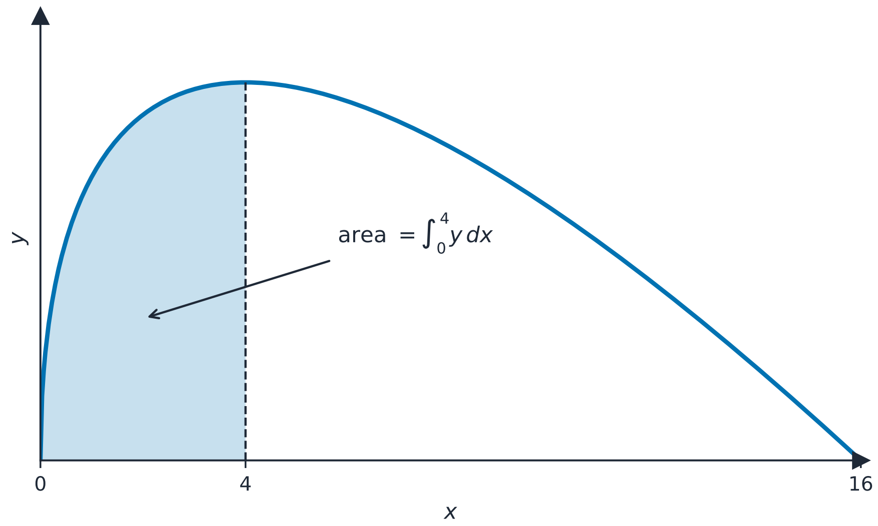
The Fundamental Theorem of Calculus and Definite Integrals

The **evaluation** form of the FTC computes a definite integral from an **antiderivative** 原函数 F (where $F' = f$):

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

So integrating a rate of change over $[a, b]$ gives the **net change** in the quantity – the single most-used result in the course.

Worked example. $\displaystyle\int_1^3 (2x+1)\,dx$: an antiderivative is $F(x) = x^2 + x$, so the value is $F(3) - F(1) = 12 - 2 = 10$.



A definite integral is the signed area between the curve and the x-axis

Finding Antiderivatives and Indefinite Integrals

An **indefinite integral** 不定积分 $\int f(x)\,dx = F(x) + C$ is the family of all antiderivatives (hence the **constant of integration** C). Reverse each derivative rule: the **power rule** becomes $\int x^n\,dx = \dfrac{x^{n+1}}{n+1} + C$ (for $n \neq -1$), with $\int \frac{1}{x}\,dx = \ln|x| + C$, and the antiderivatives of e^x, $\sin x$, $\cos x$, and $\sec^2 x$ come straight from their derivatives.

Integrating Using Substitution

u-substitution 换元积分 reverses the chain rule: choose $u = g(x)$ so that $g'(x)$ also appears, turning $\int f(g(x))g'(x)\,dx$ into $\int f(u)\,du$. Remember to convert dx to du and, for a definite integral, either change the limits to u-values or convert back to x at the end.

Integrating Using Long Division and Completing the Square

When a rational integrand is "top-heavy" (numerator degree ≥ denominator degree), **long division** rewrites it as a polynomial plus a proper fraction you can integrate. **Completing the square** in a denominator turns it into a form like $u^2 + a^2$, leading to an **arctangent** antiderivative $\frac{1}{a}\arctan\frac{u}{a} + C$.

Integrating Using Integration by Parts

Integration by parts 分部积分 reverses the product rule:

$$\int u\,dv = uv - \int v\,du.$$

Choose u to simplify when differentiated and dv to be easy to integrate (the **LIATE** guide: logs, inverse-trig, algebraic, trig, exponential). It handles products like $\int xe^x\,dx$ and $\int x\ln x\,dx$, sometimes applied twice.

Worked example. For $\int xe^x\,dx$, choose $u = x$ ($du = dx$) and $dv = e^x\,dx$ ($v = e^x$):

$$\int xe^x\,dx = xe^x - \int e^x\,dx = xe^x - e^x + C = e^x(x-1) + C.$$

Integrating Using Linear Partial Fractions

Partial fractions 部分分式 split a rational function with a factorable denominator into a sum of simpler fractions:

$$\frac{1}{(x-a)(x-b)} = \frac{A}{x-a} + \frac{B}{x-b},$$

each of which integrates to a **logarithm**. This technique is essential for the logistic differential equation in the next unit.

Evaluating Improper Integrals

An **improper integral** 反常积分 has an **infinite limit** of integration or an **infinite discontinuity** in the integrand. Evaluate it as a **limit**: $\int_a^\infty f\,dx = \lim_{b\to\infty}\int_a^b f\,dx$. If the limit is a finite number the integral **converges** 收敛; otherwise it **diverges** 发散.

Worked example. $\displaystyle\int_1^\infty \frac{1}{x^2}\,dx = \lim_{b\to\infty}\left[-\frac{1}{x}\right]_1^b = \lim_{b\to\infty}\left(1 - \frac{1}{b}\right) = 1$, so it **converges** to 1. By contrast $\int_1^\infty \frac{1}{x}\,dx$ gives $\lim_{b\to\infty}\ln b = \infty$ and **diverges** – the same integrand-shape can go either way.

Selecting Techniques for Antidifferentiation

The BC exam expects you to **recognize which method fits**: basic rules, substitution (a chain-rule pattern), by parts (a product), partial fractions (a factorable rational), or long division/completing the square. Being able to look at an integral and pick the right tool quickly is itself a tested skill.

Exam tips

- Integration is antidifferentiation; use the **power rule** $\int x^n\,dx = \frac{x^{n+1}}{n+1} + C$ and don't forget the $+C$.
- The **Fundamental Theorem** links the two: $\int_a^b f'(x)\,dx = f(b) - f(a)$, and $\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$.
- Approximate a definite integral with **Riemann sums** or the trapezoidal rule from a table of values.
- A definite integral is a signed area (below the axis counts negative); split at sign changes for total area.
- Use **u-substitution** and remember to change the limits (or back-substitute) accordingly.